Scientific Research Publishing

Advances in Pure Mathematics, 2022, 12, 436-450
https://www.scirp.org/journal/apm
ISSN Online: 2160-0384
ISSN Print: 2160-0368

Multipliers and Classification of *m*-Möbius Transformations

Dorin Ghisa[1], Eric Mikulin[2]

[1]York University, Toronto, Canada
[2]UBC, Vancouver, Canada
Email: dghisa@yorku.ca, ericmikulin@protonmail.com

How to cite this paper: Ghisa, D. and Mikulin, E. (2022) Multipliers and Classification of *m*-Möbius Transformations. *Advances in Pure Mathematics*, **12**, 436-450.
https://doi.org/10.4236/apm.2022.126033

Received: May 27, 2022
Accepted: June 27, 2022
Published: June 30, 2022

Abstract

It is known that any *m*-Möbius transformation is an ordinary Möbius transformation in every one of its variables when the other variables do not take the values a and $1/a$, where a is a parameter defining the respective *m*-Möbius transformation. For ordinary Möbius transformations having distinct fixed points, the multiplier associated with one of these points completely characterizes the nature of that transformation, *i.e.* it tells us if it is elliptic, hyperbolic or loxodromic. The purpose of this paper is to show that fixed points exist also for *m*-Möbius transformations and multipliers associated with them can be computed as well. As in the classical case, the values of those multipliers describe completely the nature of the transformations. The method we used was that of a thorough study of the coefficients of the variables involved, with which occasion we discovered surprising symmetries. These were the results allowing us to prove the main theorem regarding the fixed points of a *m*-Möbius transformation, which is the key to further developments. Finally we were able to illustrate the geometric aspects of these transformations, making the whole theory as intuitive as possible. It was as opening a window into a space of several complex variables. This allows us to prove that if a bi-Möbius transformation is elliptic or hyperbolic in z_1 at a point z_2 it will remain the same on a circle or line passing through z_2. This property remains true when we switch z_1 and z_2. The main theorem, dealing with the fixed points of an arbitrary *m*-Möbius transformation made possible the extension of this result to these transformations.

Keywords

m-Möbius Transformations, Multiplier, Elliptic, Hyperbolic, Parabolic, Loxodromic, Steiner Net

1. Introduction

The m-Möbius transformations are generated (see [1]) starting with

$$f_2(z_1,z_2)=\frac{\omega s_2-s_1+1}{s_2-s_1+\omega}=\frac{(\omega z_2-1)z_1+1-z_2}{(z_2-1)z_1+\omega-z_2}=\frac{(\omega z_1-1)z_2+1-z_1}{(z_1-1)z_2+\omega-z_1} \tag{1}$$

where $\omega\in\overline{\mathbb{C}}\setminus\{1\}$ and $s_1=z_1+z_2$, $s_2=z_1z_2$.

Sometimes it will be preferable to use instead of the parameter ω the parameter a, where $\omega=a+\frac{1}{a}-1$, $a\in\overline{\mathbb{C}}\setminus\{1\}$.

Applying recursively f_2 we get

$f_3(z_1,z_2,z_3)=f_2(f_2(z_1,z_2),z_3)=f_2(z_1,f_2(z_2,z_3))$.

An easy computation shows that

$$f_3(z_1,z_2,z_3)=\frac{(\omega+1)s_3-s_2+1}{s_3-s_1+(\omega+1)},s_3=z_1z_2z_3, \\ s_2=z_1z_2+z_1z_3+z_2z_3,s_1=z_1+z_2+z_3 \tag{2}$$

If for arbitrary m we set

$$\begin{aligned}f_m(z_1,z_2,\cdots,z_m)&=f_2(f_{m-1}(z_1,z_2,\cdots,z_{m-1}),z_m)\\&=\frac{(\omega z_m-1)f_{m-1}(z_1,z_2,\cdots,z_{m-1})+1-z_m}{(z_m-1)f_{m-1}(z_1,z_2,\cdots,z_{m-1})+\omega-z_m}\end{aligned} \tag{3}$$

this will allow the computation of f_m when f_{m-1} is known, *i.e.* when all f_k from $k=2$ to $k=m-1$ have been computed.

