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Endpoints of Multi-Valued Weak Contractions on the Metric Space of Partially Ordered Groups

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Abstract

The present work considers the endpoint in the abstract metric space. It firstly introduces the metric space of partially ordered groups and the metric space of partially ordered modules, respectively; and defines the convergence of sequences and the multi-valued weak contractions, etc., on the introduced space. And then, with the methods of functional analysis and abstract algebra, it successively establishes an endpoint theorem for the metric space of partially ordered groups and an endpoint theorem for the metric space of partially ordered modules. The contributions of this article extend the theory of cone metric space constructed by Huang and Zhang (2007) and some recent results on the fixed point and endpoint theory, such as the endpoint theorem given by Amini-Harandi (2010).

Keywords

Fixed Point, Endpoint, Metric Space Valued in Partially Ordered Group, Topological Structure, Weak Contraction

1. Introduction

Let (X,d) be a complete metric space. Denote by $CB(X)$ the class of all nonempty closed and bounded subsets of *X*. Denote by $H(A,B)$ the Hausdorff metric of *A* and *B* with respect to *d*, that is,

$$H(A,B)=\max\left\{\sup_{x\in A} d(x,B),\sup_{y\in B} d(y,A)\right\},$$

for all $A,B\in CB(X)$, where $d(x,B)=\inf_{y\in B} d(x,y)$. Further let $T:X\to 2^X$ be a multi-valued/set-valued map. A point x is called a fixed point of *T* if $x\in Tx$.

Define $Fix(T)=\{x\in X: x\in Tx\}$. A point x is called an endpoint/a stationary point of a multi-valued map T if $Tx=\{x\}$. We denote the set of all endpoints of T by $End(T)$.

The investigation of endpoint of multi-valued mappings is an important extending of the study of fixed point, which was made as early as 30 years ago, and has received great attention in recent years, see e.g. ref. [1] and ref. [2], and the references therein. In particular, Amini-Harandi [1] (2010) proved the Theorem 1.1 below.

Theorem 1.1 (Theorem 2.1 of [1]). *Let* (X,d) *be a complete metric space and* $T: X\to CB(X)$ *be a set-valued map that satisfies*

$$H(Tx,Ty)\le\psi(d(x,y)), \tag{1.1}$$

for each $x,y\in X$, *where* $\psi:[0,+\infty)\to[0,+\infty)$ *is upper semicontinuous* (u.s.c.), $\psi(t)<t$ *for each* $t>0$ *and satisfies* $\liminf_{t\to+\infty}(t-\psi(t))>0$. *Then* T *has a unique endpoint if and only if* T *has the approximate endpoint property.* (*i.e.* $\inf_{x\in X}\sup_{y\in Tx} d(x,y)=0$.)

Huang and Zhang ref. [3] (2007) introduced the concept of cone metric space, and established some fixed point theorems for contractive type maps in a normal cone metric space. Subsequently, some other authors gave many results about the fixed point theory in cone metric spaces. For example, Rezapour and Hamlbarani ref. [4] (2008) generalized some results of [3]. Raja and Vaezpour ref. [5] (2008) presented some extensions of Banach's Contraction Principle in complete cone metric spaces. Aage and Salunke ref. [6] (2011) proved some fixed point theorems for the expansion onto mappings on complete cone metric spaces. Also, many common fixed point theorems were proved for maps on cone metric spaces in some literatures, for example, see Ilić and Rakočević ref. [7] (2008); Arshad, Azam and Vetro ref. [8] (2009), whose results generalized and unified many fixed point theorems. Rezapour and Haghi ref. [9] (2009), as well as Haghi and Rezapour ref. [10] (2010) studied fixed points of multifunctions (*i.e.* multi-valued mappings) on normal cone metric spaces and on regular cone metric spaces, respectively. Moreover, Wardowski ref. [11] (2009) introduced a kind of set-valued contractions in cone metric spaces and established endpoint and fixed point theorems for his contractions.

In addition, Rezapour and Haghi [9] (2009) introduced the concept of cone topology on cone metric space. Lakshmikantham and Ćirić ref. [12] (2009) introduced the concept of a mixed g-monotone mapping and prove coupled coincidence and coupled common fixed point theorems for such nonlinear contractive mappings in partially ordered complete metric spaces. Harjani and Sadarangani ref. [13] (2009) present some fixed point theorems for weakly contractive maps in a complete metric space endowed with a partial order. And Zhang ref. [14] (2010) proved some new fixed point and coupled fixed point theorems for multivalued monotone mappings in ordered metric spaces. Finally, Amini-Harandi ref. [15] (2011) studied fixed point theorems for a kind of genera-

lized quasicontraction maps in so called the vector modular spaces.

Motivated by the contributions stated above, the present work studies the endpoint in the abstract metric space. The remainder of the paper is organized as follows. In Section 2, it introduces the metric space valued in a partially ordered group endowed with a topological structure and the metric space valued in a partially ordered module endowed with a topological structure, and establishes some fundamental concepts of analysis on the introduced spaces, such as the convergence of sequences, which extends the theory of cone metric space; the multi-valued weak contractions, and so on. In Section 3, it focuses on addressing the endpoint theory in the metric space of partially ordered group. And finally in Section 4, it focuses on addressing the endpoint theory in the metric space of partially ordered module.

2. Preliminaries

This section provides necessary preliminaries for our discussions.

We first make the following explanations. For a partial order $\preceq$ of a set, we write $a \prec b$ to indicate that $a \preceq b$ but $a \neq b$, where a and b are elements of the set. And for a group G with partial order $\preceq$, we write G_+ and G^+ to indicate respectively the sets $\{a \in G : a \succeq \theta\}$ and $\{a \in G : a \succ \theta\}$, where θ indicates the identity element of G.

