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The Infinite Polynomial Products of the Gamma and Zeta Functions

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Abstract

Starting with the binomial coefficient and using its infinite product representation, the infinite product representation of the gamma function and of the zeta function are composed of an exponential and of a trigonometric component and proved. It is proved, that all these components define imaginary roots on the critical line, if written in the form as they are in the functional equation of the zeta function.

Keywords

Gamma Function, Zeta Function, Critical Line

1. Introduction

The Riemann conjecture states that all complex roots of the zeta function are on the critical line, on the line parallel to the imaginary axis at real one half on the plane of the complex numbers, see [1].

One approach to proving this conjecture is to prove, that all these roots are in a narrowing band around the critical line. Otherwise: there are no roots outside the critical line, see [2]. The present paper takes the approach, that with Euler's formula all roots of all components of the infinite product representing the zeta function are per definition on the critical line. This is because by shifting the critical line to the imaginary axis, all these roots are on this axis.

This approach uses the functional equation of the zeta function, see [1]. Proving, that all other components of this equation—besides of the zeta function—written as infinite products define roots on the critical line, respectively on the imaginary axis, if the critical line is shifted to this axis. This proves involves the definition of the split trigonometric and split hyperbolic functions, especially of the split cosine function.

Lemma 3.1 in [3] is multiple times referenced in the following. This lemma is a generalization of Euler's formula defining for functions without real roots in finite range like ($\cos(\sigma)$) imaginary roots on the imaginary axis: ($e^{-\sigma}+e^{\sigma}=2\cdot\cosh(\sigma)=0$) defines imaginary roots for ($2\cdot\cos(i\cdot\tau)$) at odd multiples of ($\pi/2$).

Lemma 4.1 in [3] is as well referenced in the following. This lemma states, that defining complex roots for similar functions shifted to real ($\sigma=0$), to the critical line, the imaginary components of the complex roots keep their values which they have on the imaginary axis before the shifting.

This is because ($e^{\sigma-0.5}+e^{-(\sigma-0.5)}=\left[e^{\sigma-0.5}+e^{-(\sigma-0.5)}\right]\cdot\sqrt{e}=0$) defines the same imaginary components on the symmetry axis—which is now the critical line—for ($e^{0.5+i\cdot\tau}+e^{-(0.5+i\cdot\tau)}=0$), as demonstrated in **Annex 4**, Figure A8.

Equation $\left(e^{\sigma-0.5}+e^{-(\sigma-0.5)}\right)\cdot\sqrt{e}=0$ may be written as $e^{\sigma}=-e^{1-\sigma}$. In this form it is like the components of the functional equation of the zeta function, which let assume, that the functional equation defines roots for all its components on the critical line. This is in fact the case and is proved subsequently.

2. The Product Representation of the Gamma Function

The exponential function may be written as a polynomial in the form of an infinite product by the aid of the binomial coefficients. The binomial coefficients normed with their maximum and shifted to the origin give the normal distribution. It can be proved, that the normal distribution may be written with the following variables and parameters:

$$\zeta_1(j,n)=\left(j-\frac{n}{2}\right)^2\cdot\frac{2}{n};\quad \zeta_2(j,n)=\left(j-\frac{n}{2}\right)^2\cdot\sqrt{\frac{2}{n}};\quad j=1,2,\cdots,n;\quad n=1,2,\cdots,\infty \tag{2.1}$$

in the following form:

$$\lim_{n\to\infty}e^{-\zeta_1(n)}=\lim_{n\to\infty}e^{-\zeta_2(n)^2}=\lim_{n\to\infty}\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+\zeta_2(n)\cdot\sqrt{\frac{n}{2}}\right)}\cdot\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}-\zeta_2(n)\cdot\sqrt{\frac{n}{2}}\right)}\right] \tag{2.2}$$

The complete proof is not given in the present paper, only the formal identity is demonstrated in **Annex 1**. More details—with step-by-step evolution—are given in [3].

The definition of the gamma function from Gauss is:

$$\Gamma(x)=\lim_{m\to\infty}\left[m^x\cdot\frac{(m-1)!}{x\cdot(x+1)\cdot(x+2)\cdots(x+m-1)}\right]=\lim_{m\to\infty}\left(\frac{m^x}{m}\cdot\prod_{k=0}^{m-1}\frac{k+1}{k+x}\right) \tag{2.3}$$

Written for ($\left(\frac{1}{2}\right)$, $\left(\frac{1+x}{2}\right)$, $\left(\frac{1-x}{2}\right)$) and shortened gives the following quotients, both composed of an exponential part and of a trigonometric part of the relative gamma function:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+x}{2}\right)}=\lim_{n\to\infty}\left[n^{-\frac{x}{2}}\cdot\prod_{j=0}^{n}\left(1+\frac{x}{2\cdot j+1}\right)\right]=\Gamma_{exp}\left(\zeta_2(n)\right)\cdot\Gamma_{tri}\left(\zeta_2(n)\right) \tag{2.4}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)}=\lim_{n\to\infty}\left[n^{-\frac{x}{2}}\cdot\prod_{j=0}^{n}\left(1-\frac{x}{2\cdot j+1}\right)\right]=\Gamma_{exp}\left(-\zeta_2\left(n\right)\right)\cdot\Gamma_{tri}\left(-\zeta_2\left(n\right)\right) \tag{2.5}$$

Again, the step-by-step evolution of these equations is given in [3].

