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The Lebesgue Measure of the Julia Sets of Permutable Transcendental Entire Functions

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Abstract

In 1958, Baker posed the question that if f and g are two permutable transcendental entire functions, must their Julia sets be the same? In order to study this problem of permutable transcendental entire functions, by the properties of permutable transcendental entire functions, we prove that if f and g are permutable transcendental entire functions, then $mes(J(f)) = mes(J(g))$. Moreover, we give some results about the zero measure of the Julia sets of the permutable transcendental entire functions family.

Keywords

Transcendental Entire Function, Permutable Functions, Random Dynamics, Julia Set, Lebesgue Measure

1. Introduction

Let f be a transcendental entire function. We write $f^1 = f$, and $f^n = f \circ f^{n-1}$, $n \geq 2$ for the nth iterates of f. The Fatou set $F(f)$ of f consists of all z in the complex plane $\mathbb{C}$ that has a neighborhood U such that the family $\{f^n | U : n \geq 1\}$ is a normal family. The Julia set $J(f)$ of f is defined by $J(f) = \mathbb{C} \setminus F(f)$. The Julia set $J(f)$ can be characterized as the closure of the repelling periodic points of f [1]. The set $J(f)$ and $F(f)$ are completely invariant of f. For fundamental results in the iteration theory of rational and entire functions, we refer to the original papers of Fatou [2] [3] [4] [5] and Julia [6] and the books of Beardon [7], Carleson and Gamelin [8], Milnor [9], Ren [10], Zheng [11], and Qiao [12].

Two functions $f(z)$ and $g(z)$ are called permutable if

$$f(g(z)) = g(f(z)) \text{ i.e. } f \circ g(z) = g \circ f(z)$$

holds for all values of z. In 1922-23, Julia [13] and Fatou [4] independently proved that rational functions f and g of degree at least 2 such that f and g are permutable, then $J(f) = J(g)$. It is natural to consider the following open problem which was first posed in [14] by Baker.

Problem Let f and g be nonlinear entire functions. If f and g are permutable, is $J(f) = J(g)$?

In [14], Baker proved the following result:

Theorem A (Baker [14]) Suppose that f and g are transcendental entire functions such that $g(z) = af(z) + b$, where a and b are complex numbers. If g permute with f, then $J(f) = J(g)$.

Langley [15] showed that if f and g are permutable functions of finite order with no wandering domains, then $J(f) = J(g)$.

Theorem B (Langley [15]) Suppose that f and g are permutable transcendental entire functions. If both f and g have no wandering domains, then $J(f) = J(g)$.

At the same time, Bergweiler and Hinkkanen [16] introduced the so-called fast escaping set

$$A(f) := \left\{ z \in \mathbb{C} : \exists L \in N, \left| f^n(z) \right| > M\left(R, f^{n-L}\right), n > L \right\}$$

and used it to prove a result that includes the following.

Theorem C (Bergweiler and Hinkkanen [16]) If f and g are permutable transcendental entire functions such that $A(f) \subset J(f)$ and $A(g) \subset J(g)$, then $J(f) = J(g)$. In particular, this holds if f and g have no wandering domains.

For a long time, there have been many results about the problem of permutable transcendental entire functions, see [17]-[23]. However, until now, the problem has not been completely solved. In order to study the problem of permutable transcendental entire functions, by the properties of permutable transcendental entire functions, we prove that if f and g are permutable entire functions, then $mes(J(f)) = mes(J(g))$. Moreover, we give some results about the zero measure of the Julia sets of the permutable transcendental entire functions family.

2. Main Results

Write $mesE$ for the plane Lebesgue measure of a set E. Recently, various authors have studied the Lebesgue measure of Julia sets. Results on Julia sets of positive Lebesgue measures are treated in [24] [25] [26] [27]. Julia sets of Lebesgue measure zero are given in [28]. We consider the Lebesgue measure of the permutable transcendental entire functions and give some results about the zero measure of the Julia sets of the permutable transcendental entire functions family. Firstly we prove the following result.