We have proved in [1] and [2] that

$f_m(z_1,z_2,\cdots,z_m)=\frac{a_0(\omega)s_m+a_1(\omega)s_{m-1}+\cdots+a_m(\omega)}{a_m(\omega)s_m+a_{m-1}(\omega)s_{m-1}+\cdots+a_0(\omega)}$, where $a_k(\omega)$ are polynomials and s_j are symmetric sums of order j of $z_1,z_2,\cdots,z_m$. Moreover, we have shown that for $m=2k$ and $m=2k+1$ we have that $a_0(\omega)$ are polynomials of degree k and $a_m(\omega)$ are polynomials of degrees $k-1$. Let us denote by p_k,q_k, respectively p_{k-1} and q_{k-1} these polynomials, *i.e.*

$$f_{2k}(z_1,z_2,\cdots,z_{2k})=\frac{p_k(\omega)s_{2k}+\cdots+p_{k-1}(\omega)}{p_{k-1}(\omega)s_{2k}+\cdots+p_k(\omega)} \quad \text{and}$$

$$f_{2k+1}(z_1,z_2,\cdots,z_{2k+1})=\frac{q_k(\omega)s_{2k+1}+\cdots+q_{k-1}(\omega)}{q_{k-1}(\omega)s_{2k+1}+\cdots+q_k(\omega)}.$$

Let us notice that it is not obvious what should be in the blanks of these formulas and there is no way to proceed further without knowing it. The help comes from the formula (3) which implies:

$$p_{k+1}(\omega)=\omega q_k(\omega)-q_{k-1}(\omega) \tag{4}$$

and

$$(\omega-1)q_k(\omega)=\omega p_k(\omega)-p_{k-1}(\omega) \tag{5}$$

These formulas allow us to compute recursively p_k and q_k for every k. Indeed, by (1) and (2) we have: $p_1=\omega$, $p_0=1$, $q_1=\omega+1$, $q_0=1$. Using (4) we get:

$$p_2 = \omega(\omega+1)-1 = \omega^2+\omega-1 \tag{6}$$

Using (5) we get: $(\omega-1)q_2 = \omega(\omega^2+\omega-1)-\omega = \omega^3+\omega^2-2\omega$, which gives:

$$q_2 = \omega^2+2\omega \tag{7}$$

Using (4) again we obtain: $p_3 = \omega(\omega^2+2\omega)-\omega-1$, hence

$$p_3 = \omega^3+2\omega^2-\omega-1 \tag{8}$$

Using (5) again we have:

$(\omega-1)q_3 = \omega(\omega^3+2\omega^2-\omega-1)-(\omega^2+\omega-1) = \omega^4+2\omega^3-2\omega^2-2\omega+1$, thus

$$q_3 = \omega^3+3\omega^2+\omega-1 \tag{9}$$

Analogously, we compute:

$$p_4 = \omega^4+3\omega^3-3\omega \tag{10}$$

$$q_4 = \omega^4+4\omega^3+3\omega^2-2\omega-1 \tag{11}$$

These expressions agree with those found in [1] for $f_k, k=2,3,\cdots,9$. Moreover, with the notation $f_m(\mathbf{z})$ instead of $f_m(z_1,z_2,\cdots,z_m)$ we have:

$$f_4(\mathbf{z}) = \frac{p_2s_4-p_1s_3+s_2-s_1+p_1}{p_1s_4-s_3+s_2-p_1s_1+p_2} \tag{12}$$

$$f_5(\mathbf{z}) = \frac{q_2s_5-q_1s_4+s_3-s_1+q_1}{q_1s_5-s_4+s_2-q_1s_1+q_2} \tag{13}$$

$$f_6(\mathbf{z}) = \frac{p_3s_6-p_2s_5+p_1s_4-s_3+s_2-p_1s_1+p_2}{p_2s_6-p_1s_5+s_4-s_3+p_1s_2-p_2s_1+p_3} \tag{14}$$

$$f_7(\mathbf{z}) = \frac{q_3s_7-q_2s_6+q_1s_5-s_4+s_2-q_1s_1+q_2}{q_2s_7-q_1s_6+s_5-s_3+q_1s_2-q_2s_1+q_3} \tag{15}$$