The product of these above quotients eliminates the exponential components, leaving the trigonometric components, which give the known relation for the cosines function:

$$\begin{aligned}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+x}{2}\right)}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)}&=\lim_{n\to\infty}\left[\prod_{j=0}^{n}\left[\left(1+\frac{x}{2\cdot j+1}\right)\cdot\left(1-\frac{x}{2\cdot j+1}\right)\right]\right]\\&=\Gamma_{tri}\left(\zeta_2\left(n\right)\right)\cdot\Gamma_{tri}\left(-\zeta_2\left(n\right)\right)=\prod_{j=1}^{\infty}\left[1-\left(\frac{x}{2\cdot j-1}\right)^2\right]=\cos\left(x\cdot\frac{\pi}{2}\right)\end{aligned} \tag{2.6}$$

Comparing (2.2) and (2.4) for (n) growing to infinity the roots converge to multiples of (π/2). Euler already stated this. All formula and equations down to this point are based on well-known identities formulated by Euler and Gauss. They correspond to high school level. Now something new follows: The application of Euler's identities to the split trigonometric functions.

With (2.6) the trigonometric components of the relative gamma function are the split cosine functions. Because of the special properties of this function, it is rectified to use special names for them. As demonstrated in **Annex 2**, they have exponential as well as periodic properties, the names ($ec_n\left(\sigma\right)$) and ($ec_p\left(\sigma\right)$) will be used for them:

$$\Gamma_{tri}\left(\zeta_2\left(n\right)\right)=ec_n\left(\sigma\right)=\prod_{j=0}^{\infty}\left(1+\frac{\sigma}{2\cdot j+1}\right);\quad \Gamma_{tri}\left(-\zeta_2\left(n\right)\right)=ec_n\left(\sigma\right)=\prod_{j=0}^{\infty}\left(1-\frac{\sigma}{2\cdot j+1}\right) \tag{2.7}$$

As shown in **Annex 2**, the function ($ec_p\left(\sigma\right)$) has roots at odd multiples of (π/2) on the positive side of the real axis. The function ($ec_n\left(\sigma\right)$) has roots at the same values at the negative part of the real axis. They are mutually transposed. With lemma 3.1 in [3] their sum set equal to zero defines roots for their adjoint functions on the adjoint, on the imaginary axis. This in accordance with the relations of Euler.

The sine function is equal to the cosine function shifted by (π/2):

$$\begin{aligned}\sin\left(x\cdot\frac{\pi}{2}\right)&=\cos\left(\left(x-1\right)\cdot\frac{\pi}{2}\right)=\prod_{j=1}^{\infty}\left[1-\left(\frac{x-1}{2\cdot j-1}\right)^2\right]\\&=\prod_{j=1}^{\infty}\left[\frac{2\cdot j-1-\left(x-1\right)}{2\cdot j-1}\right]\cdot\prod_{j=0}^{\infty}\left[\frac{2\cdot j+1+\left(x-1\right)}{2\cdot j+1}\right]\\&=\prod_{j=1}^{\infty}\left[\frac{2\cdot j-x}{2\cdot j-1}\right]\cdot\prod_{j=0}^{\infty}\left[\frac{2\cdot j+x}{2\cdot j+1}\right]=\frac{\pi}{2}\cdot x\cdot\prod_{j=1}^{\infty}\left[1-\left(\frac{x}{2\cdot j}\right)^2\right]\end{aligned} \tag{2.8}$$

Similarly, to (2.7) the split sine functions are named as ($es_n(\sigma)$) and ($es_p(\sigma)$):

$$es_n(\sigma) = \sqrt{\frac{\pi}{2} \cdot \sigma} \cdot \prod_{j=1}^{\infty}\left(1 + \frac{\sigma}{2 \cdot j}\right); \quad es_p(\sigma) = \sqrt{\frac{\pi}{2} \cdot \sigma} \cdot \prod_{j=1}^{\infty}\left(1 - \frac{\sigma}{2 \cdot j}\right) \tag{2.9}$$

With (4.10) in [3] the infinite product polynomials of the exponential functions are:

$$\mathrm{e}^{\sigma} = e_n(\sigma) = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1 + \frac{\sigma}{a_2(k,n)}\right); \quad \mathrm{e}^{-\sigma} = e_p(\sigma) = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1 - \frac{\sigma}{a_2(k,n)}\right) \tag{2.10}$$

with

$$\left(a_2(k,n) = \frac{(2 \cdot k + n)^2}{3 \cdot n} \right). \tag{2.11}$$

The relative gamma functions (2.6) may be written with the infinite products of the exponential functions above and with the split cosine functions from (2.7) as follows:

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left[\left(1 - \frac{s \cdot \ln(n)}{2 \cdot a_2(k,n)}\right)\right] \cdot \prod_{j=0}^{n}\left[\left(1 + \frac{s}{2 \cdot j + 1}\right)\right]\right] \tag{2.12}$$

$$\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \lim_{n\to\infty}\left[\prod_{k=0}^{n}\left[\left(1 + \frac{s \cdot \ln(n)}{2 \cdot a_2(k,n)}\right)\right] \cdot \prod_{j=0}^{n}\left[\left(1 - \frac{s}{2 \cdot j + 1}\right)\right]\right]$$

The relative gamma function is compared with its infinite polynomial product representation in **Annex 3**.

