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Nilpotent Elements and Nil-Reflexive Property of Generalized Power Series Rings

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Abstract

Let R be a ring and $(S,\leq)$ a strictly ordered monoid. In this paper, we deal with a new approach to reflexive property for rings by using nilpotent elements, in this direction we introduce the notions of generalized power series reflexive and nil generalized power series reflexive, respectively. We obtain various necessary or sufficient conditions for a ring to be generalized power series reflexive and nil generalized power series reflexive. Examples are given to show that, nil generalized power series reflexive need not be generalized power series reflexive and vice versa, and nil generalized power series reflexive but not semicommutative are presented. We proved that, if R is a left *APP*-ring, then R is generalized power series reflexive, and R is nil generalized power series reflexive if and only if R/I is nil generalized power series reflexive. Moreover, we investigate ring extensions which have roles in ring theory.

Keywords

Left *APP*-Ring, Generalized Power Series Reflexive Ring, Nil Generalized Power Series Reflexive Ring, S-Quasi Armendariz Ring, Semiprime Ring, Semicommutative Ring

1. Introduction

Throughout this article, all rings are associated with identity unless otherwise stated. Any concept and notation not defined here can be found in Ribenboim ([1] [2] [3] [4]), Elliott and Ribenboim [5]. Mason introduced the reflexive property for ideals, and this concept was generalized by some authors, defining idempotent reflexive right ideals and rings, completely reflexive rings, weakly reflexive rings (see namely, [6] [7] and [8]). Let R be a ring and I be a right ideal of R. In [7], I is called a reflexive right ideal if for any $x, y \in R$, $xRy \subseteq I$ implies $yRx \subseteq I$. The reflexive right ideal concept is also specialized to the zero ideal of a ring, namely, a ring R is called reflexive [7] if its zero ideal is reflexive and a ring R is called completely reflexive if for any $a, b \in R$, $ab = 0$ implies $ba = 0$. Completely reflexive rings are called reversible by Cohn in [9] and also studied in [10]. Reduced rings are completely reflexive and every completely reflexive ring is semicommutative. The notion of Armendariz ring is introduced by Rege and Chhawchharia (see [11]). They defined a ring R to be Armendariz if $f(x)g(x) = 0$ implies $a_i b_j = 0$, for all polynomials $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \in R[x]$. In [11], a ring R is called semicommutative if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator of R is an ideal of R.

An ideal I of a ring is called semiprime if $aRa \subseteq I$ implies $a \in I$ for $a \in R$ and R is called semiprime if 0 is a semiprime ideal. Note that every semiprime ideal is reflexive by a simple computation, and so every ideal of a fully idempotent ring (*i.e.*, $I^2 = I$ for every ideal I) is reflexive by [12]. The ring R is said to be weakly reflexive if $arb = 0$ implies bra is nilpotent for $a, b \in R$ and all $r \in R$. The rings without nonzero nilpotent elements are said to be reduced rings. In [13], semicommutativity of rings is generalized to nil-semicommutativity of rings. A ring R is called nil-semicommutative if $a, b \in R$ satisfy that ab is nilpotent, then $arb \in nil(R)$ for any $r \in R$ where $nil(R)$ denotes the set of all nilpotent elements of R. Clearly, every semicommutative ring is nil-semicommutative.

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S, \leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to Elliott and Ribenboim [5].

Let $(S, \leq)$ is a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[\![R^{S,\leq}]\!]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S \mid f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[\![R^{S,\leq}]\!]$ is an abelian additive group. For every $s \in S$ and $f, g \in [\![R^{S,\leq}]\!]$, let $X_s(f, g) = \{(u, v) \in S \times S \mid u + v = s, f(u) \neq 0, g(v) \neq 0\}$. It follows from Ribenboim ([4], 4.1) that $X_s(f, g)$ is finite. This fact allows one to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u) g(v).$$

Clearly, $supp(fg) \subseteq supp(f) + supp(g)$, thus by Ribenboim ([2], 3.4) $supp(fg)$ is artinian and narrow, hence $fg \in [\![R^{S,\leq}]\!]$. With this operation, and pointwise addition, $[\![R^{S,\leq}]\!]$ becomes an associative ring, with identity element e, namely $e(0)=1$, $e(s)=0$ for every $0\neq s\in S$. Which is called the ring of generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in Ribenboim ([1] [2] [3] [4]), Elliott and Ribenboim [5] and Varadarajan ([14] [15]). For example, if $S=\mathbb{N}\cup\{0\}$ and $\leq$ is the usual order, then $[\![R^{\mathbb{N}\cup\{0\},\leq}]\!]\cong R[\![x]\!]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[\![R^{S,\leq}]\!]\cong R[S]$, the monoid ring of S over R. Further examples are given in Ribenboim [2]. To any $r\in R$ and $s\in S$, we associate the maps $c_r, e_s\in[\![R^{S,\leq}]\!]$ defined by

$$c_r(x)=\begin{cases} r, & x=0,\\ 0, & \text{otherwise,}\end{cases}\qquad e_s(x)=\begin{cases} 1, & x=s,\\ 0, & \text{otherwise.}\end{cases}$$

It is clear that $r\mapsto c_r$ is a ring embedding of R into $[\![R^{S,\leq}]\!]$, $s\mapsto e_s$, is a monoid embedding of S into the multiplicative monoid of the ring $[\![R^{S,\leq}]\!]$, and $c_re_s=e_sc_r$. Recall that a monoid S is torsion-free if the following property holds: If $s,t\in S$, if k is an integer, $k\geq 1$ and $ks=kt$, then $s=t$.

We will write monoids multiplicatively unless otherwise indicated. If R is a ring and X is a nonempty subset of R, then the left (right) annihilator of X in R is denoted by $\ell_R(X)$ ($r_R(X)$).

Motivated by the works on reflexivity, in this note we study new two concepts of reflexive property, namely, generalized power series reflexive and nilpotent property of it. Examples are given that, nil generalized power series reflexive which is neither generalized power series reflexive nor semicommutative. Since, every reversible ring is semicommutative, but the converse need not hold by ([10], Lemma 1.4 and Example 1.5), so we proved that, under sufficient conditions the converse is hold. If R is a left *APP*-ring, then R is generalized power series reflexive. Also we prove that, R is nil generalized power series reflexive if and only if R/I is nil generalized power series reflexive and R is nil generalized power series reflexive if and only if $T_n(R)$ is nil generalized power series reflexive, when n is a positive integer. Moreover, we proved that every ring with identity is weakly generalized power series reflexive and when R is a left *p.q.*-Baer ring, then R is semiprime if and only if R is generalized power series reflexive. Also as a Corollary, of a ring R is nil reflexive of generalized power series and we give a lattice structure to the right (left) annihilators of a ring.