$$f_8(\mathbf{z}) = \frac{p_4s_8-p_3s_7+p_2s_6-p_1s_5+s_4-s_3+p_1s_2-p_2s_1+p_3}{p_3s_8-p_2s_7+p_1s_6-s_5+s_4-p_1s_3+p_2s_2-p_3s_1+p_4} \tag{16}$$

$$f_9(\mathbf{z}) = \frac{q_4s_9-q_3s_8+q_2s_7-q_1s_6+s_5-s_3+q_1s_2-q_2s_1+q_3}{q_3s_9-q_2s_8+q_1s_7-s_6+s_4-q_1s_3+q_2s_2-q_3s_1+q_4} \tag{17}$$

The general forms of $f_{2k}(\mathbf{z})$ and $f_{2k+1}(\mathbf{z})$ can be easily guessed from here and then by using induction we can prove them rigorously with the help of (3):

$$f_{2k}(\mathbf{z}) = \frac{p_ks_{2k}-p_{k-1}s_{2k-1}+\cdots+(-1)^k(s_k-s_{k-1})+(-1)^k p_1s_{k-2}+\cdots-p_{k-2}s_1+p_{k-1}}{p_{k-1}s_{2k}-p_{k-2}s_{2k-1}+\cdots+(-1)^{k+1}(s_{k+1}-s_k)+(-1)^{k+1}p_1s_{k-1}+\cdots+p_k} \tag{18}$$

$$f_{2k+1}(\mathbf{z}) = \frac{q_ks_{2k+1}-q_{k-1}s_{2k}+\cdots+(-1)^k(s_{k+1}-s_{k-1})+q_1s_{k-1}+\cdots+q_{k-1}}{q_{k-1}s_{2k+1}-q_{k-2}s_{2k}+\cdots+(-1)^{k+1}(s_{k+2}-s_k)+q_1s_{k-1}+\cdots+q_k} \tag{19}$$

We skip the induction step, which is elementary.

The functions (1) have been used in the theory of Lie groups (see [3] [4]) related to actions of those groups on non orientable Klein surfaces and, in general, on non orientable n-dimensional complex manifolds.

We have proved in [1] that, considered as Möbius transformations in each one of its variables, the functions f_2, f_3, f_4 and f_5 have all the same fixed points ξ_1 and ξ_2, the roots of the equation $z^2-(\omega+1)z+1=0$, thus $\xi_1+\xi_2=\omega+1$ and $\xi_1\xi_2=1$. By using Formulas (18) and (19) we can now prove that this is true for any *m*-Möbius transformation.

2. The Fixed Points of *m*-Möbius Transformations

Theorem 1. For every $k\ge 2$ the following identities are true:

$$p_{k+1}+p_{k-1}=(\omega+1)p_k \tag{20}$$

$$q_{k+1}+q_{k-1}=(\omega+1)q_k \tag{21}$$

Proof. The relations (20) and (21) are obvious for $k=2$. For an arbitrary *k* we proceed by induction supposing that (20) and (21) are true for every subscript $j<k$. Then we have by (4):

$$\begin{aligned}p_{k+1}+p_{k-1}&=\omega(q_k+q_{k-2})-(q_{k-1}+q_{k-3})\\&=\omega(\omega+1)q_{k-1}-(\omega+1)q_{k-2}\\&=(\omega+1)(\omega q_{k-1}-q_{k-2})\\&=(\omega+1)p_k.\end{aligned}$$

Also, we have by (5):

$$\begin{aligned}(\omega-1)(q_{k+1}+q_{k-1})&=\omega(p_{k+1}+p_{k-1})-(p_k+p_{k-2})\\&=\omega(\omega+1)p_k-(\omega+1)p_{k-1}\\&=(\omega+1)(\omega p_k-p_{k-1})\\&=(\omega+1)(\omega-1)q_k,\end{aligned}$$

which implies (21). Hence (20) and (21) are true for every *k*.

Theorem 2 (The Main Theorem). For every $m\ge 2$ the function f_m, considered as a Möbius transformation in any one of its variables, has the same fixed points ξ_1 and ξ_2, which are the solutions of the equation
$z^2-(1+\omega)z+1=0$.