Definition 2.1. Let G be an abelian/a commutative group with partial order $\preceq$. We call G a $\preceq$-partially ordered group, a partially ordered group for simplicity, if $\prec$ satisfies the law (g1) $a \prec b \Rightarrow a + c \prec b + c, \forall a, b, c \in G$. Let further G be an R-module and the integral ring R be a $\leq$-partially ordered group. Assume that the partial order $<$ satisfies the law (r1): $1 > 0$, where 1 and 0 are the unit element and the identity element of R, respectively. Assume also that the partial orders $<$ and $\prec$ satisfy the law (m1): $a \prec b \Rightarrow ra \prec rb, \forall a, b \in G$ and $\forall r \in R^+$ (*i.e.* $r > 0$). Then we call G an $(R, \leq, \preceq)$-partially ordered module, a partially ordered module for simplicity.

Remark 2.2. 1) For convenience, we focus our attention to study under the assumption that there exist non-identity elements in group G below. 2) Note that each element of a group has an inverse element. From (g1), we can easily obtain the order relation: (g1)' $a \preceq b \Leftrightarrow a + c \preceq b + c, \forall a, b, c \in G$. In addition, from (m1), we can easily obtain the order relation: (m1)' $a \preceq b \Rightarrow ra \preceq rb$, $\forall a, b \in G$ and $\forall r \in R_+$. 3) From (m1), we can also obtain the order relations: (m2) $r < s \Rightarrow ra \prec sa, \forall r, s \in R$ and $\forall a \in G^+$; and (m2)' $r \leq s \Rightarrow ra \preceq sa$, $\forall r, s \in R$ and $\forall a \in G_+$. In fact, let $r < s$, then, by (g1), we have $0 < s - r$. Let also $a \in G^+$, *i.e.* $\theta \prec a$. Then, by (m1), we have $(s-r)\theta \prec (s-r)a$ $\Rightarrow \theta \prec (s-r)a$. From (g1), this leads to $ra \prec ra + (s-r)a = sa$. So we have (m2). Finally, from (m2), it is obvious that we have (m2)'. 4) It is obvious that the partially ordered module is a special kind of the partially ordered group.

Example 2.3. Let E be a Banach space over the real field $\mathbb{R}$ and P be a subset of E. P is called a cone if and only if: 1) P is closed, nonempty, and $P \neq \{\theta\}$; 2)

$r,s\in\mathbb{R}_+, a,b\in P\Rightarrow ra+sb\in P$; 3) $a\in P$ and $-a\in P\Rightarrow a=\theta$. Here $\mathbb{R}_+$ denotes all the non-negative real numbers. For a given cone P of E, define the partial order $\preceq$ on E by $x\preceq y$ if and only if $y-x\in P$, see [3]. Then it can be easily verified that E is an $(\mathbb{R},\le,\preceq)$-partially ordered module, and therefore, of course, is a $\preceq$-partially ordered group. Here $\le$ is the usual order of $\mathbb{R}$.

In the following part of this section G is supposed either of a $\preceq$-partially ordered group and an $(R,\le,\preceq)$-partially ordered module unless otherwise specified.

Definition 2.4. Let $\ll$ be a non-empty relation of G. $\ll$ is called an analytic topological structure of partially ordered group G if it satisfies: (t1) $a\ll b\Rightarrow a\prec b,\forall a,b\in G$; (t2) $a\preceq b, b\ll c\Rightarrow a\ll c$; (t3) $a\ll b\Rightarrow a+c\ll b+c,\forall a,b,c\in G$; (t4) $\theta\preceq a\ll\varepsilon,\forall\varepsilon\gg\theta\Rightarrow a=\theta$; and (t5) $\forall\varepsilon\gg\theta$, there exists $\eta\gg\theta$ such that $\eta\ll\varepsilon$. $\ll$ is called an analytic topological structure of partially ordered module G if it also satisfies: (t6) $a\ll b\Rightarrow ra\ll rb,\forall a,b\in G$ and $\forall r\in R^+$.

Remark 2.5. In the definition above, for $\ll$ is non-empty, there are actually infinite elements ε such that $\varepsilon\gg\theta$ in G. In fact, since $\ll$ is a non-empty relation, there exist at least two elements a and b such that $a\ll b$. By (t3), we have $\theta\ll b-a$. Thus, according to (t5), the result holds.

Example 2.6. For the partially ordered module E of Example 2.3, define the relation $\ll$ by $x\ll y$ if and only if $y-x\in intP$, where $intP$ denotes the interior of P, see [3] and [4]. Then we can verify that $\ll$ is an analytic topological structure of E. In fact, it is obvious that $\ll$ satisfies (t1), (t3), (t5) and (t6). To prove (t2), let $a\preceq b$ and $b\ll c$. Then, from $b\ll c$, we have $\theta\ll(c-b)$. So there is an $r\in\mathbb{R}^+$ such that $N((c-b),r)\subset P$, where $N((c-b),r)=\{x\in E:\|x-(c-b)\|<r\}$ and $\|x\|$ indicates the norm of x. Consider $N((c-a),r)$. Let $u\in N((c-a),r)$. Then $\|u-(b-a)-(c-b)\|=\|u-(c-a)\|<r$. This implies $u-(b-a)\in N((c-b),r)\subset P$. On the other hand, $(b-a)\in P$ for $a\preceq b$. So $u=u-(b-a)+(b-a)\in P$. Namely $N((c-a),r)\subset P$. Hence $\theta\ll c-a$, e.g. $a\ll c$, that is, (t2) holds. To prove (t4), assume $\theta\preceq a\ll\varepsilon$, $\forall\varepsilon\gg\theta$. Let $c\gg\theta$. Then $\frac{c}{n}\gg\theta$. By regarding $\frac{c}{n}$ as ε, we have $\frac{c}{n}\gg a\Rightarrow\frac{c}{n}-a\gg\theta\Rightarrow\frac{c}{n}-a\in P$ for all $n\in\mathbb{N}$, where $\mathbb{N}$ represents all the natural numbers. This leads to $-a\in P$ because $\frac{c}{n}\to\theta$ (in norm) and P is closed. So, by $a\in P$, we have $a=\theta$. That is, (t4) holds. Therefore, $\ll$ is an analytic topological structure of E.