In what follows, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the ring of integers, and for a positive integer n, $\mathbb{Z}_n$ is the ring of integers modulo n. For a positive integer n, let $Mat_n(R)$ denote the ring of all $n\times n$ matrices and $T_n(R)$ the ring of all $n\times n$ upper triangular matrices with entries in R. We write $R[x]$, $P(R)$, and $S_n(R)$, for the polynomial ring over a ring R, the prime radical of R, and the subring consisting of all upper triangular matrices over a ring R with equal main diagonal entries.

2. Reflexive Rings of Generalized Power Series

According to [16], a ring R is called to be quasi-Armendariz if whenever polynomials $f(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$, $g(x)=b_0+b_1x+b_2x^2+\cdots+b_nx^n\in R[x]$ satisfy $f(x)R[x]g(x)=0$, then $a_iRb_j=0$ for each i,j. It was proved in ([10], Proposition 2.4) that if R is an Armendariz ring, then R is completely reflexive if and only if $R[x]$ is completely reflexive. In [17], for a torsion-free and cancellative monoid, a ring R is said to be S-quasi-Armendariz, if whenever $f,g\in[\![R^{S,\le}]\!]$ satisfy $f[\![R^{S,\le}]\!]g=0$, then $f(u)Rg(v)=0$ for each $u,v\in S$. We start by the first concept in this paper.

Definition 2.1. *Let S be a torsion-free and cancellative monoid, ≤ a strict order on S. A ring R is called generalized power series reflexive, if whenever* $f,g\in[\![R^{S,\le}]\!]$ *satisfy* $f[\![R^{S,\le}]\!]g=0$, *then* $g[\![R^{S,\le}]\!]f=0$.

The following result appeared in ([18], Lemma 2.1).

Lemma 2.2. *Let S be a torsion-free and cancellative monoid, ≤ a strict order on S. Then* $[\![R^{S,\le}]\!]$ *is reduced if and only if R is reduced.*

In [18], A ring R is called S-Armendariz ring, if for each $f,g\in[\![R^{S,\le}]\!]$ such that $fg=0$ implies that $f(u)g(v)=0$ for each $u,v\in S$ and it was shown that generalized power series rings over semicommutative rings are semicommutative. By ([17], Proposition 2.4) and ([18], Proposition 2.7), respectively.

Lemma 2.3. *Let* $(S,\le)$ *be a strictly ordered monoid and R be an S-Armendariz ring. Then R is semicommutative if and only if* $[\![R^{S,\le}]\!]$ *is semicommutative.*

Lemma 2.4. *Let S be a torsion-free and cancellative monoid, ≤ a strict order on S and R a reduced ring. Then R is an S-quasi-Armendariz.*

A ring R is symmetric if for all $a,b,c\in R$ we have $abc=0$ implies that $acb=0$. A ring R is called reversible if for all $a,b\in R$ we have $ab=0$ if and only if $ba=0$. Reversible rings were defined by Cohn in [9]. He shows that Kothe's conjecture is true for the class of reversible rings. Reversible rings are clearly reflexive. It is shown by ([8], Lemma 2.1) that a ring R is reflexive if and only if $IJ=0$ implies $JI=0$ for all ideals I,J of R. These arguments naturally give rise to extending the study of symmetric ring property to the lattice of ideals. A generalization of symmetric rings was defined by Camillo, Kwak and Lee in [19]. A ring R is called ideal-symmetric if $IJK=0$ implies $IKJ=0$ for all ideals I,J,K of R. It is obvious that semiprime rings are ideal-symmetric.

Theorem 2.5. *Let R be a ring,* $(S,\le)$ *be a strictly ordered monoid. Assume that R is a reduced S-quasi-Armendariz. Then we have:*

(1) R is reflexive if and only if $[\![R^{S,\le}]\!]$ is reflexive;

(2) R is ideal-symmetric if and only if $[\![R^{S,\le}]\!]$ is ideal-symmetric.

Proof. We only prove (2), because the proof of the other case is similar. Assume that R is ideal-symmetric and $f_1,f_2,f_3\in A=[\![R^{S,\le}]\!]$ are such that $f_1Af_2Af_3=0$. Since R is a reduced S-quasi-Armendariz, hence by Lemma 2.2 and Lemma 2.4, we have $f_1(u)Rf_2(v)Rf_3(w)=0$ for all $u,v,w\in S$. Since R is ideal-symmetric, we have $f_1(u)Rf_3(w)Rf_2(v)=0$ for each $u,v,w\in S$. Now, reduced of R implies that, $f_1Af_3Af_2=0$. Hence $[\![R^{S,\le}]\!]$ is ideal-symmetric. Conversely, suppose that A is ideal-symmetric. Let $aRbRc=0$ for all $a,b,c\in R$.

Since R is reduced, $caAcbAcc=0$. Thus $caAccAcb=0$ and $aRcRb=0$ for all $a,b,c\in R$. Therefore, R is ideal-symmetric. □

Corollary 2.6. *Let S be a torsion-free and cancellative monoid, ≤ a strict order on S, and R a reduced ring. Then R is generalized power series reflexive rings.*

Proposition 2.7. *Let S be a torsion-free and cancellative monoid, ≤ a strict order on S. If R is reduced semicommutative ring, then R is an S-Armendariz if and only if R is generalized power series reflexive rings.*

Proof. Apply Lemma 2.3 and Lemma 2.4. □

Corollary 2.8. *Let* $(S,\le)$ *a strictly totally ordered monoid. A ring R is a completely reflexive ring if and only if R is generalized power series semicommutative reflexive.*

An ideal I of R is said to be right *s*-unital if, for each $a\in I$ there exists an element $e\in I$ such that $ae=a$. Note that if I and J are right *s*-unital ideals, then so is $I\bigcap J$ (if $a\in I\bigcap J$, then $a\in aIJ\subseteq a(I\bigcap J)$).

The following result follows from Tominaga ([20], Theorem 1).

Lemma 2.9. *An ideal I of a ring R is left* (*resp. right*) *s-unital if and only if for any finitely many elements* $a_1,a_2,\cdots,a_n\in I$, *there exists an element* $e\in I$ *such that* $a_i=ea_i$ (*resp.* $a_i=a_ie$) *for each* $i=1,2,\cdots,n$.

Clark defined quasi-Baer rings in [21]. A ring R is called quasi-Baer if the left annihilator of every left ideal of R is generated by an idempotent. Note that this definition is left-right symmetric. Some results of a quasi-Baer ring can be found in [21] and [22] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. A ring R is called a right (resp., left) *PP*-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). A ring R is called a *PP*-ring (also called a Rickart ring ([23], p. 18])) if it is both right and left *PP*. We say a ring R is a left *APP*-ring if the left annihilator $l_R(Ra)$ is right *s*-unital as an ideal of R for any element $a\in R$.