Proof: The affirmation of this theorem may come as a surprise: there is no obvious reason why these functions depending on m independent complex variables should display such a strong property. As it will appear next, this is a result of the symmetry of coefficients appearing in Formulas (18) and (19). By (18), the equality

$$f_{2k}(z_1,z_2,\cdots,z_{2k})=z_{2k} \tag{22}$$

is true if and only if

$$\begin{aligned}&p_k s_{2k}-p_{k-1}s_{2k-1}+\cdots+(-1)^k(s_k-s_{k-1})-p_1 s_{k-2}+\cdots+p_{k-1}\\&=z_{2k}\left[p_{k-1}s_{2k}-p_{k-2}s_{2k-1}+\cdots+(-1)^{k+1}(s_{k+1}-s_k)+p_1 s_{k-1}-p_2 s_{k-2}+\cdots+p_k\right].\end{aligned}$$

Taking into account the fact that $s_{2k}=s_{2k-1}z_{2k}$ and for every $j<2k$ we can replace s_j by $z_{2k}s_{j-1}+s_j$, where on the left hand side s_j are the symmetric sums of order *j* of $z_1,z_2,\cdots,z_{2k}$ and on the right hand side s_{j-1} are the symmetric sums of order $j-1$ of $z_1, z_2, \cdots, z_{2k-1}$, this last equality is:

$$\begin{aligned}
&p_k s_{2k-1} z_{2k} - p_{k-1}\left(s_{2k-2} z_{2k} + s_{2k-1}\right) + \cdots + (-1)^k \left[\left(s_k - s_{k-1}\right) z_{2k} + s_{k-1} - s_{k-2}\right] \\
&+(-1)^k p_1\left(s_{k-3} z_{2k} + s_{k-2}\right) + \cdots - p_{k-2}\left(z_{2k} + s_1\right) + p_{k-1} \\
&= z_{2k}\left\{p_{k-1} s_{2k-1} z_{2k} - p_{k-2}\left(s_{2k-2} z_{2k} + s_{2k-1}\right) + \cdots + (-1)^k p_1\left(s_{k+1} z_{2k} + s_{k+2}\right)\right. \\
&\quad + (-1)^k \left[\left(s_k - s_{k-1}\right) z_{2k} + s_{k+1} - s_k\right] + \cdots + p_{k-2}\left(z_{2k} s_1 + s_2\right) \\
&\quad \left. - p_{k-1}\left(z_{2k} + s_1\right) + p_k\right\}
\end{aligned}$$

This is a second degree equation in z_{2k}. An easy computation shows that the coefficient of z_{2k}^2 is:

$$\begin{aligned}
P(\mathbf{z}) = {} & p_{k-1}\left(s_{2k-1} - 1\right) - p_{k-2}\left(s_{2k-2} - s_1\right) \cdots + (-1)^k p_1\left(s_{k+1} - s_{k-2}\right) \\
& + (-1)^{k+1}\left(s_k - s_{k-1}\right),
\end{aligned}$$

and it is the same as the constant term of the equation.

Taking into account the Equality (20) we obtain for the coefficient of z_{2k} the expression $-(\omega+1) P(\mathbf{z})$ therefore we have for the fixed points of $f_{2k}(\mathbf{z})$ when considered as a function of z_{2k} the equation:

$$P(\mathbf{z})\left[z_{2k}^2 - (\omega+1) z_{2k} + 1\right] = 0 \tag{23}$$

where $\mathbf{z} = \left(z_1, z_2, \cdots, z_{2k-1}\right)$.

If $P(\mathbf{z}) = 0$ then (22) is satisfied independently of the values of z_{2k}, hence every point z_{2k} is a fixed point of $f_{2m}(\mathbf{z})$ considered as Möbius transformation in z_{2k} and this is a trivial situation. Otherwise, if $P(\mathbf{z}) \neq 0$ then the fixed points are ξ_1 and ξ_2 such that $\xi_1 + \xi_2 = \omega + 1$ and $\xi_1 \xi_2 = 1$. Due to the symmetry of f_{2k} (it depends only of symmetric sums of $z_1, z_2, \cdots, z_{2k}$), this is true for every variable z_j.