Definition 2.7. Let $\ll$ be an analytic topological structure of G and $a\in G_+$. A sequence $\{a_n\}$ of G_+ is said to be convergent to a (in $\ll$) if $\forall\varepsilon\gg\theta$, there is a natural number N such that $\theta\preceq a_n-a\ll\varepsilon$ for all $n>N$, denoted by $a_n\to a$ or $\lim_{n\to\infty}a_n=a$.

Remark 2.8. 1) Let $\ll$ be an analytic topological structure of G, which is

different from the $\prec$. Suppose $\prec$ is also an analytic topological structure of G, and sequence $\{a_n\}$ of G_+ converges to θ in $\prec$. Then we can easily know that a_n converges to θ in $\ll$. (In fact, let $\varepsilon \gg \theta$. Then, from (t5), there exists $\eta \gg \theta$ such that $\eta \ll \varepsilon$. For the η, since $a_n \to \theta$ in $\prec$, there is a natural number N such that $a_n \prec \eta$ for all $n > N$. By (t2) and $\eta \ll \varepsilon$, this leads to $a_n \ll \varepsilon$ for all $n > N$. Hence $a_n \to \theta$ in $\ll$.) That is, the convergence in $\prec$ is stronger than in $\ll$. So, in the case, the convergence in $\ll$ can be regarded as a kind of weak convergence. 2) For the analytic topological structure $\ll$ of the partially ordered module E in Example 2.6, it can be easily verified that $\ll$ is different from the $\prec$ if E is a two-dimensional Euclidean space and $P = \{(x, y) : x \geq 0, y \geq 0\}$. 3) It can be easily verified that for an analytic topological structure $\ll$ of G, $a_n \to a \Leftrightarrow \forall b \gg a \succeq \theta$, there is a natural number N such that $a \preceq a_n \ll b$ for all $n > N$. In fact, assume $a_n \to a$. $\forall b \gg a$, let $\varepsilon = b - a$. Then, by (t3), we have $\varepsilon \gg \theta$. So, there is a natural number N such that $\theta \preceq a_n - a \ll \varepsilon$ for all $n > N$. From (g1)' and (t3), this leads to $a \preceq a_n \ll b$ for all $n > N$. Conversely, $\forall \varepsilon \gg \theta$, let $b = a + \varepsilon$. Then $b \gg a \succeq \theta$. So, there is a natural number N such that $a \preceq a_n \ll b \Rightarrow \theta \preceq a_n - a \ll \varepsilon$ for all $n > N$. Note that $a \succeq \theta$. This shows $a_n \to a$.

Remark 2.9. For the E and the analytic topological structure $\ll$ of Example 2.6, let $\{a_n\}$ be a sequence in E_+. Assume $a_n \to \theta$ in norm. Then, $\forall \varepsilon \gg \theta$, there exists $r \in \mathbb{R}^+$ such that $N(\varepsilon, r) \subset P$. Due to $a_n \to \theta$ in norm, there exists also a natural number N such that $\|a_n\| < r$ for all $n > N$. Therefore, $\|(\varepsilon - a_n) - \varepsilon\| < r \Rightarrow (\varepsilon - a_n) \in N(\varepsilon, r) \Rightarrow (\varepsilon - a_n) \in intP$, that is, $a_n \ll \varepsilon$, for all $n > N$. This implies that $a_n \to \theta$ in $\ll$ if $a_n \to \theta$ in norm.

G always associates with an analytic topological structure $\ll$ and the convergence of the sequences of G_+ is in $\ll$ are assumed below.

Lemma 2.10. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of G_+. We have the three conclusions as follows.* 1) *If $a_n \to \theta$, then $\lim_{n\to\infty} a_n$ is unique.* 2) *If $a_n \to \theta$ and $b_n \to \theta$, then $a_n + b_n \to \theta$.* 3) *If $b_n \succeq a_n \succeq a \succeq \theta$ for all $n \in \mathbb{N}$ and $b_n \to a$, then $(b_n - a_n) \to \theta$.*

Proof. Proving 1). Let $\lim_{n\to\infty} a_n = a$. Then there is a natural number N_1 such that $a_n \succeq a$ for all $n > N_1$. On the other hand, $\forall \varepsilon \gg \theta$, since $a_n \to \theta$, there is a natural number N_2 such that $\varepsilon \gg a_n \succeq \theta$ for all $n > N_2$. Let $n = \max\{N_1, N_2\} + 1$. Then, $\varepsilon \gg a_n$ and $a_n \succeq a \succeq \theta$. From (t2), this leads to $\varepsilon \gg a \succeq \theta$. By virtue of (t4), we have $a = \theta$. Hence 1) holds.

Proving 2). Let $\varepsilon \gg \theta$. By (t5), there exists $\eta \gg \theta$ such that $\varepsilon - \eta \gg \theta$. For $a_n \to \theta$ and $b_n \to \theta$, there are natural numbers N_1 and N_2 such that $a_n \ll \eta, \forall n > N_1$ and $b_n \ll \varepsilon - \eta, \forall n > N_2$. Put $N = \max\{N_1, N_2\}$. We have: $a_n + b_n \ll \eta + \varepsilon - \eta = \varepsilon, \forall n > N$. Hence, $a_n + b_n \to \theta$. That is 2) holds.