Proposition 2.10. *Let* $(S,\le)$ *a strictly totally ordered monoid. If R is a reduced left APP-ring, then R is generalized power series reflexive.*

Proof. Let $0\ne f,g\in[\![R^{S,\le}]\!]$ with $f[\![R^{S,\le}]\!]g=0$. We use the transfinite induction to show that $f(u)Rg(v)=0$ for all $u,v\in S$. Assume that $\pi(f)=u_0$, $\pi(g)=v_0$. Let $(u,v)\in X_{u_0+v_0}(f,g)$. So $u_0\le u$ and $v_0\le v$. If $u_0<u$, then $u_0+v_0<u+v_0\le u+v=u_0+v_0$, a contradiction. Thus $u=u_0$. Similarly, $v=v_0$. So $X_{u_0+v_0}(f,g)=\{(u_0,v_0)\}$. Hence for any $t\in R$, from $f[\![R^{S,\le}]\!]g=0$ we have,

$$0=(fc_tg)(u_0+v_0)=\sum_{(u,v)\in X_{u_0+v_0}(f,c_tg)}f(u)tg(v)=f(u_0)tg(v_0).$$

So $f(u_0)Rg(v_0)=0$. Now, let $\lambda\in S$ with $u_0+v_0\le\lambda$ and assume that for any $u\in supp(f)$ and any $v\in supp(g)$, if $u+v<\lambda$, then $f(u)Rg(v)=0$. We claim that $f(u)Rg(v)=0$, for each $u\in supp(f)$ and each $v\in supp(g)$ with $u+v=\lambda$. For convenience, we write

$X_\lambda(f,g)=\{(u,v)\mid u+v=\lambda, u\in supp(f), v\in supp(g)\}$ as $\{(u_i,v_i)\mid i=1,2,\cdots,n\}$ such that $u_1<u_2<\cdots<u_n$, where n is a positive integer (Note that if $u_1=u_2$, then from $u_1+v_1=u_2+v_2$ we have $v_1=v_2$, and then $(u_1,v_1)=(u_2,v_2)$). Since $f[[R^{S,\le}]]g=0$, for any $t\in R$ we have:

$$0=(fc_tg)(\lambda)=\sum_{(u,v)\in X_\lambda(f,c_tg)} f(u)tg(v)=\sum_{i=1}^{n} f(u_i)tg(v_i). \tag{3}$$

Let $e_{u_1}\in t_R(f(u_1)R)$. So $f(u_1)Re_{u_1}=0$ and which implies $f(u_1)Re_{u_1}g(v_1)=0$. Let $t'\in R$ be an arbitrary element. Then we have $f(u_1)t'e_{u_1}g(v_1)=0$. Take $t=t'e_{u_1}$ in Equation (3). Thus,

$$\sum_{i=2}^{n} f(u_i)t'e_{u_1}g(v_i)=0.$$

Note that $u_1+v_i<u_i+v_i=\lambda$ for any $i\ge 2$. So by compatibility and induction hypothesis, $f(u_1)Rg(v_i)=0$ for each $i\ge 2$. Since R is right *APP*, $r_R(f(u_1)R)$ is left *s*-unital. So without lose of generality and using Lemma 2.9, we can assume that $g(v_i)=e_{u_1}g(v_i)$, for each $i\ge 2$. Therefore

$$\sum_{i=2}^{n} f(u_i)t'g(v_i)=0. \tag{4}$$

Let $e_{u_2}\in r_R(f(u_2)R)$. So $f(u_2)Re_{u_2}=0$ and then $f(u_2)Re_{u_2}g(v_2)=0$. This implies $f(u_2)Re_{u_2}g(v_2)=0$.

Let $q\in R$ be an arbitrary element. So $f(u_2)qe_{u_2}g(v_2)=0$. Also note that $u_2+v_i<u_i+v_i=\lambda$ for any $i\ge 3$. So by induction hypothesis, $f(u_2)Rg(v_i)=0$. Therefore $g(v_i)\in r_R(f(u_2)R)$, for each $i\ge 3$. Since $r_R(f(u_2)R)$ is left *s*-unital, without lose of generality and using Lemma 2.9, again we can assume that $g(v_i)=e_{u_2}g(v_i)$, for each $i\ge 3$. Take $t'=qe_{u_2}$ in Equation (4), so we have:

$$\sum_{i=2}^{n} f(u_i)qe_{u_2}g(v_i)=0. \tag{5}$$

Continuing in this manner, we have $f(u_n)pg(v_n)=0$, where p is an arbitrary element of R. Thus $f(u_n)Rg(v_n)=0$. Hence $f(u_{n-1})Rg(v_{n-1})=0$, $\cdots$, $f(u_2)Rg(v_2)=0$, $f(u_1)Rg(v_1)=0$. Therefore, by transfinite induction, $f(u)Rg(v)=0$ for any $u,v\in S$. By Lemma 2.2, $g(v)Rf(u)=0$. Thus $g[[R^{S,\le}]]f=0$, the proof is done. □

Corollary 2.11. *Let* $(S,\le)$ *a strictly totally ordered monoid. If I is a finitely generated left ideal of R then for all* $a\in l_R(I)$, $a\in al_R(I)$. *So R is generalized power series reflexive.*

Proof. By Proposition 2.10 and ([24], Proposition 2.6). □

Corollary 2.12. *Let* $(S,\le)$ *a strictly totally ordered monoid. If R is a Baer ring. Then R is generalized power series reversible if and only if R is generalized power series reflexive.*

It is obvious that commutative rings are symmetric and symmetric rings are reversible, but the converses do not hold by ([25], Examples I.5 and II.5) and ([26], Examples 5 and 7). Every reversible ring is semicommutative, but the converse need not hold by ([10], Lemma 1.4 and Example 1.5). On the other hand, it can be easily checked that reversible rings are reflexive, and hence there exists a reflexive and semicommutative ring which is not symmetric by ([26], Examples 5 and 7). However, we have the following which is a direct consequence of routine computations.

Proposition 2.13. *Let* $(S,\leq)$ *a strictly totally ordered monoid. Then R is generalized power series semicommutative and reflexive if and only if R is generalized power series reversible.*

A ring R is called semiprime if for any $a \in R$, $aRa = 0$, implies $a = 0$. Let R be a ring and $(S,\leq)$ a strictly totally ordered monoid. A ring R is called *S*-semiprime if $f[[R^{S,\leq}]]f = 0$, then $f = 0$ for each $f \in [[R^{S,\leq}]]$.