Theorem 1. If f and g are permutable transcendental entire functions, then

$$mes\left(J\left(f\right)\right)=mes\left(J\left(g\right)\right).$$

Let $f_1, f_2, \cdots, f_M$ be entire functions. Put $\mathcal{F}=\{f_1, f_2, \cdots, f_M\}$, and

$$\Sigma_M=\left\{\left(j_1, j_2, \cdots, j_n, \cdots\right) \mid j_i \in \{1,2,\cdots,M\}, i=1,2,\cdots,n,\cdots\right\}.$$

For $\sigma=\left(j_1, j_2, \cdots, j_n, \cdots\right)\in\Sigma_M$, we define $\left\{W_\sigma^n\left(z\right)\right\}_{n=1}^{\infty}$ as following

$$W_\sigma^1\left(z\right)=f_{j_1}\left(z\right),$$

$$W_\sigma^2\left(z\right)=f_{j_2}\circ f_{j_1}\left(z\right),$$

$$\cdots,$$

$$W_\sigma^n\left(z\right)=f_{j_n}\circ f_{j_{n-1}}\circ\cdots\circ f_{j_1}\left(z\right),$$

$$\cdots.$$

We also define the inverse of $\left\{W_\sigma^n\left(z\right)\right\}_{n=1}^{\infty}$ as following:

$$W_\sigma^{-n}\left(z\right)=\left(W_\sigma^n\right)^{-1}\left(z\right)=f_{j_1}^{-1}\circ f_{j_2}^{-1}\circ\cdots\circ f_{j_n}^{-1}\left(z\right), \text{for } n\in\mathbb{N}.$$

A point $z\in\mathbb{C}$ is said to be a normal point of $\mathcal{F}$. If there exists a neighborhood U of z such that $\left\{W_\sigma^n\left(z\right)\right\}$ is a normal family on U for each $\sigma\in\Sigma_M$. The set of normal points is called the Fatou set of $\mathcal{F}$, denoted by $F\left(\mathcal{F}\right)$, and its complement in $\mathbb{C}$, denoted by $J\left(\mathcal{F}\right)$, is called the Julia set of $\mathcal{F}$. The Fatou set $F\left(\mathcal{F}\right)$ is open and forward invariant and Julia set $J\left(\mathcal{F}\right)$ is closed and backward invariant. More information about the random dynamical system can be found in [10] [29] [30].

In this paper, we study the random dynamics of entire functions family of which the orbits of singularity stay away from the Julia set. Let

$$S_i=\left\{z:\text{some branch of } f_i^{-1} \text{ has a singularity at } z\right\}, i=1,2,\cdots,M,$$

and $P_i=\bigcup_{j=1}^{\infty}f_i^j\left(S_i\right)$. If $\delta_0>0$, put

$$C_i=\left\{\text{entire function } f_i, d\left(\overline{P_i}, J\left(f_i\right)\right)>\delta_0\right\}, i=1,2,\cdots,M.$$

McMullen [31] proved the following theorem.

Theorem D (McMullen [31]) If $f\in C_i$, then for $z\in J\left(f\right)$ we have

$$\left|\left(f^n\right)'\left(z\right)\right|\to\infty, \text{as } n\to\infty.$$

Let $P=\bigcup_{\sigma\in\Sigma_M}\bigcup_{n>0}W_\sigma^n\left(\bigcup_{i=1}^{M}P_i\right)$ and

$$C=\left\{\text{entire function family } \mathcal{F}, d\left(\overline{P}, J\left(\mathcal{F}\right)\right)>C_0>0\right\}.$$

We prove the following result.

Theorem 2. If $f_i\in\mathcal{F}\in C$, $z\in\bigcap_{i=1}^{M}J\left(f_i\right)$, then for any $\sigma\in\Sigma_M$,

$\left|\left(W_\sigma^n\right)'(z)\right| \to \infty$, as $n \to \infty$.

McMullen [31] gave the following notion. A plane set E is called thin at ∞, if its density is bounded away from 1 in all sufficiently large discs, that is, if there exist positive R and ε such that all complex z and every discs $D(z,r)$ of center z and radius $r > R$.

$$dens\left(E, D(z,r)\right) = \frac{mes\left(E \cap D(z,r)\right)}{mes\left(D(z,r)\right)} < 1 - \varepsilon.$$

In [31], McMullen proved the following result.