Proving 3). Arguing by contradiction, assume $\lim_{n\to\infty}(b_n - a_n) \neq \theta$. Then there exists a $\delta \gg \theta$ and a subsequence $\{b_{n_i} - a_{n_i}\}$ such that $(b_{n_i} - a_{n_i}) \ll \delta$ does not hold for all $i \in \mathbb{N}$. From $a_{n_i} \succeq a$, by (g1)', we have $b_{n_i} - a_{n_i} \preceq b_{n_i} - a$. This implies that $b_{n_i} - a \ll \delta$ does not hold for all $i \in \mathbb{N}$. (In fact, if for some

$i \in \mathbb{N}$, $b_{n_i} - a \ll \delta$, then, from (t2) and $b_{n_i} - a_{n_i} \preceq b_{n_i} - a$, we have $b_{n_i} - a_{n_i} \ll \delta$, which contradicts that $\left(b_{n_i} - a_{n_i}\right) \ll \delta$ does not hold.) Hence $\lim_{n\to\infty} b_n \neq a$. The contradiction shows 3) holds. □

Definition 2.11. G is called regular if every decreasing sequence $\{a_n\}$ of G_+ is convergent. That is, if a sequence $\{a_n\}$ of G_+ satisfies $a_n \preceq a_{n+1}$ for all $n \in \mathbb{N}$, then exists a $a \in G_+$ such that a_n converges to a.

Remark 2.12. 1) Let $A \subseteq G_+, A \neq \varnothing$ and $a \in G_+$. a is called the infimum of A if and only if a is a lower bound of A, and $c \preceq a$ for each lower bound c of A, denoted by $infA$. 2) It is obvious that there is at most one infimum for each subset of G_+. In fact, for any $A \subseteq G_+$, let a and b be two infimums of A. Then both $a \preceq b$ and $b \preceq a$ hold. Hence $a = b$. This shows that A has at most one infimum. 3) In particular, G is regular if for each non-empty subset A of G_+, $infA$ exists and there exists a sequence $\{a_n\}$ of A such that a_n converges to $infA$. Actually, let $\{a_n\}$ be a decreasing sequence of G_+. Then, in the case, $inf\{a_n\}$ exists and there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $\{a_{n_i}\}$ converges to $inf\{a_n\}$. Since $\{a_{n_i}\}$ converges to $inf\{a_n\}$, $\forall \varepsilon \gg \theta$, there is a natural number I such that $\varepsilon \gg a_{n_i} - inf\{a_n\} \succeq \theta$ for all $i \geq I$. For $\{a_n\}$ decreasing, this leads to $\varepsilon \gg a_n - inf\{a_n\} \succeq \theta$ for all $n \geq n_I$. That is, $\{a_n\}$ converges to $inf\{a_n\}$. Hence G is regular.

Definition 2.13. Let X be a non-empty set. Suppose the mapping $d : X \times X \to G$ satisfies

(d1) $d(x,y) \succeq \theta$ for all $x, y \in X$ and $d(x,y) = \theta$ if and only if $x = y$,

(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(d3) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a metric (on X) valued in partially ordered group G, and (X,d) is called a metric space valued in partially ordered group G, when G is a partially ordered group; a metric of group and a metric space of group for simplicity, respectively. (Then d is called a metric valued in partially ordered module G, and (X,d) is called a metric space valued in partially ordered module G, when G is a partially ordered module.)

In the rest of this section, we always assume that (X,d) is either of a metric space valued in partially ordered group G and a metric space valued in partially ordered module G.

Definition 2.14. For given (X,d), let $x \in X$ and $\{x_n\}$ be a sequence in X.

1) We call that $\{x_n\}$ converges to x if and only if $d(x_n, x) \to \theta$, denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

2) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to \theta$, that is, $\forall \varepsilon \gg \theta$, there is a natural number N such that $d(x_n, x_m) \ll \varepsilon$ for all $n, m \geq N$.

3) (X,d) is complete if and only if every Cauchy sequence is convergent.

4) (X,d) is regular if and only if G is regular.

Remark 2.15. The relation between the regular space and the complete space is an interesting question for further research.

Definition 2.16. Given (X,d), let $T : X \to \left(2^X - \varnothing\right)$ be a multi-valued

mapping and $\varphi: X \times X \to G_+$ be a mapping with $\varphi(x,y) \prec d(x,y)$ for all $d(x,y) \succ \theta$. T is called a multi-valued (φ-)weak contraction on (X,d) if, for all different $x, y \in X$, $\forall x' \in Tx$, there exists $\bar{y} \in Ty$ such that

$$d(x', \bar{y}) \preceq \varphi(x,y). \tag{2.1}$$

T is called a global multi-valued (φ-)weak contraction on (X,d) if, for all different $x, y \in X$, we have

$$d(x', y') \preceq \varphi(x,y), \forall x' \in Tx, \forall y' \in Ty. \tag{2.2}$$

The weak contraction T is called to satisfy C-condition (convergence condition) if $d(x_n, y_n) - \varphi(x_n, y_n) \to \theta$, then $d(x_n, y_n) \to \theta$, where x_n and y_n are two sequences of X.

Remark 2.17. 1) It is obvious that a global multi-valued weak contraction is a multi-valued weak contraction. 2) The weak contraction T is called to satisfy C'-condition if $d(x_n, x_m) - \varphi(x_n, x_m) \to \theta (n \neq m)$, that is, $\forall \varepsilon \gg \theta$, there is a N such that $d(x_n, x_m) - \varphi(x_n, x_m) \ll \varepsilon$ for all $n, m > N$ and $n \neq m$, then $d(x_n, x_m) \to \theta$. 3) If T satisfies C-condition, then it also satisfies C'-condition. In fact, for the set $\{(n,m): n, m \in \mathbb{N}, n \neq m\}$ is countable, it can be rewritten as the sequence $\{(x_i', y_i')\}$. Assume $d(x_n, x_m) - \varphi(x_n, x_m) \to \theta (n \neq m)$. Let $\varepsilon \gg \theta$. Then there exists a natural number N such that $d(x_n, x_m) - \varphi(x_n, x_m) \ll \varepsilon$ whenever $n, m > N$ and $n \neq m$. Because the set $\{(n,m): n, m \leq N\}$ is finite, there is a natural number I such that if $i > I$ and $(x_i', y_i') = (x_n, x_m)$, then $n, m > N$. This implies $\forall i > I$, we have $d(x_i', y_i') - \varphi(x_i', y_i') \ll \varepsilon$. Hence $d(x_i', y_i') - \varphi(x_i', y_i') \to \theta$. For T satisfies C-condition, we have $d(x_i', y_i') \to \theta$. Further, due to $d(x_i', y_i') \to \theta$, $\forall \varepsilon \gg \theta$, there is a natural number I' such that $d(x_i', y_i') \ll \varepsilon$ whenever $i > I'$. Since the set $\{i : i \leq I'\}$ is finite, there is a natural number N' such that if $n, m > N'$, $n \neq m$ and $(x_n, x_m) = (x_i', y_i')$, then $i > I'$. That is, $\forall n, m > N'$, $n \neq m$, we have $d(x_n, x_m) \ll \varepsilon$. Note that $d(x_n, x_m) = \theta$ when $n = m$. This leads to $d(x_n, x_m) \to \theta$.