The following result appeared in ([27], Lemma 2.7).

Lemma 2.14. *Let R be a ring and* $(S,\leq)$ *a strictly totally ordered monoid. Then R is a semiprime ring if and only if* $[[R^{S,\leq}]]$ *is a semiprime ring.*

Proposition 2.15. *Let* $(S,\leq)$ *be a strictly totally ordered monoid. Assume that R is semiprime. Then R is reflexive ring if and only if* $[[R^{S,\leq}]]$ *is reflexive.*

Proof. Since, semiprime is quasi-Armendriz, and so reflexive. Thus by Lemma 2.14 and ([8], Lemma 2.1), the proof it follows from Theorem 2.5. □

Corollary 2.16. ([7], *Proposition* 3.2) *Let R be a quasi-Armendariz ring, then the following statements are equivalent*:

(1) *R is reflexive.*

(2) $R[x]$ *is reflexive.*

(3) $R[x;x^{-1}]$ *is reflexive.*

Theorem 2.17. *Let* $(S,\leq)$ *be a strictly totally ordered monoid and R be a left p.q.-Baer ring. Then the following conditions are equivalent*:

(1) *R is a semiprime ring*;

(2) *R is generalized power series reflexive ring*;

(3) *R is a right idempotent generalized power series reflexive ring*;

(4) *R is a left idempotent generalized power series reflexive ring*; *and*

(5) $S_l(R) = B(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) are obvious. (3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (5) analog with the proof of ([8], Proposition 3.15). □

Let I be an index set and R_i be a ring for each $i \in I$. Let $(S,\leq)$ be a strictly ordered monoid, if there is an injective homomorphism $f: R \to \prod_{i\in I} R_i$ such that, for each $j \in I$, $\pi_j f: R \to R_j$ is a surjective homomorphism, where $\pi_j: \prod_{i\in I} R_i \to R_j$ is the *j*th projection. We have the following.

Proposition 2.18. *Let* R_i *be a ring,* $(S,\leq)$ *a strictly totally ordered monoid, for each i in a finite index set I. If* R_i *is generalized power series reflexive ring. for each i, then* $R = \prod_{i\in I} R_i$ *is generalized power series reflexive ring.*

Proof. Let $R = \prod_{i\in I} R_i$ be the direct product of rings $(R_i)_{i\in I}$ and R_i is generalized power series reflexive, for each $i \in I$. Denote the projection $R \to R_i$ as Π_i. Suppose that $f, g \in [[R^{S,\leq}]]$ are such that $f[[R^{S,\leq}]]g = 0$. Set $f_i = \prod_i f$, $g_i = \prod_i g$ and $h_i = \prod_i h$. Then $f_i, g_i \in [[R_i^{S,\leq}]]$. For any $u, v \in S$, assume $f(u) = (a_i^u)_{i\in I}$, $g(v) = (b_i^v)_{i\in I}$. Now, for any $h \in [[R^{S,\leq}]]$, any $r \in R$ and any

$s \in S$,

$$\begin{aligned}(fc_r g)(s) &= \sum_{(u,v)\in X_s(f,c_r g)} f(u) r g(v) \\ &= \sum_{(u,v)\in X_s(f,c_r g)} \left(a_i^u\right)_{i\in I} \left(r_i\right)_{i\in I} \left(b_i^v\right)_{i\in I} \\ &= \sum_{(u,v)\in X_s(f,c_r g)} \left(\left(a_i^u\right) r_i \left(b_i^v\right)\right)_{i\in I} \\ &= \sum_{(u,v)\in X_s(f,c_r g)} \left(f_i(u) r_i g_i(v)\right)_{i\in I} \\ &= \left(\sum_{(u,v)\in X_s(f_i,c_{r_i} g_i)} f_i(u) r_i g_i(v)\right)_{i\in I} \\ &= \left(\left(f_i h_i g_i\right)(s)\right)_{i\in I}.\end{aligned}$$

Since $(fc_r g)(s) = 0$ we have

$$\left(f_i c_{r_i} g_i\right)(s) = 0.$$

Thus, $f_i h_i g_i = 0$. Now it follows $f_i(u) r_i g_i(v) = 0$ for any $r \in R$, any $u, v \in S$ and any $i \in I$, since R_i is generalized power series reflexive. Hence, for any $u, v \in S$,

$$f(u) r g(v) = \left(f_i(u)(r_i) g_i(v)\right)_{i\in I} = 0$$

since I is finite. Thus, $f(u) R g(v) = 0$. Then by reflexive ring, we have

$$\left(g_i(v)(r_i) f_i(u)\right)_{i\in I} = g(v) R f(u) = 0.$$

This means that $g\left[\!\left[R^{S,\le}\right]\!\right] f = 0$. Therefore R is generalized power series reflexive. □

In the following we study ring theoretic properties and extensions related to the right (left) annihilators of a generalized power series reflexive rings.

Let $\gamma = C(f)$ be the content of *f*, *i.e.*, $C(f) = \{f(u) \mid u \in supp(f)\} \subseteq R$. Since, $R \simeq c_R$ we can identify, the content of *f* with

$$c_{C(f)} = \left\{c_{f(u_i)} \mid u_i \in supp(f)\right\} \subseteq \left[\!\left[R^{S,\le}\right]\!\right].$$

Lemma 2.19. ([28], *Lemma* 2.1) *Let R be a ring, S a strictly ordered monoid,* $\left[\!\left[R^{S,\le}\right]\!\right]$ *the generalized power series ring and* $U \subseteq R$. *Then*

$$\left[\!\left[R^{S,\le}\right]\!\right] \ell_R(U) = \ell_{\left[\!\left[R^{S,\le}\right]\!\right]}(U), \left(r_R(U)\left[\!\left[R^{S,\le}\right]\!\right] = r_{\left[\!\left[R^{S,\le}\right]\!\right]}(U)\right).$$

By Lemma 2.19 we have two maps $\phi : rAnn_R(id(R)) \to rAnn_{\left[\!\left[R^{S,\le}\right]\!\right]}\left(id\left(\left[\!\left[R^{S,\le}\right]\!\right]\right)\right)$ and $\psi : lAnn_R(id(R)) \to lAnn_{\left[\!\left[R^{S,\le}\right]\!\right]}\left(id\left(\left[\!\left[R^{S,\le}\right]\!\right]\right)\right)$ defined by $\phi(I) = I\left[\!\left[R^{S,\le}\right]\!\right]$ and $\psi(J) = \left[\!\left[R^{S,\le}\right]\!\right] J$ for every $I \in rAnn_R(id(R)) = \{r_R(U) \mid U \text{ is an ideal of } R\}$ and $J \in lAnn_R(id(R)) = \{l_R(U) \mid U \text{ is an ideal of } R\}$, respectively. Obviously, ϕ is injective. In the following Theorem we show that ϕ and ψ are bijective maps if and only if R is generalized power series reflexive.