Now, taking into account (21) we can draw the same conclusion for f_{2k+1}, except that this time instead of $P(\mathbf{z})$ we have $Q(\mathbf{z}) = q_k\left(s_{2k} - 1\right) - q_{k-1}\left(s_{2k-1} - s_1\right) + \cdots + (-1)^{k-1}\left(s_{k+1} - s_{k-1}\right)$, which completely proves the theorem.

This theorem states that for every $j = 1, 2, \cdots, 2k$, if $P(\mathbf{z}) \neq 0$, $\mathbf{z} = \left(z_1, z_2, \cdots, z_{j-1}, z_{j+1}, \cdots, z_{2k}\right)$ then we have $f_{2k}\left(z_1, z_2, \cdots, z_{j-1}, \xi_l, z_{j+1}, \cdots, z_{2k}\right) = \xi_l$, $l = 1, 2$ and for every $j = 1, 2, \cdots, 2k+1$, if $Q(\mathbf{z}) \neq 0$, $\mathbf{z} = \left(z_1, z_2, \cdots, z_{j-1}, z_{j+1}, \cdots, z_{2k+1}\right)$ then we have $f_{2k+1}\left(z_1, z_2, \cdots, z_{j-1}, \xi_l, z_{j+1}, \cdots, z_{2k}\right) = \xi_l$, $l = 1, 2$. In other words, if we let constant the variables $z_1, z_2, \cdots, z_{j-1}, z_{j+1}, \cdots, z_m$ then $w = f_m(\mathbf{z})$ is a Möbius transformation of the (z_j)-plane having the fixed points ξ_1 and ξ_2. Then the Steiner net (see [1]) determined by these fixed points is mapped by f_m onto a Steiner net in the (w)-plane. The pre-image by f_m of this last Seiner net is an object in $\overline{\mathbb{C}}^m$ whose projections on every (z_k)-plane, $k = 1, 2, \cdots, m$ is the Steiner net determined by the same points ξ_1 and ξ_2 from the respective plane. We will prove next that there is a unique Mö bius transformation of the (z_j)-plane onto the (z_k)-plane which carries those Steiner nets one into the other.

3. Omitted Values

It can be easily checked (see [2]) that

$$f_2(z_1,a)=a,\ f_2(z_1,1/a)=1/a \tag{24}$$

for every $z_1\in\overline{\mathbb{C}}$, where $a+1/a=\omega+1$, hence if we want f_2 to be a Möbius transformation in z_1 we need to require that z_2 is omitting the values a and $1/a$. Analogously, $f_2(a,z_2)=a$ and $f_2(1/a,z_2)=1/a$ for every $z_2\in\overline{\mathbb{C}}$, thus f_2 is a Möbius transformation in z_2 if and only if z_1 is different of a and $1/a$. However, the two functions are defined for every $z_1\in\overline{\mathbb{C}}$, respectively for every $z_2\in\overline{\mathbb{C}}$. Moreover, these equalities show that a and $1/a$ are in fact the fixed points of f_2 and we can choose, for example $\xi_1=a$ and $\xi_2=1/a$. Obviously, the fixed points belong to the domain of each function and they are omitted only when the variable is considered as a parameter. On the other hand, the identity $f_2(a,1/a)=f_2(1/a,a)=1$ shows that indeed we need to omit those values for the parameter z_2, since otherwise the relations (24) cannot be true. It also results from the Equation (3) that $f_m(z_1,z_2,\cdots,z_m)=a$ if and only if at least one of the variables is a and no other variable is $1/a$. Similarly, $f_m(z_1,z_2,\cdots,z_m)=1/a$ if and only if at least one of the variables is $1/a$ and no other variable is a.

4. Multipliers and Classification of *m*-Möbius Transformations

Let us deal first with the bi-Möbius transformations $w=f_2(z_1,z_2)=\dfrac{(\omega z_2-1)z_1+1-z_2}{(z_2-1)z_1+\omega-z_2}$, which is a Möbius transformations in z_1 for every $z_2\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ and $w=f_2(z_1,z_2)=\dfrac{(\omega z_1-1)z_2+1-z_1}{(z_1-1)z_2+\omega-z_1}$, which is a Möbius transformation in z_2 for every $z_1\in\overline{\mathbb{C}}\setminus\{a,1/a\}$. By Theorem 2 they have the same fixed points ξ_1 and ξ_2, solutions of the equation $z^2-(1+\omega)z+1=0$.