Definition 2.18. A map $T: X \to (2^X - \varnothing)$ on (X,d) is said to have approximate endpoint property if there exist a sequence $\{x_n\}$ of X and a sequence $\{a_n\}$ of G_+ with $a_n \to \theta$ such that

$$d(x_n, x_n') \preceq a_n, \forall x_n' \in Tx_n, \tag{2.3}$$

for all $n \in \mathbb{N}$.

Remark 2.19. When (X,d) is the usual complete metric space, it can be easily verified that T has the approximate endpoint property in Theorem 1.1 and T has the approximate endpoint property defined in Definition 2.18 are equivalent. (In fact, if T has the approximate endpoint property in Theorem 1.1, that is, $\inf_{x \in X} \sup_{y \in Tx} d(x,y) = 0$, then there is a sequence $\{x_n\}$ of X such that $\sup_{y \in Tx_n} d(x_n, y) \to 0$. Let $a_n = \sup_{y \in Tx_n} d(x_n, y)$. Then $d(x_n, x_n') \leq a_n, \forall x_n' \in Tx_n$ and $a_n \to 0$. This shows that T has the approximate endpoint property defined in Definition 2.18. On the other hand, if T has the approximate endpoint property defined in Definition 2.18, that is, there exist a sequence $\{x_n\}$ of X and a se-

quence $\{a_n\}$ of $\mathbb{R}_+$ with $a_n \to 0$ such that $d(x_n, x_n') \le a_n, \forall x_n' \in Tx_n$, then $\sup_{y \in Tx_n} d(x_n, y) \le a_n$ for all $n \in \mathbb{N}$. This implies that $\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0$, namely, T has the approximate endpoint property in Theorem 1.1. Hence we have the equivalence stated above.)

Lemma 2.20. *Let T be a multi-valued weak contraction on (X, d). Then we have the following two conclusions.* 1) *T has approximate endpoint property if T has endpoints.* 2) *T has one endpoint at most.* (*i.e.* $|End(T)| \le 1$. *Here* $|End(T)|$ *denotes the cardinal number of* $End(T)$.)

Proof. 1) is obvious. In fact, let x be an endpoint of T. Put $x_n = x$ and $a_n = \theta$ for all $n \in \mathbb{N}$. Then $a_n \to \theta$ and (2.3) holds for all $n \in \mathbb{N}$. Hence T has the approximate endpoint property. To prove 2), assume $|End(T)| > 1$. Then, there exist $x, y \in End(T)$ such that $x \ne y$. From (2.1), we have $d(x, y) \preceq \varphi(x, y)$. Note that $\varphi(x, y) \prec d(x, y)$ for any $d(x, y) \succ \theta$. This implies $d(x, y) = \theta$. Hence, from (d1), we have $x = y$. This contradicts $x \ne y$. So $|End(T)| \le 1$, that is, 2) holds. □

3. Main Results

In this section, we always assume that (X, d) is a metric space valued in partially ordered group G.

Now we are ready to prove our main results. We first present the following Theorem 3.1, which extends Theorem 1.1 (Theorem 2.1 of Amini-Harandi [1]) to the case of the metric space of group.

Theorem 3.1. *Let T be a multi-valued weak contraction on complete (X, d) and satisfy C-condition. Then T has a unique endpoint if and only if it has the approximate endpoint property.*

Proof. The necessity is clear from the 1) of Lemma 2.20. Next we prove the sufficiency.

Since T has the approximate endpoint property, there exist sequences $\{x_n\}$ of X and $\{a_n\}$ of G_+ satisfying (2.3) and $a_n \to \theta$. If there exists a subsequence $\{x_{n_i}\}$ such that x_{n_i} being the same point x of X for all $i \in \mathbb{N}$, then we can easily know that x is an endpoint of T from (2.3). (In fact, for any given $x' \in Tx$, we have $d(x, x') \preceq a_{n_i}$ for all $i \in \mathbb{N}$. Since $a_n \to \theta$, we have $a_{n_i} \to \theta (i \to \infty)$. So we have $\theta \preceq d(x, x') \ll \varepsilon$ for all $\varepsilon \gg \theta$. This implies $d(x, x') = \theta$ from (t4), that is, $x = x'$. Hence x is an endpoint of T.) Otherwise, without loss of generality, we can assume $x_n \ne x_m$ whenever $n \ne m$ and continue to prove as follows.