Theorem 2.20. *Let R be a reduced ring, S a strictly ordered monoid and* $\left[\!\left[R^{S,\le}\right]\!\right]$ *the generalized power series. Then the following are equivalent*:

(1) *R is generalized power series reflexive ring.*

(2) *The function* $\phi: rAnn_R(id(R)) \to rAnn_{[\![R^{S,\le}]\!]}(id([\![R^{S,\le}]\!]))$ *is bijective, where* $\phi(I) = I[\![R^{S,\le}]\!]$.

(3) *The function* $\psi: lAnn_R(id(R)) \to lAnn_{[\![R^{S,\le}]\!]}(id([\![R^{S,\le}]\!]))$ *is bijective, where* $\psi(J) = [\![R^{S,\le}]\!]J$.

Proof. (1) $\Rightarrow$ (2) Let $Y \subseteq [\![R^{S,\le}]\!]$ and $\gamma = \bigcup_{f\in Y} C(f)$. From Lemma 2.19 it is sufficient to show that $r_{[\![R^{S,\le}]\!]}(f) = r_R C(f)[\![R^{S,\le}]\!]$ for all $f \in Y$. In fact, let $g \in r_{[\![R^{S,\le}]\!]}(f)$ and for any $h \in [\![R^{S,\le}]\!]$. Then $fhg = 0$ and by assumption $f(u_i)tg(v_j) = 0$ for each $u_i \in supp(f), t \in R$ and each $v_j \in supp(g)$. Then for a fixed $u_i \in supp(f), t \in R$ and each $v_j \in supp(g)$, $0 = f(u_i)tg(v_j) = (c_{f(u_i)}c_t g)(v_j)$ and it follows that $g \in r_R \bigcup_{u_i\in supp(f)} c_{f(u_i)}c_t[\![R^{S,\le}]\!] = r_R C(f)[\![R^{S,\le}]\!]$. So $r_{[\![R^{S,\le}]\!]}(f) \subseteq r_R C(f)[\![R^{S,\le}]\!]$.

Conversely, let $g \in r_R C(f)[\![R^{S,\le}]\!]$, then $c_{f(u_i)}c_t g = 0$ for each $u_i \in supp(f), t \in R$. Hence, $0 = (c_{f(u_i)}c_t g)(v_j) = f(u_i)tg(v_j)$ for each $u_i \in supp(f), t \in R$ and $v_j \in supp(g)$. Thus,

$$(fhg)(s) = \sum_{(u_i,v_j)\in X_s(f,c_t g)} f(u_i)tg(v_j) = 0$$

and it follows that $g \in r_{[\![R^{S,\le}]\!]}(f)$. Hence $r_R C(f)[\![R^{S,\le}]\!] \subseteq r_{[\![R^{S,\le}]\!]}(f)$ and it follows that $r_R C(f)[\![R^{S,\le}]\!] = r_{[\![R^{S,\le}]\!]}(f)$. So

$$r_{[\![R^{S,\le}]\!]}(Y) = \bigcap_{f\in Y} r_{[\![R^{S,\le}]\!]}(f) = \bigcap_{f\in Y} r_R C(f)[\![R^{S,\le}]\!] = r_R(\gamma)[\![R^{S,\le}]\!].$$

(2) $\Rightarrow$ (1) Suppose that $f, g \in [\![R^{S,\le}]\!]$ be such that $f[\![R^{S,\le}]\!]g = 0$. Then $g \in r_{[\![R^{S,\le}]\!]}(f)$ and by assumption $r_{[\![R^{S,\le}]\!]}(f) = \gamma[\![R^{S,\le}]\!]$ for some right ideal γ of *R*. Consequently, $0 = fc_t c_{g(v_j)}$ and for any $u_i \in supp(f)$, $0 = (fc_t c_{g(v_j)})(u_i) = f(u_i)tg(v_j)$ for each $u_i \in supp(f), t \in R$ and $v_j \in supp(g)$. Thus by reduced ring, $g(v_j)tf(u_i) = 0$, then $g[\![R^{S,\le}]\!]f = 0$. Hence, R is generalized power series reflexive ring. The proof of (1) $\Leftrightarrow$ (3) is similar to the proof of (1) $\Leftrightarrow$ (2). □

Definition 2.21. *A submodule N of a left R-module M is called a pure submodule if* $L \otimes_R N \to L \otimes_R M$ *is a monomorphism for every right R-module L. By* ([29], *Proposition* 11.3.13), *for an ideal I, the following conditions are equivalent*:

(1) *I is right s-unital*;

(2) *R*/*I is flat as a left R-module*;

(3) *I is pure as a left ideal of R.*

Theorem 2.22. *Let R be a reduced ring,* $(S,\le)$ *a strictly totally ordered monoid. Then the following statements are equivalent*:

(1) $r_R(aR)$ *is pure as a right ideal in R for any element* $a \in R$;

(2) $r_{[\![R^{S,\le}]\!]}(f[\![R^{S,\le}]\!])$ *is pure as a right ideal in* $[\![R^{S,\le}]\!]$ *for any element* $f \in [\![R^{S,\le}]\!]$.

In this case R is generalized power series reflexive ring.

Proof. Assume that the condition (1) holds. Firstly, by using the same method of the proof of Proposition 2.10 we can proved that R is generalized power series reflexive. Finally, by using Lemma 2.9 we can see that the condition (2) holds.

Conversely, suppose that the condition (2) holds. Let a be an element of R. Then $r_{[\![R^{S,\leq}]\!]}\left(a[\![R^{S,\leq}]\!]\right)$ is left s-unital. Hence, for any $b \in r_R(aR)$, there exists an element $f \in [\![R^{S,\leq}]\!]$ such that $bf = b$. Let $f(0)$ be the constant term of f. Then $f(0) \in r_R(aR)$ and $f(0)b = b$. This implies that $r_R(aR)$ is left s-unital. Therefore condition (1) holds. □

3. Nil Generalized Power Series Reflexive Rings

In this section, we first give the following concept, so called nil generalized power series reflexive, that is a generalization of generalized power series reflexive rings and study the relations between nil generalized power series reflexive and some certain classes of rings.