The sum and the difference of the split cosine functions ($ec_n(\sigma,n)$) and ($ec_p(\sigma,n)$) in (2.7) and the corresponding split sine functions (2.9) result the corresponding hyperbolic functions ($ecch(\sigma,n)$) and ($ecsh(\sigma,n)$). These functions are equal to the cosine and sine hyperbolic functions with (k_{cosh})-fold arguments. This factor is dependent on the number of the components (n) of the infinite products:

$$ecch(n) = \frac{1}{2}\left(ec_p(n) + ec_n(n)\right) = \cosh\left(k_{cosh} \cdot \sigma\right) = \frac{1}{2} \cdot \left(\mathrm{e}^{k_{cosh} \cdot \sigma} + \mathrm{e}^{-k_{cosh} \cdot \sigma}\right) \tag{2.13}$$

$$ecsh(n) = \frac{1}{2}\left(ec_p(n) - ec_n(n)\right) = \sinh\left(k_{cosh} \cdot \sigma\right) = \frac{1}{2} \cdot \left(\mathrm{e}^{k_{cosh} \cdot \sigma} - \mathrm{e}^{-k_{cosh} \cdot \sigma}\right)$$

The dependence of the factor (k_{cosh}) on the number of the components within the infinite products (n) is analyzed in **Annex 2**: The factor is evaluated up to the number of components equal to ($n = 2 \times 10^8$). The effective values of the factor are approximated by the following formula:

$$k_{cosh}(n) = \ln\left(\mathrm{e}^{\frac{3}{4}} \cdot \sqrt{n}\right); \quad k_{cosh}(n) = \frac{3}{4} + \frac{\ln(n)}{2} \tag{2.14}$$

The product of the split cosine functions ($ec_n(\sigma,n)$) and ($ec_p(\sigma,n)$) result the cosine function ($\cos(\sigma)$):

$$\prod_{j=0}^{n}\left[\left(1+\frac{s}{2\cdot j+1}\right)\right]\cdot\prod_{j=0}^{n}\left[\left(1-\frac{s}{2\cdot j+1}\right)\right]=\prod_{j=0}^{n}\left[\left(1-\left(\frac{s}{2\cdot j+1}\right)^2\right)\right]$$
$$=ec_n(\sigma,n)\cdot ec_p(\sigma,n)=\cos(\sigma) \quad (2.15)$$

The relations for the exponential function with real roots—written as infinite polynomial product—result as adjoint function the trigonometric function the cosine function—as infinite polynomial product—with roots exclusively on the imaginary axis (see lemma 3.1 in [3]), corresponding to the relation of Euler:

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left(1-\frac{\sigma}{a_2(k,n)}\right)+\lim_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{\sigma}{a_2(k,n)}\right) = \lim_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{\tau^2}{c_1(k,n)}\right) \quad (2.16)$$
$$P_{p\_p\_e}(\sigma,n)+P_{p\_n\_e}(\sigma,n)=2\cdot Q_{n\_e}(\tau,i\cdot C_1);$$
$$e^{\sigma}+e^{-\sigma}=2\cdot\cosh(\sigma)=2\cdot\cos(i\cdot\tau)$$

The Euler relation extended to the split cosine function gives:

$$\lim_{n\to\infty}\left(ec_p(\sigma,n)+ec_n(\sigma,n)\right)=\frac{1}{2}\cdot\left(e^{-k_{cosh}\cdot\sigma}+e^{k_{cosh}\cdot\sigma}\right)$$
$$=ecch(\sigma,n)=\cosh(k_{cosh}\cdot\sigma)=\cos(i\cdot k_{cosh}\cdot\tau) \quad (2.17)$$

The split cosine functions with roots on the real axis are like both, the exponential functions, and the trigonometric functions: their sum defines a function like the cosine hyperbolic function, their product results the cosine function. Herewith the sum and difference of the split components of the cosine functions with real roots defines roots for their adjoint functions ($ecch(i\cdot\tau)$) and ($ecsh(i\cdot\tau)$) on the adjoint, on the imaginary axis:

$ec_p(\sigma)=-ec_n(\sigma)$ defines roots for $ecch(i\cdot\tau)\approx\cos(k_{cosh}\cdot i\cdot\tau)$ (2.18)

$ec_p(\sigma)=ec_n(\sigma)$ defines roots for $ecsh(i\cdot\tau)\approx\sin(k_{cosh}\cdot i\cdot\tau)$

$ecch(i\cdot\tau)=\frac{1}{2}\cdot\left(ec_p(i\cdot\tau)+ec_n(i\cdot\tau)\right)=0$ defines roots for $ec_p(\sigma)\approx\cos(k_{cosh}\cdot\sigma)$ (2.19)

$ecsh(i\cdot\tau)=\frac{1}{2\cdot i}\cdot\left(ec_p(i\cdot\tau)-ec_n(i\cdot\tau)\right)=0$ defines roots for $es_p(\sigma)\approx\sin(k_{cosh}\cdot\sigma)$

The placement of the roots exclusively on the adjoint axis is independent of the number of factors k_{cosh} applied in the infinite products (2,13): they may influence only the value of the roots. Because the present paper concerns only the placement of the roots, the effect of the factor (k_{cosh}) may be neglected.

In **Figure A9** the value of the roots of the split cosine functions with complex arguments on the critical line approaches with rising number of the components (n) within the infinite product polynomials the value given by the cosine hyperbolic function with complex arguments: for sufficient identity in the figure, it is risen to the tenfold value in comparison with the other figures earlier in **Annex**

4. The value of the roots of the cosine hyperbolic function may be reached by (n) growing to infinity. This shift on the critical line does not influence the fact, which the roots are on the critical line.

At the same time the rising number of the components considered in the infinite polynomial products the function value between the roots—with intermittent sign—is rising too.

Herewith the following lemma is formulated:

Lemma 2.1:

The split components of the cosine function shifted on the real axis by (1/2) define the roots for the adjoint functions on the critical line, nearing the same values, which they have on the imaginary axis before the shifting.