For any different $n, m \in \mathbb{N}$, let $x_n' \in Tx_n$. Then

$$d(x_n, x_m) \preceq d(x_n, x_n') + d(x_n', x_m). \tag{3.1}$$

Since $x_n \ne x_m$, according to (2.1), there exists $\bar{x}_m \in Tx_m$ such that

$$d(x_n', \bar{x}_m) \preceq \varphi(x_n, x_m).$$

Using this and (3.1), we further obtain

$$\begin{aligned} d(x_n, x_m) &\preceq d(x_n, x_n') + d(x_n', \bar{x}_m) + d(\bar{x}_m, x_m) \\ &\preceq d(x_n, x_n') + \varphi(x_n, x_m) + d(\bar{x}_m, x_m). \end{aligned} \tag{3.2}$$

By (2.3), we have $d(x_n, x_n') \preceq a_n$ and $d(\bar{x}_m, x_m) \preceq a_m$. From (3.2), this leads to

$$\begin{aligned} & d(x_n, x_m) \preceq a_n + \varphi(x_n, x_m) + a_m \\ & \Rightarrow d(x_n, x_m) - \varphi(x_n, x_m) \preceq a_n + a_m, n \neq m. \end{aligned} \tag{3.3}$$

For $x_n \neq x_m$, we have $d(x_n, x_m) \succ \theta$. So, $d(x_n, x_m) - \varphi(x_n, x_m) \succeq \theta$. On the other hand, noting that $a_n \to \theta$, following the proof on the 2) of Lemma 2.10, we can easily know that $a_n + a_m \to \theta$. Thus we can obtain $d(x_n, x_m) - \varphi(x_n, x_m) \to \theta (x_n \neq x_m)$ from (3.3). This implies $d(x_n, x_m) \to \theta$ for T satisfies the C-condition, which leads to φ satisfies the C'-condition, see the 2) and 3) of Remark 2.17. Hence, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there is a $x \in X$ such that $x_n \to x$.

We show x is an endpoint of T below.

Since $x_n \neq x_m$ whenever $n \neq m$, without loss of generality, we can assume $x_n \neq x$ for any $n \in \mathbb{N}$. Let $x' \in Tx$. Then, for all $n \in \mathbb{N}$, we have

$$d(x', x) \preceq d(x', x_n) + d(x_n, x). \tag{3.4}$$

For $x_n \neq x$, by (2.1), there exists $\bar{x}_n \in Tx_n$ such that

$$d(x', \bar{x}_n) \preceq \varphi(x, x_n) \prec d(x, x_n). \tag{3.5}$$

In terms of (3.4), (3.5) and (2.3), we obtain

$$\begin{aligned} d(x', x) &\preceq d(x', \bar{x}_n) + d(\bar{x}_n, x_n) + d(x_n, x) \\ &\preceq \varphi(x, x_n) + a_n + d(x, x_n) \\ &\preceq d(x, x_n) + a_n + d(x, x_n). \end{aligned}$$

Since $d(x, x_n) \to \theta$ and $a_n \to \theta$, by the 2) of Lemma 2.10, we obtain $d(x, x_n) + d(x, x_n) + a_n \to \theta$. So, $d(x, x') \ll \varepsilon$ for all $\varepsilon \gg \theta$. Note also that $d(x, x') \succeq \theta$. From (t4), we have $d(x, x') = \theta$. Hence $x = x'$. That is, $x \in End(T)$.

Finally, the uniqueness is directly obtained from the 2) of Lemma 2.20. The proof completes. □

Remark 3.2. Here we make a simple explanation for Theorem 3.1 extending Theorem 1.1. Firstly, it is obvious that for the usual order ≤ of the real field $\mathbb{R}$, $\mathbb{R}$ is a ≤-partially ordered group with analytic topological structure >. Further, due that (X, d) in Theorem 1.1 is a complete metric space, it is a complete metric space of the group $\mathbb{R}$ with analytic topological structure >. That is, (X, d) satisfies the requirement of Theorem 3.1. Secondly, for the $\psi(t)$ in Theorem 1.1, let $\varphi(x, y) = \psi(d(x, y))$, then $\varphi(x, y)$ is a mapping from $X \times X$ to $\mathbb{R}_+$ and $\varphi(x, y) < d(x, y)$ for all $d(x, y) > \theta$. For the mapping T in Theorem 1.1 and the φ defined above, we have $H(Tx, Ty) \leq \varphi(x, y)$ for all different $x, y \in X$, that is,

$$\max\left\{\sup_{x'\in Tx} d\left(x',Ty\right),\sup_{y'\in Ty} d\left(Tx,y'\right)\right\}\le\varphi\left(x,y\right).$$

This leads to $\sup_{x'\in Tx} d\left(x',Ty\right)\le\varphi\left(x,y\right)\Rightarrow d\left(x',Ty\right)\le\varphi\left(x,y\right),\forall x'\in Tx$. Thus, for Ty is closed and bounded, $\forall x'\in Tx$, there is a $\bar{y}\in Ty$ such that $d\left(x',\bar{y}\right)\le\varphi\left(x,y\right)$. This shows that T is a multi-valued weak contraction on the space $\left(X,d\right)$. Thirdly, if $d\left(x_n,y_n\right)-\varphi\left(x_n,y_n\right)\to 0$, then $d\left(x_n,y_n\right)\to 0$, that is, T satisfies the C-condition. In fact, if $d\left(x_n,y_n\right)$ does not converge to 0, then there exist a $\delta>0$ and a subsequence $\left\{d\left(x_{n_i},y_{n_i}\right)\right\}$ such that

$$d\left(x_{n_i},y_{n_i}\right)-\psi\left(d\left(x_{n_i},y_{n_i}\right)\right)>\delta$$

for all $i\in\mathbb{N}$. 「 We show the fact is true as follows. Let $d_n=d\left(x_n,y_n\right)$ and d_n do not converge to 0. If $\left\{d_n\right\}$ is unbounded, without loss of generality, we can assume that $\left\{d_n\right\}$ increases and converges to $+\infty$. For

$$\liminf_{t\to+\infty}\left(t-\psi\left(t\right)\right)=\lim_{t\to+\infty}\left[\inf\left\{\left(s-\psi\left(s\right)\right):s>t\right\}\right]>0$$

and

$$\inf\left\{\left(d_k-\psi\left(d_k\right)\right):k>n\right\}\ge\inf\left\{\left(s-\psi\left(s\right)\right):s>d_n\right\},$$

we have

$$\begin{aligned}&\liminf_{n\to\infty}\left(d_n-\psi\left(d_n\right)\right)=\lim_{n\to\infty}\left[\inf\left\{\left(d_k-\psi\left(d_k\right)\right):k>n\right\}\right]\\&\ge\lim_{n\to\infty}\left[\inf\left\{\left(s-\psi\left(s\right)\right):s>d_n\right\}\right]=\lim_{t\to+\infty}\left[\inf\left\{\left(s-\psi\left(s\right)\right):s>t\right\}\right]>0.\end{aligned}$$