Definition 3.1. *Let $(S,\leq)$ be a strictly ordered monoid. A ring R is called nil generalized power series reflexive if whenever $f,g \in [\![R^{S,\leq}]\!]$ satisfy $fhg \in [\![nil(R)^{S,\leq}]\!]$ implies $ghf \in [\![nil(R)^{S,\leq}]\!]$ for each $h \in [\![R^{S,\leq}]\!]$.*

Let $S = (\mathbb{N}\bigcup\{0\},+)$, and $\leq$ is the usual order. Then $[\![R^{S,\leq}]\!] \cong R[\![x]\!]$. So the ring R is nil generalized power series reflexive if and only if R is nil power series reflexive. Hence a nil generalized power series reflexive is a generalization of nil power series reflexive and power series reflexive. In the next, we provide some examples for nil generalized power series reflexive rings. It is show that, nil generalized power series reflexive need not be generalized power series reflexive. In ([8], Theorem 2.6), Kwak and Lee proved that R is a reflexive ring if and only if $Mat_n(R)$ is a reflexive ring for all $n \geq 1$. However, this is not the case in nil generalized power series reflexive of R. There are nil generalized power series reflexive over which matrix rings need not be nil generalized power series reflexive as shown below.

Example 3.2. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. Then*

(1) *If R be a ring with $nil(R)$ an ideal of R. Then R is nil generalized power series reflexive.*

(2) *For any reduced ring R, the ring $T_n(R)$ is nil generalized power series reflexive. However, the ring of all 2×2 matrices over any field and satisfying the condition that $0 \leq s$ for every $s \in S$ is not nil generalized power series reflexive.*

(3) *For R be a reduced ring. Consider the ring*

$$S_n(R) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R; 1 \leq i,j \leq n \right\}.$$

Then $S_n(R)$ is not generalized power series reflexive, when $n \geq 4$, but $S_n(R)$ and R are nil generalized power series reflexive for all $n \geq 1$.

Proof. (1) Assume that $f,g \in [\![R^{S,\le}]\!]$, with fhg is nilpotent for all $h \in [\![R^{S,\le}]\!]$. So there exists a positive integer n such that $(fhg)^n = 0$. Therefore $\left(f(u)h(w)g(v)\right)^n = 0$, for any $u,v,w \in S$. Then $f(u)h(w)g(v) \in nil(R)$ and so $g(v)h(w)f(u)g(v)h(w)f(u) \in nil(R)$. Hence $g(v)h(w)f(u)$ is nilpotent. Thus, ghf is nilpotent.

(2) For a ring R, by [30],

$$nil\left(T_n(R)\right) = \begin{pmatrix} nil(R) & R & R & \cdots & R \\ 0 & nil(R) & R & \cdots & R \\ 0 & 0 & nil(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & nil(R) \end{pmatrix}.$$

Let R be a reduced ring. Then $nil(R) = 0$ and so $nil\left(T_n(R)\right)$ is an ideal. By (1), $T_n(R)$ is nil generalized power series reflexive. Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in Mat_2(F),$$

where F is a field. For any $C \in Mat_2(F)$, $0 \ne s \in S$. $ACB = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$ is nilpotent, but $BCA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not nilpotent. Therefore $Mat_2(F)$ is not nil generalized power series reflexive.

(3) By the same argument as in ([8], Example 2.3) that $S_n(R)$ is not generalized power series reflexive when $n \ge 4$. Since R is reduced, R is nil generalized power series reflexive. Note that

$$nil\left(S_n(R)\right) = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a \in nil(R), a_{ij} \in R; 1 \le i,j \le n \right\}.$$

The ring R being reduced implies that $nil\left(S_n(R)\right)$ is an ideal. By (1), $S_n(R)$ is nil generalized power series reflexive. □

By Example 3.2(2), for n by n upper triangular matrix ring over R. It is easy to verify the next proposition.

Proposition 3.3. *Let R be a ring, $(S,\le)$ a strictly ordered monoid. A ring R is nil generalized power series reflexive if and only if $T_n(R)$ is nil generalized power series reflexive, for any positive integer n.*

Proof. Suppose that $T_n(R)$ is nil generalized power series reflexive. Note that R is isomorphic to the subring of $T_n(R)$. Thus R is nil generalized power series reflexive, since each subring of nil generalized power series reflexive ring is also nil generalized power series reflexive. For the forward implication, let $f,g \in [\![T_n(R)^{S,\le}]\!]$ such that $fhg \in [\![nil\left(T_n(R)^{S,\le}\right)]\!]$, where $f = f_{ij}(u), h = h_{ij}(w)$ and $g = g_{ij}(v)$, for all $u,w,v \in S$ and (i,j)th entry of

$f(u), h(w)$ and $g(v)$ respectively. Since $\left[\!\left[nil\left(T_n(R)^{S,\leq}\right) \right]\!\right] = \left\{ (a_{ij}) \mid a_{ij} \in nil(R) \right\}$, then we have $f_{ii}h_{ii}g_{ii} \in nil(R)$ for each $1 \leq i \leq n$. Since R is nil generalized power series reflexive, there exists some positive integer $m_{u,w,v,i}$ such that $\left(f_{ii}(u)h_{ii}(w)g_{ii}(v)\right)^{m_{u,w,v,i}} = 0$. Let $m_{u,w,v} = \max\left\{ m_{u,w,v,i} \mid 1 \leq i \leq n \right\}$. Then $f_{ii}(u)h_{ii}(w)g_{ii}(v)$ is nilpotent and so $g_{ii}(v)h_{ii}(w)f_{ii}(u)$ is nilpotent. Therefore, $T_n(R)$ is nil generalized power series reflexive. □

Now we shall give an example to show that there exists a nil generalized power series reflexive ring which is not generalized power series reflexive. Also generalized power series reflexive rings may not be nil generalized power series reflexive either as shown below.

Example 3.4. *There exists a nil generalized power series reflexive ring which is neither generalized power series reflexive nor semicommutative.*

Proof. Let R be a reduced ring. By Examples 3.2(2), $T_2(R)$ is nil generalized power series reflexive. On the other hand, $nil(T_2(R)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}$. Consider $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in T_2(R)$. Then $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R & R \\ 0 & R \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq 0$ for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in R$. This shows that $T_2(R)$ is not generalized power series reflexive. $T_2(R)$ is also not semicommutative (by Lemma 2.3). For if, $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T_2(R)$, then $AB = 0$, but $ACB \neq 0$. □

It is shown by ([8], Lemma 2.1) that a ring R is reflexive if and only if $IJ = 0$ implies $JI = 0$ for all ideals I, J of R.

Lemma 3.5. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. For a ring R, consider the following conditions.*

(1) *R is nil generalized power series reflexive.*

(2) *If ARB is a nil set, then so is BRA for any subsets A, B of R.*

(3) *If IJ is nil, then JI is nil for all right (or left) ideals I, J of R.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Assume that R is nil generalized power series reflexive. Let A, B be two nonempty subsets of R with ARB is a nil set. For any $f \in A$ and $g \in B$, fhg is nilpotent for all $h \in R$. Then ghf is nilpotent. This implies that BRA is nil.