Theorem E (McMullen [31]) If $f \in C_i$, E is a measurable completely invariant subset of $J(f)$ such that E is thin at ∞, then $mesE = 0$.

We consider the entire function family in C, and show the following results.

Theorem 3. If $f_i \in \mathcal{F} \in C$, E is a measurable completely invariant set of $f_i, i = 1, 2, \cdots, M$, and $E \subset \bigcup_{i=1}^{M} J(f_i)$ such that E is thin at ∞, then $mesE = 0$.

For the permutable transcendental entire functions family, we prove the following result.

Theorem 4. If $f_i \in \mathcal{F} \in C$, $f_i \circ f_j = f_j \circ f_i$, for $i, j \in \{1, 2, \cdots, M\}$ and exits $i \in \{1, 2, \cdots, M\}$ such that $mes\left(J(f_i)\right) = 0$, then $mes\left(J(f_j)\right) = 0$, for any $j \in \{1, 2, \cdots, M\}$.

Remark. By using theorem 1, we can remove the special condition of the transcendental entire functions family in theorem 4. Let $\mathcal{F}$ be a permutable transcendental entire functions family and exits a $i \in \{1, 2, \cdots, M\}$ such that $mes\left(J(f_i)\right) = 0$, then $mes\left(J(f_j)\right) = 0$, for any $j \in \{1, 2, \cdots, M\}$.

3. Proofs of Theorems 1, 2, 3 and 4

The following well-known result is needed in the proof of theorems (see [32] Lemma 4.1).

Lemma 1 (Baker [32]) If f and g are permutable transcendental entire functions, then

$$g\left(J(f)\right) \subset J(f).$$

3.1. Proof of Theorem 1

Since f, g are permutable transcendental entire functions, Lemma 1 imply that $f\left(J(g)\right) \subset J(g)$, and hence that $J(g) \subset f^{-1}\left(J(g)\right)$. By the complete invariance of $J(g)$ we have

$$g^{-1}\left(J(g)\right) \subset J(g) \subset f^{-1}\left(J(g)\right). \tag{1}$$

So

$$mes\left(g^{-1}\left(J(g)\right)\right) \le mes\left(J(g)\right) \le mes\left(f^{-1}\left(J(g)\right)\right). \tag{2}$$

Similarly we have

$$g\left(J(f)\right)\subset J(f),\ J(f)\subset g^{-1}\left(J(f)\right),$$

then

$$f^{-1}\left(J(f)\right)\subset J(f)\subset g^{-1}\left(J(f)\right). \tag{3}$$

So

$$mes\left(f^{-1}\left(J(f)\right)\right)\le mes\left(J(f)\right)\le mes\left(g^{-1}\left(J(f)\right)\right) \tag{4}$$

If $mes\left(J(f)\right)<mes\left(J(g)\right)$, we have a contradiction with (1) and (2);

If $mes\left(J(f)\right)>mes\left(J(g)\right)$, we have a contradiction with (3) and (4).

So $mes\left(J(f)\right)=mes\left(J(g)\right)$.

3.2. Proof of Theorem 2

Since $\mathcal{F}\in C$, hence

$$d\left(\overline{P_i},J(\mathcal{F})\right)>C_0>0.$$

Since

$$\bigcup_{i=1}^{M}J(f_i)\subset J(\mathcal{F}),$$

then

$$d\left(\overline{P_i},J(f_i)\right)>C_0>0.$$

so f_i have uniform expansion, that is, for all $i\in\{1,2,\cdots,M\}$, exist a number α such that $\left|(f_i)'(z)\right|>\alpha>1$, where $z\in J(f_i)$. Since $z\in\bigcap_{i=1}^{M}J(f_i)$, for any $\sigma\in\Sigma_M$,

$$\left|\left(W_\sigma^n\right)'(z)\right|=\prod_{i=1}^{n}\left|f'_{j_i}\left(f_{j_{i-1}}\circ f_{j_{i-2}}\circ\cdots\circ f_{j_1}\right)(z)\right|>\alpha^n,$$

$$\left|\left(W_\sigma^n\right)'(z)\right|\to\infty,\ \text{as}\ n\to\infty.$$

3.3. Proof of Theorem 3

If $E\subset J(f_i)$, for some $i\in\{1,2,\cdots,M\}$, by the theorem E, we have $mes(E)=0$.