Hence there exist a $\delta>0$ and a subsequence $\left\{d_{n_i}\right\}$ such that $d_{n_i}-\psi\left(d_{n_i}\right)>\delta$ for all $i\in\mathbb{N}$, that is, the fact is true. If $\left\{d_n\right\}$ is bounded, without loss of generality, we assume that $\left\{d_n\right\}$ increases and converges to a point $t'>0$. Then, for ψ is u.s.c. at t', *i.e.* $\limsup_{t\to t'}\psi\left(t\right)\le\psi\left(t'\right)$, and $\psi\left(t'\right)<t'$, we have

$$\liminf_{t\to t'}\left(t-\psi\left(t\right)\right)=t'-\limsup_{t\to t'}\psi\left(t\right)\ge t'-\psi\left(t'\right)>0.$$

Note that $\left\{d_n\right\}$ increases and

$$\lim_{\Delta t\to 0}\left[\inf\left\{\left(t-\psi\left(t\right)\right):0<\left|t-t'\right|<\Delta t\right\}\right]=\liminf_{t\to t'}\left(t-\psi\left(t\right)\right).$$

We have

$$\begin{aligned}&\liminf_{n\to\infty}\left(d_n-\psi\left(d_n\right)\right)=\lim_{n\to\infty}\left[\inf\left\{\left(d_k-\psi\left(d_k\right)\right):k>n\right\}\right]\\&\ge\lim_{n\to\infty}\left[\inf\left\{\left(t-\psi\left(t\right)\right):d_n<t<t'\right\}\right]=\lim_{\Delta t\to 0}\left[\inf\left\{\left(t-\psi\left(t\right)\right):0<t-t'<\Delta t\right\}\right]\\&\ge\lim_{\Delta t\to 0}\left[\inf\left\{\left(t-\psi\left(t\right)\right):0<\left|t-t'\right|<\Delta t\right\}\right]=\liminf_{t\to t'}\left(t-\psi\left(t\right)\right)>0.\end{aligned}$$

Hence the fact is also true. 」 That is, $d\left(x_{n_i},y_{n_i}\right)-\varphi\left(x_{n_i},y_{n_i}\right)>\delta$ for all $i\in\mathbb{N}$. This contradicts $d\left(x_n,y_n\right)-\varphi\left(x_n,y_n\right)$ converges to 0. Hence T satisfies the C-condition. Finally, for the T has the approximate endpoint property of

Theorem 1.1, from Remark 2.19, it has the approximate endpoint property (defined in Definition 2.18). Hence we can directly obtain Theorem 1.1 from Theorem 3.1.

Next we further present the following Theorem 3.3, which shows, in the setting that (X,d) is complete and regular, if the global multi-valued weak contraction satisfies C-condition, then it has the approximate endpoint property, so has a unique endpoint from Theorem 3.1.

Theorem 3.3. *Let (X,d) be complete and regular, T be a global multi-valued weak contraction on (X,d) and satisfy C-condition. Then T has a unique endpoint.*

Proof. We first prove the existence of endpoints.

Arguing by contradiction, assume T has no endpoint. Then for any $x \in X$, there is at least one $y \in Tx$ such that $y \neq x$. Hence there must be a sequence $\{y_n\}$ of X such that $y_{n+1} \in Ty_n$ and $y_{n+1} \neq y_n$ for all $n \in \mathbb{N}$. Note T is a global multi-valued weak contraction. In terms of $y_{n+1} \in Ty_n$, $y_{n+1} \neq y_n$, (2.2) and $\varphi(x,y) \prec d(x,y)$ for $d(x,y) \succ \theta$, we have

$$d(y_{n+1}, y_{n+2}) \preceq \varphi(y_n, y_{n+1}) \prec d(y_n, y_{n+1}) \tag{3.6}$$

for all $n \in \mathbb{N}$. Hence the sequence $\{d(y_n, y_{n+1})\}$ is decreasing. So, for G is regular, there exists $a \in G_+$ such that $d(y_n, y_{n+1}) \to a$. Hence, from (3.6), we have $a \preceq \varphi(y_n, y_{n+1}) \prec d(y_n, y_{n+1})$. Further, according to the 3) of Lemma 2.10, we obtain $d(y_n, y_{n+1}) - \varphi(y_n, y_{n+1}) \to \theta$. For T satisfies C-condition, this leads to $d(y_n, y_{n+1}) \to \theta$. And by (3.6) we further have $\varphi(y_n, y_{n+1}) \to \theta$. Now let $x_n = y_{n+1}$ and $a_n = \varphi(y_n, y_{n+1})$. Then we have $d(x_n, x_n') \preceq a_n, \forall x_n' \in Tx_n$ and $a_n \to \theta$. That is, T has the approximate endpoint property. Thus, by Theorem 3.1, T has endpoints. This contradicts our assumption. Hence the existence of endpoints is true.

Finally, the uniqueness follows directly from the 2) of Lemma 2.20. This ends the proof. □

For the single-valued weak contraction can be regarded as a kind of specific global multi-valued weak contraction, from Theorem 3.3, we can immediately derive the Corollary 3.4 below, which generalizes Lemma 2.4 and Corollary 2.5 of [1].

Corollary 3.4. *Let (X,d) be complete and regular, f be a single-valued weak contraction on (X,d) and satisfy C-condition. Then T has a unique fixed point.*

Remark 3.5. The multi-valued weak contraction cannot have the approximate endpoint property, even in the usual metric space, for instance, see the Example 2.3 of [1].