(2) $\Rightarrow$ (3) Let I and J be any right ideals of R such that IJ is nil. Since $IR \subseteq I$, IRJ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. Assume that I and J be any left ideals of R such that IJ is nil. Since $RJ \subset J$ and then $IRJ \subseteq IJ$, IRJ is nil. By (2), JRI is nil. Since $JI \subseteq JRI$, we get JI is nil. □

Lemma 3.6. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S. The following conditions are equivalent for a ring $R, u \in S$.*

(1) *$f(u)R \subseteq nil(R)$ for any $f(u) \in nil(R)$.*

(2) $Rf(u) \subseteq nil(R)$ *for any* $f(u) \in nil(R)$.

Proof. (1) $\Rightarrow$ (2) Assume that $f(u)h(w) \in nil(R)$ for all $h(w) \in R$, for any $u,v,w \in S$, for any $f(u) \in nil(R)$. Let $(f(u)h(w))^n = 0$ for some positive integer *n*. Then $(h(w)f(u))^{n+1} = 0$, hence $h(w)f(u)$ is nilpotent. Thus $Rf(u) \subseteq nil(R)$. Similarly, we can show (2) $\Rightarrow$ (1). □

The next result gives a source of nil generalized power series reflexive.

Proposition 3.7. *Let S be a torsion-free and cancellative monoid,* ≤ *a strict order on S and R be a ring such that* $f(u)R \subseteq nil(R)$ *for any* $f(u) \in nil(R)$, *for each* $u \in S$. *Then R is nil generalized power series reflexive.*

Proof. Assume that $f,g \in [[R^{S,\le}]]$, with $fhg \in [[nil(R)^{S,\le}]]$ for any $h \in [[R^{S,\le}]]$. So there exists a positive integer *n* such that $(fhg)^n = 0$. Therefore $(f(u)h(w)g(v))^n = 0$, for any $u,v,w \in S$. So $f(u)h(w)g(v) \in nil(R)$, by hypothesis, $f(u)g(v)R \subseteq nil(R)$. Hence $g(v)h(w)f(u) \in nil(R)$ for any $h(w) \in R$. Thus, $ghf \in [[nil(R)^{S,\le}]]$. □

According to ([31], Lemma 2.7). If *R* is S_i-compatible for each *i*, then *R* is *S*-compatible. By ([32], Lemma 3.1), in a semicommutative ring *R*, $nil(R)$ is an ideal of *R*. In ([7], Example 2.1) shows that any semicommutative ring need not be reflexive, but this is not the case when we deal with nil-reflexive rings. It can be observed that every semicommutative ring is nil generalized power series reflexive as a consequence of Proposition 3.7. But we give its direct proof in the next.

In [7], weakly reflexive rings are studied in detail for rings having an identity. However weakly reflexive rings are nothing but all rings with identity as it is shown below.

Lemma 3.8. *Let S be a torsion-free and cancellative monoid,* ≤ *a strict order on S. Then every ring with identity is nil generalized power series reflexive.*

Proof. Let $f,g \in [[R^{S,\le}]]$, with $fhg = 0$, for all $h \in [[R^{S,\le}]]$. Then $f(u)h(w)g(v) = 0$, for any $u,v,w \in S$. and so $(g(v)h(w)f(u))^2 = g(v)h(w)(f(v)g(u))h(w)f(u) = 0$ for all $h(w) \in R$. Hence $g(v)h(w)f(u)$ is nilpotent for all $h(w) \in R$. Thus, $ghf = 0$. Therefore, *R* is nil generalized power series reflexive. □

Proposition 3.9. *Let S be a torsion-free and cancellative monoid,* ≤ *a strict order on S. Then every semicommutative ring is nil generalized power series reflexive.*

Proof. Let *R* be semicommutative ring. Let $f,g \in [[R^{S,\le}]]$ with $fhg \in [[nil(R)^{S,\le}]]$ for all $h \in [[R^{S,\le}]]$, and any $u,w,v \in S$, define the operation of convolution:

$$(fhg)(s) = \sum_{(u,w,v) \in X_s(f,h,g)} f(u)h(w)g(v) = 0.$$

By Lemma 3.6 we have $f(u)g(v) \in nil(R)$. So $g(v)f(u) \in nil(R)$, for any $u,v \in S$. Since *R* is semicommutative, $g(v)h(w)f(u) \in nil(R)$ for any $h(w) \in R$. Thus $ghf \in [[nil(R)^{S,\le}]]$. Therefore *R* is nil generalized power series reflexive. □

Proposition 3.10. *Let* $(S,\le)$ *be a strictly ordered monoid. If R is finite subdirect product of nil generalized power series reflexive rings, then R is nil generalized power series reflexive ring.*

Proof. Let $I_k\ (k=1,\cdots,l)$ be ideals of R such that R/I_k is nil generalized power series reflexive and $\bigcap_{k=1}^{l} I_k = 0$. Let f and g be in $[\![R^{S,\le}]\!]$ with $fhg \in [\![nil(R)^{S,\le}]\!]$, for all $h \in [\![R^{S,\le}]\!]$. Clearly $\bar{f}\bar{h}\bar{g} \in [\![nil(R/I_k)^{S,\le}]\!]$. Since R/I_k is nil generalized power series reflexive, we have $(f(u)h(w)g(v))^{r_{u,w,v,k}} \in I_k$, for each $u,w,v \in S$ and $k=1,\cdots,l$. Assume that $r_{u,w,v} = \max\{r_{u,w,v,k} \mid k=1,\cdots,l\}$. So $(f(u)h(w)g(v))^{r_{u,w,v}} \in \bigcap_{k=1}^{l} I_k = 0$. Hence $f(u)h(w)g(v) \in nil(R)$, for each $u,w,v \in S$, then $g(v)h(w)f(u) \in nil(R)$. Thus, $ghf \in [\![nil(R)^{S,\le}]\!]$, and we are done. □

In the next result it is presented that any corner ring of nil generalized power series reflexive ring inherits the nil generalized power series reflexivity property. But the nil generalized power series reflexivity property is not Morita invariant because of Examples 3.2.

Proposition 3.11. *Let* $(S,\le)$ *a strictly ordered monoid. Let R be a ring and* $e^2 = e \in R$. *If R is nil generalized power series reflexive, then so is* eRe.