Theorem 3. If $w=f_2(z_1,z_2)$ has distinct fixed points ξ_1 and ξ_2, then for every $z_2\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ there is a number $\mu_1=\mu_1(z_2)\in\mathbb{C}$ such that $\dfrac{w-\xi_1}{w-\xi_2}=\mu_1\dfrac{z_1-\xi_1}{z_1-\xi_2}$ and for every $z_1\in\overline{\mathbb{C}}\setminus\{a,1/a\}$ there is a number $\mu_2=\mu_2(z_1)\in\mathbb{C}$ such that $\dfrac{w-\xi_1}{w-\xi_2}=\mu_2\dfrac{z_2-\xi_1}{z_2-\xi_2}$.

Proof: The existence of those numbers is guaranteed by the following: If we set $\zeta=\varphi(z_1)=\dfrac{z_1-\xi_1}{z_1-\xi_2}$, which carries $z_1=\xi_1$ into 0 and $z_1=\xi_2$ into ∞, then $M_1(\zeta)=\varphi\circ f_2\circ\varphi^{-1}(\zeta)$, where f_2 is considered as a function of z_1 depending on the parameter z_2, is a Möbius transformation having the fixed points 0 and ∞. The only Möbius transformations satisfying such a property are those of the form $M_1(\zeta)=\mu_1\zeta$ for some complex number μ_1. Then:

$$\mu_1 \frac{z_1-\xi_1}{z_1-\xi_2} = \varphi \circ f_2 \circ \varphi^{-1}(\zeta) = \frac{f_2 \circ \varphi^{-1}(\zeta)-\xi_1}{f_2 \circ \varphi^{-1}(\zeta)-\xi_2} = \frac{f_2(z_1,z_2)-\xi_1}{f_2(z_1,z_2)-\xi_2} = \frac{w-\xi_1}{w-\xi_2} \tag{25}$$

We call the number μ_1 the multiplier of this transformation associated with ξ_1 (see [5], p. 166). This is the multiplier of f_2 as a function of z_1 and it depends on z_2. Similarly, for $\zeta = \psi(z_2) = \frac{z_2-\xi_1}{z_2-\xi_2}$, which carries $z_2 = \xi_1$ into 0 and $z_2 = \xi_2$ into ∞, we have again that $M_2(\zeta) = \psi \circ f_2 \circ \psi^{-1}(\zeta)$ is a Möbius transformation having the fixed points 0 and ∞ and therefore $M_2(\zeta) = \mu_2 \zeta$, where this time $\mu_2 = \mu_2(z_1)$. It is obvious that the multipliers of f_2 associated to ξ_2 are respectively $1/\mu_1$ and $1/\mu_2$.

Since $f_2(\infty, z_2) = \frac{\omega z_2 - 1}{z_2 - 1}$, the multiplier of f_2 (as a Möbius transformation in z_1) associated with ξ_1 is obtained replacing $z_1 = \infty$ in (25), *i.e.*

$\mu_1 = \mu_1(z_2) = \left[\frac{\omega z_2 - 1}{z_2 - 1} - \xi_1\right] \Big/ \left[\frac{\omega z_2 - 1}{z_2 - 1} - \xi_2\right]$, or

$$\mu_1(z_2) = \frac{(\omega z_2 - 1)-(z_2-1)\xi_1}{(\omega z_2 - 1)-(z_2-1)\xi_2} = \frac{(\omega-\xi_1)z_2 + (\xi_1 - 1)}{(\omega-\xi_2)z_2 + (\xi_2 - 1)} \tag{26}$$

and analogously,

$$\mu_2(z_1) = \frac{(\omega z_1 - 1)-(z_1-1)\xi_1}{(\omega z_1 - 1)-(z_1-1)\xi_2} = \frac{(\omega-\xi_1)z_1 + (\xi_1 - 1)}{(\omega-\xi_2)z_1 + (\xi_2 - 1)} \tag{27}$$