Proof: With lemma 3.1 in [3] the shifting of the roots of the cosine and of the sine functions by (1/2) on the real axis shifts the roots of the adjoint functions to the critical line, leaving the value of the imaginary part of the complex roots unchanged.

Because the arguments of the adjoint functions of the split components of the cosine hyperbolic function are proportional to the arguments of the adjoint functions of the cosine hyperbolic function, the placement of the imaginary components of the complex roots on the critical line are not only similar, but are approaching—with the number of the components of the infinite products (n) growing to infinity—their corresponding values, as stated in the lemma and concluding the proof.

The functional equation of the Riemann-function (see [1]) is written as follows:

$$\xi(s)=\Gamma\left(\frac{s}{2}\right)\cdot\pi^{-\frac{s}{2}}\cdot\zeta(s)=\Gamma\left(\frac{1-s}{2}\right)\cdot\pi^{-\frac{1-s}{2}}\cdot\zeta(1-s)=\xi(1-s) \tag{2.20}$$

The exponential component is written in the following form, defining complex roots on the critical line:

$$\begin{gathered}\mathrm{e}^{-\sigma\cdot\ln\left(\sqrt{\pi}\right)}=\mathrm{e}^{-(1-\sigma)\cdot\ln\left(\sqrt{\pi}\right)};\ \ \mathrm{e}^{-\sigma\cdot\ln\left(\sqrt{\pi}\right)}\cdot\mathrm{e}^{\frac{1}{2}\cdot\ln\left(\sqrt{\pi}\right)}=\mathrm{e}^{-(1-\sigma)\cdot\ln\left(\sqrt{\pi}\right)}\cdot\mathrm{e}^{\frac{1}{2}\cdot\ln\left(\sqrt{\pi}\right)}\\ \mathrm{e}^{-\left(\sigma-\frac{1}{2}\right)\cdot\ln\left(\sqrt{\pi}\right)}=\mathrm{e}^{\left(\sigma-\frac{1}{2}\right)\cdot\ln\left(\sqrt{\pi}\right)};\ \ \mathrm{e}^{\sigma_c\cdot\ln\left(\sqrt{\pi}\right)}=\mathrm{e}^{-\sigma_c\cdot\ln\left(\sqrt{\pi}\right)}\end{gathered} \tag{2.21}$$

The corresponding equation of the infinite polynomial products corresponding to lemma 4.1 in [3] is:

$$\lim_{n\to\infty}\prod_{k=0}^{n}\left(1-\frac{s}{i\cdot\dfrac{a_2(k,n)}{\ln\left(\sqrt{\pi}\right)}}\right)=\lim_{n\to\infty}\prod_{k=0}^{n}\left(1+\frac{1-s}{i\cdot\dfrac{a_2(k,n)}{\ln\left(\sqrt{\pi}\right)}}\right) \tag{2.22}$$

The gamma component of the functional equation of the zeta function has with (2.4) two parts, the exponential part, and the trigonometric part. Similarly, to (2.19) the exponential part—in the form written in the functional equation of the zeta function—define with (2.21) roots on the critical line. This allows to formulate the following lemma:

Lemma 2.2:

The roots of the split cosine functions on one axis define roots for the adjoint functions on the adjoint axis.

Proof: The second-degree split polynomials () and () are mutually transpose, with roots on the real axis and correspond in all aspect to infinite polynomials with monotonously rising positive components, defined in Lemma 2.1 in [3]. Therefore, the sum and difference of the second-degree split polynomials define adjoint polynomials with roots on the adjoint axis, on the imaginary axis (see (6.7) in [3]), as stated in the lemma and concluding the proof.

The trigonometric components of the relative gamma function being the split components of the cosine function, with lemma 2.1, in case their roots on the real axis are shifted by (/) the roots of their adjoint functions are shifted to the critical line and they preserve the placement of their roots on the imaginary axis as components of the complex roots on the critical line.

This corresponds to the shifting to the critical line of the cosine function, which is the product of its split components:

$$\begin{aligned} & Q_{p\_n\_e\left[(\sigma-0.5)\cdot\frac{\pi}{2}\right]}\cdot Q_{p\_p\_e\left[(\sigma-0.5)\cdot\frac{\pi}{2}\right]} \\ & =\lim_{n\to\infty}\prod_{j=1}^{n}\left[\left(1+\frac{\sigma-0.5}{2\cdot j-1}\right)\right]\cdot\prod_{j=0}^{n}\left[\left(1-\frac{\sigma-0.5}{2\cdot j-1}\right)\right] \\ & =\lim_{n\to\infty}\prod_{j=1}^{n}\left[\left(1-\left(\frac{\sigma-0.5}{2\cdot j-1}\right)^{2}\right)\right]=\cos(\sigma-0.5) \end{aligned} \tag{2.23}$$

Herewith both components of the gamma function define—written in the form of the functional equation of the zeta function—roots on the critical line, as demonstrated in **Annex 4**.

3. Infinite Polynomial Product for the Riemann Zeta Function

The zeta function ($\zeta(s)$) written for the series of primes ($P_{(n)}$) as infinite product, composed of an exponential and a trigonometric part is:

$$\begin{aligned} \zeta(s) & =\lim_{n\to\infty}\prod_{k=0}^{n}\left[1-\frac{1}{\left[P_{(k)}\right]^{s}}\right]^{-1}=\lim_{n\to\infty}\prod_{k=0}^{n}\frac{1}{1-\mathrm{e}^{-s\cdot\ln\left[P_{(k)}\right]}} \\ & =\lim_{n\to\infty}\prod_{k=0}^{n}\frac{\mathrm{e}^{\frac{s}{2}\cdot\ln\left[P_{(k)}\right]}}{\mathrm{e}^{\frac{s}{2}\cdot\ln\left[P_{(k)}\right]}-\mathrm{e}^{-\frac{s}{2}\cdot\ln\left[P_{(k)}\right]}}=\zeta_{exp}(s)\cdot\zeta_{trig}(s) \end{aligned} \tag{3.1}$$

The functional equation of the zeta function (2.20) is:

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\cdot\frac{\pi^{-\frac{s}{2}}}{\pi^{-\frac{1-s}{2}}}=\frac{\zeta(1-s)}{\zeta(s)} \tag{3.2}$$

Setting equal the numerator and the denominator of the gamma and of the pi functions to each other they define complex roots exclusively on the critical line, respectively on the imaginary axis, if shifted there.