4. Endpoint Theory for the Metric Space of Module

Note that a metric space of module is a special metric space of group. As applications of the results proved above, this section discusses the endpoint theory for

the metric space of module. We always assume (X,d) is a metric space of module G in this section.

By regarding $\alpha(x,y)d(x,y)$ as $\varphi(x,y)$, we can easily derive the next Theorem 4.1 from Theorem 3.1 and Theorem 3.3.

Theorem 4.1. *Let (X,d) be complete, $T: X \to (2^X - \varnothing)$ be a multi-valued mapping. Let also $\alpha: X \times X \to [0,1)(=\{r \in R: 0 \le r < 1\})$ be a mapping, which satisfies: for any two sequences $\{x_n\}$ and $\{y_n\}$ of X with $[1-\alpha(x_n,y_n)]d(x_n,y_n) \to \theta$, there is an $\alpha \in [0,1)$ such that $\alpha(x_n,y_n) \le \alpha$ for all $n \in \mathbb{N}$, as well as $(1-\alpha)$ has multiplicative inverse $(1-\alpha)^{-1}$ and $(1-\alpha)^{-1} > 0$. Then we have the following two conclusions.*

1) *Suppose for all different $x,y \in X$, $\forall x' \in Tx$, there exists $\bar{y} \in Ty$ such that $d(x',\bar{y}) \preceq \alpha(x,y)d(x,y)$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

2) *Let (X,d) be regular. Suppose for all different $x,y \in X$, we have $d(x',y') \preceq \alpha(x,y)d(x,y)$, $\forall x' \in Tx, \forall y' \in Ty$. Then T has a unique endpoint.*

Proof. Let $\varphi(x,y) = \alpha(x,y)d(x,y)$. Then $\varphi: X \times X \to G_+$ is a mapping. Since $\alpha(x,y) \prec 1$, by (m2), see the 3) of remark 2.2, we have $\varphi(x,y) = \alpha(x,y)$ $d(x,y) \prec d(x,y)$ for all $d(x,y) \succ \theta$. We prove the statement that if $d(x_n,y_n) - \varphi(x_n,y_n) \to \theta$, then $d(x_n,y_n) \to \theta$ below.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X. Then

$$d(x_n,y_n) - \varphi(x_n,y_n) = [1-\alpha(x_n,y_n)]d(x_n,y_n).$$

So, if $d(x_n,y_n) - \varphi(x_n,y_n) \to \theta$, then

$$[1-\alpha(x_n,y_n)]d(x_n,y_n) \to \theta. \tag{4.1}$$

Hence there exists an $\alpha \in [0,1)$ such that $\alpha(x_n,y_n) \le \alpha, \forall n \in \mathbb{N}$. By (g1)', this further leads to $(1-\alpha) \le [1-\alpha(x_n,y_n)]$. Since also $d(x_n,y_n) \succeq \theta$, by (m2)', we have

$$(1-\alpha)d(x_n,y_n) \preceq [1-\alpha(x_n,y_n)]d(x_n,y_n), \forall n \in \mathbb{N}. \tag{4.2}$$

In terms of (4.1) and (4.2), we obtain $(1-\alpha)d(x_n,y_n) \to \theta$. For $\alpha < 1$, we have $1-\alpha > 0$. Let $\varepsilon \gg \theta$. By (t6), we have $(1-\alpha)\varepsilon \gg \theta$. Hence, there exists a natural N such that $(1-\alpha)d(x_n,y_n) \ll (1-\alpha)\varepsilon$ for all $n > N$. For $(1-\alpha)^{-1} > 0$, from (t6), we obtain also $d(x_n,y_n) \ll \varepsilon$ for all $n > N$. This implies $d(x_n,y_n) \to \theta$.

For conclusion 1), by $\varphi(x,y) = \alpha(x,y)d(x,y) \prec d(x,y)$ for all $d(x,y) \succ \theta$ and the statement proved above, it is obvious that T is a multi-valued weak contraction on the space (X,d) and satisfies C-condition. Hence we can immediately know that the conclusion is true from Theorem 3.1. For conclusion 2), T is clearly a global multi-valued weak contraction and satisfies C-condition. Note that (X,d) is regular. We can immediately know that the conclusion is true from Theorem 3.3. This completes the proof. □

Replacing, in Theorem 4.1, $\alpha(x,y)$ by α, we directly obtain the following Corollary 4.2.

Corollary 4.2. *Let* (X,d) *be complete,* $T: X \to (2^X - \varnothing)$ *be a multi-valued mapping. Let also* $\alpha \in [0,1)$ *with* $(1-\alpha)^{-1}$ *and* $(1-\alpha)^{-1} > 0$. 1) *Suppose* T *satisfies for all different* $x, y \in X$, $\forall x' \in Tx$, *there exists* $\bar{y} \in Ty$ *such that* $d(x', \bar{y}) \preceq \alpha d(x, y)$. *Then* T *has a unique endpoint if and only if* T *has the approximate endpoint property.* 2) *Let* (X,d) *be regular. Suppose* T *satisfies for all different* $x, y \in X$, $d(x', y') \preceq \alpha d(x, y), \forall x' \in Tx, \forall y' \in Ty$. *Then* T *has a unique endpoint.*

Proof. Let $\alpha(x, y) = \alpha$. And then applying Theorem 4.1, we obtain the Corollary instantly. □

Finally, in the 2) of Corollary 4.2, replacing also multi-valued mapping by single-valued mapping, we obtain Corollary 4.3 below.

Corollary 4.3. *Let* (X,d) *be complete and regular,* $T: X \to X$ *be a single-valued mapping,* $\alpha \in [0,1)$ *with* $(1-\alpha)^{-1}$ *and* $(1-\alpha)^{-1} > 0$. *Suppose* T *satisfies for all different* $x, y \in X$, $d(Tx, Ty) \preceq \alpha d(x, y)$. *Then* T *has a unique fixed point.*

Remark 4.4. In particular, when (X,d) is the usual complete metric space, Corollary 4.3 is just the famous Banach fixed point theorem.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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