Let us notice that the discriminant of the linear-fractional functions $\mu_1(z_2)$ and $\mu_2(z_1)$ is $(\omega-1)(\xi_2-\xi_1)$ and it is different of 0 since $\omega \neq 1$ and $\xi_1 \neq \xi_2$, hence they are Möbius transformations, which means that there is a one-to-one correspondence between μ_1 and z_2, respectively μ_2 and z_1. On the other hand, it is known that the nature of a non parabolic Möbius transformation is completely characterized by the values of its multiplier (see [5], page 164), namely it is *elliptic*, *hyperbolic* or *loxodromic* when the multiplier is respectively $e^{i\theta}, \theta \in \mathbb{R}$, or it is a real number $\rho \neq 1$, or the product $\rho e^{i\theta}, \rho \neq 1$. Since the inverse transformations of (26) and (27) are

$$z_2 = \frac{(\xi_2-1)\mu_1 - (\xi_1-1)}{(\xi_2-\omega)\mu_1 + (\omega-\xi_1)} \tag{28}$$

$$z_1 = \frac{(\xi_2-1)\mu_2 - (\xi_1-1)}{(\xi_2-\omega)\mu_2 + (\omega-\xi_1)} \tag{29}$$

we can state the following:

Theorem 4. The non parabolic bi-Möbius transformation (1) having distinct fixed points ξ_1 and ξ_2 regarded as a Möbius transformation in each one of its variables is elliptic, hyperbolic or loxodromic when the other variable takes the values

$$\chi(\mu) = \left[(\xi_2-1)\mu - (\xi_1-1)\right] \Big/ \left[(\xi_2-\omega)\mu + (\omega-\xi_1)\right] \tag{30}$$

where μ is respectively $e^{i\theta}$, $\theta \in \mathbb{R}$, or $\rho \neq 1$, $\rho \in \mathbb{R}$, or the product of two such numbers.

Proof: This is a straightforward result from Felix Klein classification (see [5], page 163) of classical Möbius transformations. We notice that $\chi(\mu)$ is a Möbius transformation and therefore it carries the unit circle $\mu = e^{i\theta}$ into a circle or a straight line. An easy computation shows that $\chi(1) = 1$ and $\chi(-1) = -1$, therefore if the image of the unit circle by $\chi(\mu)$ is a strait line, this line should be the real axis, otherwise it is a circle passing through 1 and −1. The function $\chi(\mu)$ also carries the real axis into a circle or a straight line. Checking if it is a straight line comes to see if the denominator of $\chi(\mu)$ cancels for a real μ. It cancels for $\mu = (\xi_1 - \omega)/(\xi_2 - \omega) = \xi_1\xi_2 - (\xi_1 + \xi_2)\omega + \omega^2 = 1 - \omega$ and this is real when ω is real. Otherwise, the image by $\chi(\mu)$ of the real axis is a circle passing through −1 and 1. We conclude that the non parabolic bi-Möbius transformation (1) is elliptic in every variable on a given generalized circle in the plane of the other variable and it is a hyperbolic bi-Möbius transformation in one variable when the other variable describes another given generalized circle. In all the other cases (1) is loxodromic.

Formula (30) describes the way the Steiner nets from the (z_1)-plane and respectively the (z_2)-plane (see [1]) corresponding to the fixed points ξ_1 and ξ_2 are moved by $f_2(z_1, z_2)$ into a Stainer net in the (w)-plane, when this last one is identified with the (z_1)-plane, respectively with the (z_2)-plane. Namely, when $\mu = e^{i\theta}$, $\theta \in \mathbb{R}$ the net is moved alongside every Apollonius circle (clockwise for the circles around ξ_2 and counterclockwise for the circles around ξ_1, see **Figure 1** below), while when $\mu \in \mathbb{R}$, $\mu \neq 1$ it is moved alongside every circle passing through ξ_1 and ξ_2 (see **Figure 2** below). For a loxodromic transformation the motion is spiral-like alongside a double spiral issuing from ξ_1 and ending in ξ_2 (see [5], page 165 and 166).