Therefore, this equation states, that the same is valid for the zeta function as well: it defines complex roots and/or poles exclusively on the critical line:

$$\Gamma\left(\frac{s}{2}\right)=\Gamma\left(\frac{1-s}{2}\right);\ \ \mathrm{e}^{-s\cdot\ln\left(\sqrt{\pi}\right)}=\mathrm{e}^{-(1-s)\cdot\ln\left(\sqrt{\pi}\right)};\ \ \zeta(1-s)=\zeta(s) \tag{3.3}$$

$$\zeta_{exp}(1-s)\cdot\zeta_{trig}(1-s)=\zeta_{exp}(s)\cdot\zeta_{trig}(s);\ \ \zeta_{exp}(1-s)=\zeta_{exp}(s);$$
$$\zeta_{trig}(1-s)=\zeta_{trig}(s)$$

Thus, it is valid for both components of the zeta functions as well:

$$\mathrm{e}^{s\cdot\ln\left(\sqrt{P_{(k)}}\right)}=\mathrm{e}^{(1-s)\cdot\ln\left(\sqrt{P_{(k)}}\right)} \tag{3.4}$$
$$\mathrm{e}^{s\cdot\ln\left(\sqrt{P_{(k)}}\right)}-\mathrm{e}^{-s\cdot\ln\left(\sqrt{P_{(k)}}\right)}=\mathrm{e}^{(1-s)\cdot\ln\left(\sqrt{P_{(k)}}\right)}-\mathrm{e}^{-(1-s)\cdot\ln\left(\sqrt{P_{(k)}}\right)}$$

In this form the trigonometric part of the zeta function is just a repetition of the exponential part:

$$\mathrm{e}^{s\cdot\ln\left(\sqrt{P_{(k)}}\right)}=\mathrm{e}^{(1-s)\cdot\ln\left(\sqrt{P_{(k)}}\right)};\ \ \mathrm{e}^{-s\cdot\ln\left(\sqrt{P_{(k)}}\right)}=\mathrm{e}^{-(1-s)\cdot\ln\left(\sqrt{P_{(k)}}\right)}$$

If Equation (3.4) defines complex roots and/or poles exclusively on the critical line, then multiplying both sides by ($\mathrm{e}^{\frac{1}{2}\cdot\ln\left(\sqrt{P_{(k)}}\right)}$) and replacing (σ) by ($\sigma_c+\frac{1}{2}$) shifts the symmetry axis with the roots from the critical line to the imaginary axis for the exponential part:

$$\mathrm{e}^{\sigma\cdot\ln\left[\sqrt{P_{(k)}}\right]}\mathrm{e}^{\left(-\frac{1}{2}\right)\ln\left[\sqrt{P_{(k)}}\right]}=\mathrm{e}^{(1-\sigma)\cdot\ln\left[\sqrt{P_{(k)}}\right]}\mathrm{e}^{\left(-\frac{1}{2}\right)\ln\left[\sqrt{P_{(k)}}\right]} \tag{3.6}$$
$$\mathrm{e}^{-\left(\frac{1}{2}-\sigma\right)\cdot\ln\left[\sqrt{P_{(k)}}\right]}=\mathrm{e}^{\left(\frac{1}{2}-\sigma\right)\cdot\ln\left[\sqrt{P_{(k)}}\right]};\ \ \mathrm{e}^{-\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]}=\mathrm{e}^{\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]}$$
$$\mathrm{e}^{\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]}-\mathrm{e}^{-\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]}=\frac{2}{i}\sinh\left[\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]\right]$$

In this form setting equal to zero defines the sine function with roots exclusively on the imaginary axis. Therefore, before the shifting—with ($\sigma=\sigma_c+\frac{1}{2}$)—these roots are exclusively on the critical line with roots and/or poles at even multiples of ($\pi/2$):

$\frac{2}{i}\cdot\sinh\left[\sigma_c\cdot\ln\left[\sqrt{P_{(k)}}\right]\right]=0$ defines $2\cdot\sin\left[i\cdot\tau_c\cdot\ln\left[\sqrt{P_{(k)}}\right]\right]$ with roots

$$\tau_c(n)=\frac{2\cdot n\cdot\pi}{\ln\left[\sqrt{P_{(k)}}\right]} \tag{3.7}$$

$\frac{2}{i}\cdot\sinh\left[\left(\sigma-\frac{1}{2}\right)\cdot\ln\left[\sqrt{P_{(k)}}\right]\right]=0$ defines $2\cdot\sin\left[\left(\frac{1}{2}+i\cdot\tau_c\right)\cdot\ln\left[\sqrt{P_{(k)}}\right]\right]$ with roots $\tau_c(n)=\frac{2\cdot n\cdot\pi}{\ln\left[\sqrt{P_{(k)}}\right]}$

The fact, that the shifting of the symmetry axis does not change the placement of the roots on the imaginary axis, is demonstrated in **Annex 4**.