Proof. Let $efe, ege \in [\![(eRe)^{S,\le}]\!]$ with $efe(ehe)ege \in [\![nil(eRe)^{S,\le}]\!]$ for all $ehe \in [\![(eRe)^{S,\le}]\!]$. Let e be an idempotent of R. It is easy to see that c_e is an idempotent element of $[\![(eRe)^{S,\le}]\!]$ and $c_e g = g c_e$ for every $g \in [\![R^{S,\le}]\!]$. Then $(c_e f)(c_e g) \in [\![nil(eR)^{S,\le}]\!]$. Since R is nil generalized power series reflexive, we have $fhg \in [\![nil(R)^{S,\le}]\!]$, and so $ghf \in [\![nil(R)^{S,\le}]\!]$. Then there exists $m \in \mathbb{N}$ such that $(efe(ehe)ege)^m = 0$. Hence $ege(ehe)efe \in [\![nil(eRe)^{S,\le}]\!]$. □

Corollary 3.12. *Let R be a ring,* $(S,\le)$ *a strictly ordered monoid. For a central idempotent e of a ring* R, eR *and* $(1-e)R$ *are nil generalized power series reflexive if and only if R is nil generalized power series reflexive.*

Proof. Assume that eR and $(1-e)R$ are nil generalized power series reflexive. Since the nil generalized power series reflexivity property is closed under finite direct products, $R \cong eR \times (1-e)R$ is nil generalized power series reflexive. The converse is trivial by Proposition 3.11. □

In the next, we investigate the relations between a ring R and R/I for some ideal I of R in terms of nil generalized power series reflexivity. By Theorem 2.5, symmetric ring is nil generalized power series reflexive.

Theorem 3.13. *Let R be a ring,* $(S,\le)$ *a strictly ordered monoid. If I be an ideal of R contained in* $nil(R)$. *Then R is nil generalized power series reflexive if and only if R/I is nil generalized power series reflexive.*

Proof. (1) "$\Rightarrow$" Let $\bar{f}, \bar{g} \in [\![(R/I)^{S,\le}]\!]$ with $\bar{f}\bar{h}\bar{g} \in [\![nil(R/I)^{S,\le}]\!]$ for all $\bar{h} \in [\![(R/I)^{S,\le}]\!]$. By hypothesis, the order $(S,\le)$ can be refined to a strict total order $\le$ on S. We will use transfinite induction on the strictly totally ordered set $(S,\le)$ to show that $\bar{g}\bar{h}\bar{f} \in [\![nil(R/I)^{S,\le}]\!]$. Firstly, by transfinite induction to show $g(v)h(w)f(u) \in nil(R)$ for any $u \in supp(f)$ and any $v \in supp(g)$. Since $supp(f)$ and $supp(g)$ are nonempty subsets of S, the set of minimal elements of $supp(f)$ and $supp(g)$, respectively, are finite and non-empty.

Let u_0 and v_0 denote the minimum elements of $supp(f)$ and $supp(g)$ in the $\leq$ order, respectively. By analogy with the proof of Proposition 2.10, we can show that $f(u_0)Rf(v_0)=0$. Therefore, by transfinite induction, we can proof that $f(u)h(w)f(v)=0$. Since $\overline{f}\,\overline{h}\,\overline{g}\in[\![nil(R/I)^{S,\leq}]\!]$. Then there exists $n\in\mathbb{N}$ such that $(\overline{f}\,\overline{h}\,\overline{g})^n=\overline{0}$. So $(f(u)h(w)g(v))^n\in I$, for any $u,w,v\in S$. Since $I\subseteq nil(R)$, $(f(u)h(w)g(v))^n=0$. Hence $f(u)h(w)g(v)\in nil(R)$, so $g(v)h(w)f(u)\in nil(R)$, by R is nil generalized power series reflexive, $ghf\in[\![nil(R)^{S,\leq}]\!]$. Thus $\overline{g}\,\overline{h}\,\overline{f}\in[\![nil(R/I)^{S,\leq}]\!]$ for all $\overline{h}\in[\![(R/I)^{S,\leq}]\!]$. Therefore R/I is nil generalized power series reflexive.

"$\Leftarrow$" Let $f,g\in[\![R^{S,\leq}]\!]$ and suppose that $fhg\in[\![nil(R)^{S,\leq}]\!]$ for all $h\in[\![R^{S,\leq}]\!]$. Then $\overline{f}\,\overline{h}\,\overline{g}\in[\![nil(R/I)^{S,\leq}]\!]$ and so $\overline{g}\,\overline{h}\,\overline{f}\in[\![nil(R/I)^{S,\leq}]\!]$ since R/I is nil generalized power series reflexive. There exists $m\in\mathbb{N}$ such that $(\overline{g}\,\overline{h}\,\overline{f})^m=\overline{0}$. This shows that $(g(v)h(w)f(u))^m\in I$. Since $I\subseteq nil(R)$, $(g(v)h(w)f(u))^m\in nil(R)$. So there exists $n\in\mathbb{N}$ such that $\left((g(v)h(w)f(u))^m\right)^n=0$ and so $g(v)h(w)f(u)\in nil(R)$. Thus, $ghf\in[\![nil(R)^{S,\leq}]\!]$. Therefore, R is nil generalized power series reflexive. □

Now we give some characterizations of nil generalized power series reflexivity by using the prime radical of a ring.

Corollary 3.14. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid. A ring R is nil generalized power series reflexive if and only if $R/P(R)$ is nil generalized power series reflexive.*

Proof. Since every element of $P(R)$ is nilpotent, it follows from Theorem 3.13. □

We close this section by determining abelian semiperfect nil generalized power series reflexive rings.

Theorem 3.15. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid. Consider the following statements.*

(1) *R is a finite direct sum of local nil generalized power series reflexive rings.*

(2) R is *a semiperfect nil generalized power series reflexive ring.*

Then (1) $\Rightarrow$ (2) *If R is abelian, then* (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) Assume that R is a finite direct sum of local nil generalized power series reflexive rings. Then R is semiperfect because local rings are semiperfect and a finite direct sum of semiperfect rings is semiperfect, and moreover R is nil generalized power series reflexive by Proposition 2.18.

(2) $\Rightarrow$ (1) Suppose that R is an abelian semiperfect nil generalized power series reflexive ring. Since R is semiperfect, R has a finite orthogonal set $\{e_1,e_2,\cdots,e_n\}$ of local idempotents whose sum is 1 by ([33], Theorem 27.6), say $1=e_1+e_2+\cdots+e_n$ such that each e_iRe_i is a local ring where $1\leq i\leq n$. The ring R being abelian implies $e_iRe_i=e_iR$. Each e_iR is a nil generalized power series reflexive by Proposition 3.11. Hence R is nil generalized power series reflexive by Proposition 2.18. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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