If $E\not\subseteq J(f_i)$, for any $i\in\{1,2,\cdots,M\}$, put $E_i=\{z:z\in J(f_i)\cap E\}$. If $z\in E_i$, then $z\in J(f_i)$ and $z\in E$. By the completely invariant of $J(f_i)$ and E, we have $f_i(z)\in J(f_i)$ and $f_i(z)\in E$, so that, $f_i(z)\in J(f_i)\cap E$. By the definition of E_i, $f_i(z)\in E_i$, then $f_i(E_i)\subset E_i$. On the other hand, by the definition of E_i and E, $J(f_i)$ are completely invariant sets, then

$$f_i^{-1}(E_i)=f_i^{-1}\left(J(f_i)\cap E\right)\subset f_i^{-1}\left(J(f_i)\right)\subset J(f_i),$$

So $f_i^{-1}(E_i)\subset J(f_i)$. Similarly

$$f_i^{-1}(E_i)=f_i^{-1}\left(J(f_i)\cap E\right)\subset f_i^{-1}(E)\subset E,$$

So $f_i^{-1}(E_i) \subset E$. Thus

$$f_i^{-1}(E_i) \subset J(f_i) \cap E = E_i.$$

So $f_i^{-1}(E_i) \subset E_i$, E_i are all completely invariant sets of f_i. Since E is thin at ∞, then E_i is thin at ∞. By theorem E and $E_i \subset J(f_i)$ we have $mes(E_i) = 0$.

Since

$$E \subset \bigcup_{i=1}^{M} J(f_i),$$

and

$$E_i = \{z : z \in J(f_i) \cap E\},$$

so

$$E = \bigcup_{i=1}^{M} E_i,$$

$$mes(E) \le \sum_{i=1}^{M} mes(E_i) = 0.$$

Therefore

$$mes(E) = 0.$$

3.4. Proof of Theorem 4

If $J(f_j)$ is not thin at ∞, then from the definition of E is thin at ∞, for any $R > 0$ and $\varepsilon > 0$, exists $z \in \mathbb{C}$ and $r > R$, such that

$$\frac{mes\left(J(f_j) \cap D(z,r)\right)}{mes\left(D(z,r)\right)} \ge 1 - \varepsilon.$$

So that

$$\begin{aligned} mes\left(J(f_j) \cap D(z,r)\right) &\ge mes\left(D(z,r)\right)(1-\varepsilon) \\ &= \pi r^2 (1-\varepsilon) > \pi R^2 (1-\varepsilon) \to \infty, R \to \infty. \end{aligned}$$

Then

$$mes\left(J(f_i) \cap D(z,r)\right) \to \infty, R \to \infty,$$

$$mes\left(J(f_j)\right) \to \infty, R \to \infty.$$

Since for any $i, j \in \{1, 2, \cdots, M\}$,

$$f_i \circ f_j = f_j \circ f_i,$$

by Lemma 1,

$$f_i\left(J(f_j)\right) \subset J(f_j),$$

hence

$$J(f_j) \subset f_i^{-1}\left(J(f_j)\right), \tag{5}$$

and

$$f_j^{-1}\left(J\left(f_j\right)\right) \subset J\left(f_j\right) \subset f_i^{-1}\left(J\left(f_j\right)\right) \tag{6}$$

(5) and (6) contradiction with $mes\left(J\left(f_i\right)\right)=0$ and $mes\left(J\left(f_j\right)\right)=\infty$, so $J\left(f_j\right)$ is thin at ∞. By Theorem 3, we have $mes\left(J\left(f_j\right)\right)=0$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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