4. Conclusion

Herewith all sub equations with components of the functional equation define roots on the critical line. Thus, the functional equation of the zeta function defines all complex roots on the critical line.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Annex 1: The Normal Distribution and the Exponential Function

The normal distribution and the exponential functions are shown in Figure A1 below for the ranges ($n = 400$), for ($j = 1,\cdots,n$) and for ($\sigma = -4,-3.9,\cdots,4$). The complex variables are defined in (2.1):

$$\zeta_1(j,n) = \left(j - \frac{n}{2}\right)^2 \cdot \frac{2}{n};\quad \zeta_2(j,n) = \left(j - \frac{n}{2}\right) \cdot \sqrt{\frac{2}{n}} \tag{A1.1}$$

The infinite polynomial products for the normal distribution and for the exponential functions as well as their roots are defined in (2.10) and (2.11):

$$a_2(k,n) = \frac{(2\cdot k + n)^2}{3\cdot n};\quad \lim_{n\to\infty} a_2(k,n) = \infty \tag{A1.2}$$

$$F_1(j,n) = \prod_{k=0}^{n}\left(1 - \frac{\zeta_1(j,n)}{a_2(k,n)}\right);\quad F_2(\sigma) = \prod_{k=0}^{n}\left(1 - \frac{\sigma^2}{a_2(k,n)}\right)$$

$$P_{p\_n\_e} = \prod_{k=0}^{n}\left(1 + \frac{\sigma}{a_2(k,n)}\right);\quad P_{p\_p\_e} = \prod_{k=0}^{n}\left(1 - \frac{\sigma}{a_2(k,n)}\right)$$

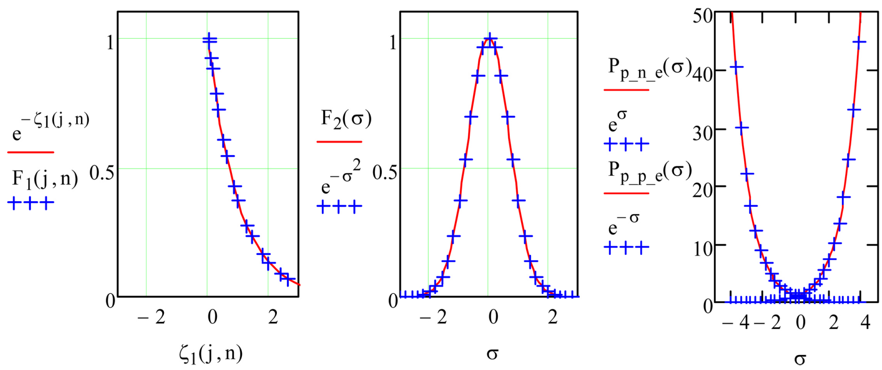

Figure A1. Normal distribution and exponential function, compared with their infinite product representation.

Annex 2: The Split Trigonometric Components of the Gamma Function

The second-degree split polynomials ($ec_p(\sigma,n)$), ($ec_n(\sigma,n)$) and their product, the cosine function are with (2.6) and (2.7):

$$ec_p(\sigma,n) = \prod_{j=1}^{n}\left(1 - \frac{\sigma}{2\cdot j - 1}\right);\quad ec_n(\sigma,n) = \prod_{j=1}^{n}\left(1 + \frac{\sigma}{2\cdot j - 1}\right) \tag{A2.1}$$

$$ec(\sigma,n) = ec_p(\sigma,n)\cdot ec_n(\sigma,n) = \cos\left(\frac{\pi}{2}\cdot\sigma\right)$$

The absolute value of the split components of the cosines function are shown in Figure A2 below with the parameter ($n = 50000$), for the range ($\sigma = -10,-9.8,\cdots,10$), as well as their product. The split component ($ec_p(\sigma,n)$) of the cosine function has only positive real roots at odd multiples of (π/2) and

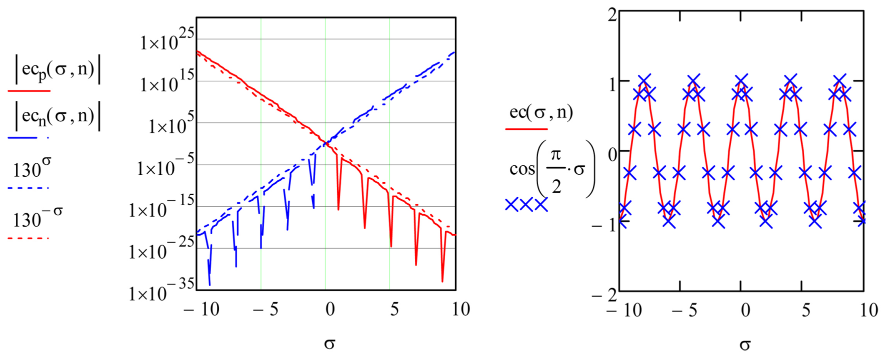

Figure A2. The split periodic components of the gamma polynomial with all real roots.

its absolute value is decreases, the other split component with only negative real roots. The split components are mutually transpose. Both components are equal to unity at zero: ($ec_n(0,n)=1$, $ec_p(0,n)=1$).

The detail of the split cosine function in the range ($\sigma\sigma=0,0.1,\cdots,4$) is shown in **Figure A3** below. The function changes its sign at odd multiples of (π/2), therefore it has roots at these points.

This figure shows that the function ($ec_p(\sigma\sigma,n)$) changes the sign at ($\sigma=1$) and at ($\sigma=3$). Therefore, the function has roots everywhere, where the function cosine has roots on the positive part of the real axis.

Similarly, the function ($ec_n(\sigma\sigma,n)$) has roots everywhere, where the cosine function has roots on the negative part of the real axis.

With (2.19) the sum and the difference of these functions define the first-degree split functions, which formally correspond to the cosine hyperbolic and sine hyperbolic functions:

$$ecch(\sigma,n)=\frac{1}{2}\left(ec_p(\sigma,n)+ec_n(\sigma,n)\right);\quad ecsh(\sigma,n)=\frac{1}{2}\left(ec_p(\sigma,n)-ec_n(\sigma,n)\right) \tag{A2.2}$$

The factor ($k_{cosh}(n)$) defined in (2.14) is evaluated for the range up to the number of components ($n_{lim}=2\times10^8$) once and the results are written to files, because the evaluation is quite time consuming. These files are read:

$$n_{lim}=\text{READPRN}(\text{"prod\_lim.prn"});\quad k_{kosh\_eff}=\text{READPRN}(\text{"factor\_cosh.prn"})$$

The number of the evaluated effective values of the factors is:
($length(n_{lim})-1=2\times10^8$).

The approximating function (2.12) is evaluated for the same numbers of the factors and the results are compared with the effective values in the range ($j=1,\cdots,length(n_{lim})-1$) in **Figure A4** below. The evaluation of the effective values above the limit of (n_{lim}) is very time-consuming. The approximation formula given below is valid in the observed range.

$$k_{cosh\_appr_{(j)}}=\frac{3}{4}+\ln\left[\sqrt{n_{lim_{(j)}}}\right];\quad k_{cosh}(n)=\frac{3}{4}+\ln\left[\sqrt{n}\right] \tag{A2.3}$$

With the parameter ($n=40000$) and for the range ($\sigma=-1,-0.9,\cdots,1$) they are shown for real arguments in **Figure A5** below. In fact, they are proportional to the cosine hyperbolic and sine hyperbolic functions, with the factor of proportionality of the arguments ($k = k_{kosh}(n) = 6.048$) corresponding to the applied number of components of the infinite products ($n = 40000$).

In the range ($\sigma = -0.5, -0.48, \cdots, 0.5 \ldots; \ldots \tau = -0.5, -0.48, \cdots, 0.5$) he split sine hyperbolic and the split cosine hyperbolic function sin central positions are shown in **Figure A6** below.

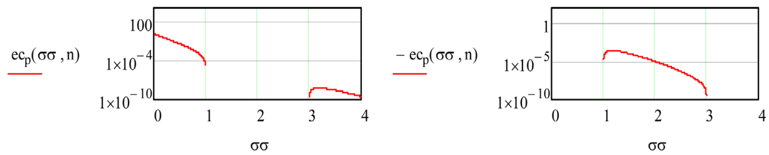

Figure A3. A section of the roots of the split cosine function.

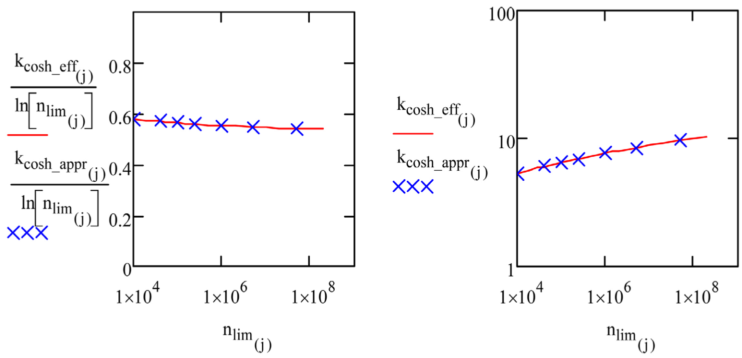

Figure A4. The factor between the split cosine hyperbolic and the hyperbolic functions.

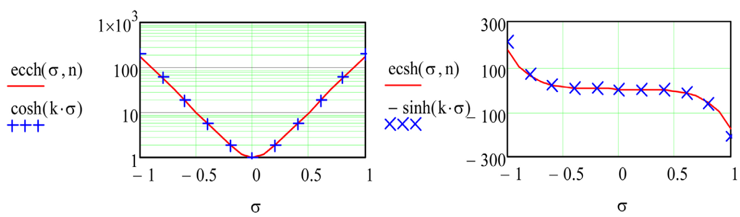

Figure A5. The sum and difference of the split cosine function for real arguments.

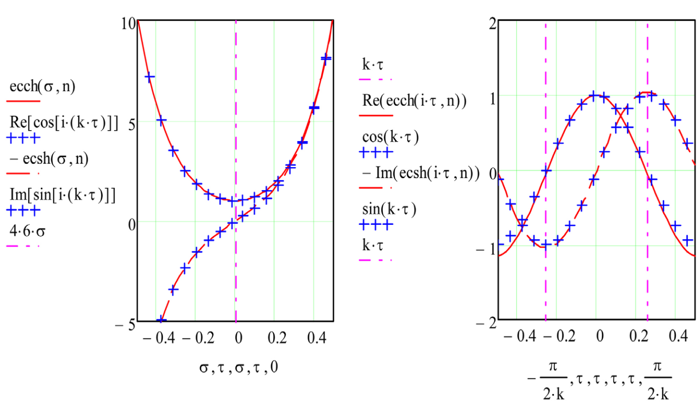

Figure A6. Split cosine and split sine hyperbolic functions and their adjoint function.

Annex 3: The Relative Gamma Function

With (2.12) the approximation of the gamma function by infinite products for reel arguments, as a polynomial is:

$$\Gamma_{rel\_appr}(x,m) = \prod_{k=0}^{m}\left(1 - \frac{x \cdot \ln(m)}{2 \cdot a_2(k,m)}\right) \cdot \prod_{k=0}^{m}\left(1 + \frac{x}{2 \cdot j + 1}\right) \tag{A3.1}$$

In the range ($j = 1, \cdots, 150$) for real arguments ($x_{(j)} = \frac{j-50}{10}$) and for the number of quotients considered ($m = 5000$) the relative gamma function and its approximation as infinite product are compared in **Figure A7** below. Since the evaluation is quite time consuming, the results are written to a file. They are read in case of the evaluation of the present paper:

$$F_{gamma_{(j)}} = \Gamma_{rel\_appr\left[x_{(j)},m\right]}; \quad \text{WRITEPRN}(\text{"GAMMA\_appr.PRN"}) = F_{gamma} \tag{A3.2}$$

$$F_{gamma} = \text{READPRN}(\text{"GAMMA\_appr.PRN"});$$

$$kk = 1, \cdots, length\left(F_{gamma}\right) - 1: \quad x_{(kk)} = \frac{kk-50}{10}$$

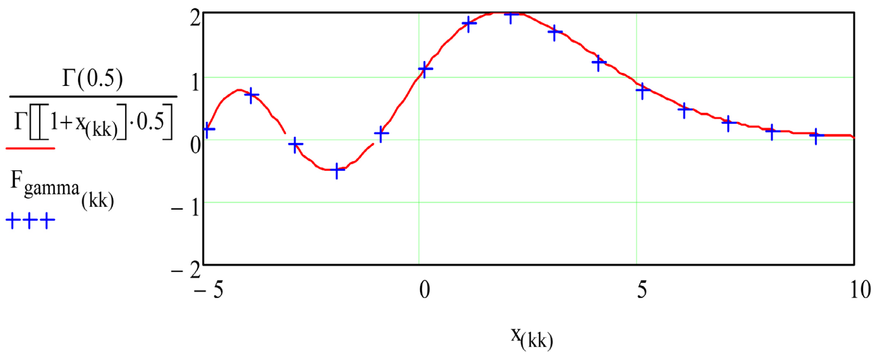

Figure A7. The relative gamma function expressed as infinite product.

Annex 4: Hyperbolic and Trigonometric Functions Shifted to the Critical Line

In the range ($\sigma = -3, -2.8, \cdots, 3$, $\tau = -3, -2.8, \cdots, 3$) the sine hyperbolic and the cosine hyperbolic functions in central position are in **Figure A6** and in shifted position in **Figure A8** below.

The split sine hyperbolic and the split cosine hyperbolic functions are shown in shifted position in **Figure A9** below, in the range ($\sigma = -0.5, -0.48, \cdots, 1.5$, $\tau = -0.5, -0.48, \cdots, 1.5$), with the parameter ($n = 400000$), and the factor of proportionality of the arguments ($k = k_{cosh}(n) = 7.2$):

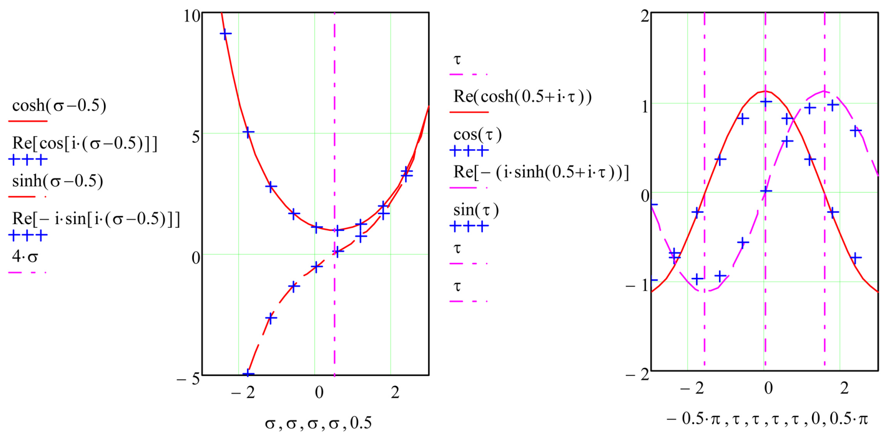

Figure A8. The cosine and the sine hyperbolic functions shifted to the critical line.

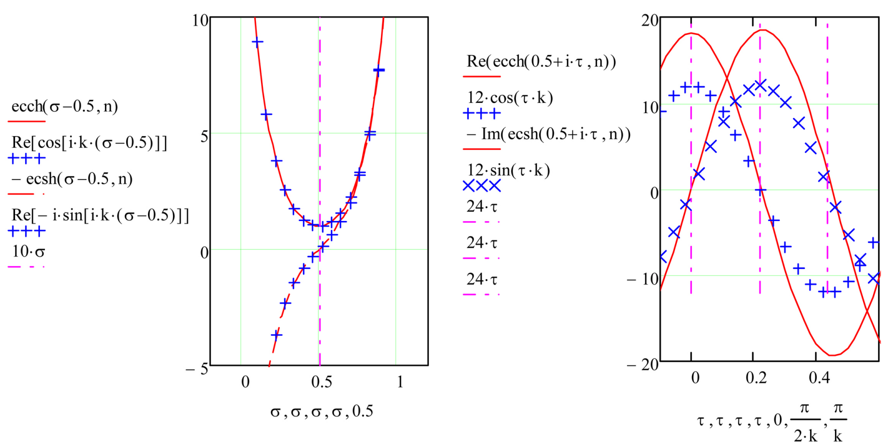

Figure A9. Split cosine and split sine hyperbolic functions shifted to the critical line.