When representing these motions on the Riemann sphere, it can be easily seen that the loxodromic motion is obtained by composing in any order the elliptic

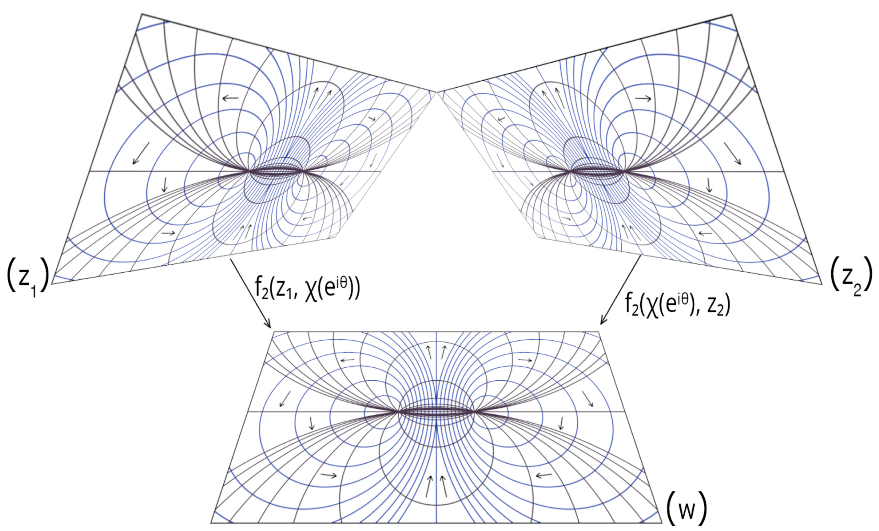

Figure 1. Moving elliptic Steiner nets by f_2 from the coordinate planes to the image plane.

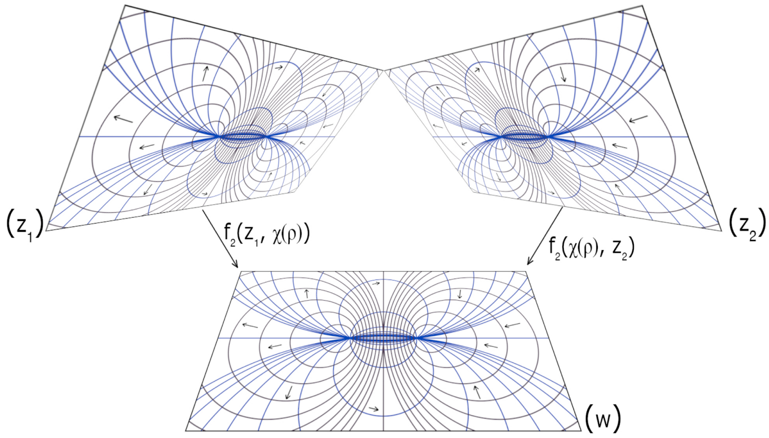

Figure 2. Moving hyperbolic Steiner nets by f_2 from the coordinate planes to the image plane.

and the hyperbolic corresponding motions (see [5], page 153). On the other hand, we have seen in [1] that given $w \in \overline{\mathbb{C}}$ there is a unique Möbius transformation $z_2 = h(z_1) = \dfrac{(1-w)z_1 + \omega w - 1}{(\omega - w)z_1 + w - 1}$ such that $f_2(z_1, h(z_1)) = f_2(h^{-1}(z_2), z_2) = w$, therefore every Steiner net from the (w)-plane is the image by f_2 of a couple of Steiner nets from the (z_1)-plane, respectively (z_2)-plane. These last nets are the image of each other by h, respectively by h^{-1} (see the figures below).

The generalization of this theory to m-Möbius transformations for $m > 2$ can be easily done by using the recurrence formula (3).

Theorem 5. If the m-Möbius transformation (3) has distinct fixed points ξ_1 and ξ_2, then it is elliptic, hyperbolic or loxodromic in each one of its variables when the multiplier μ is $e^{i\theta}$, $\theta \in \mathbb{R}$, or it is real different of 1, or respectively the product of two such numbers.

Proof: By Theorem 2, $f_m(z_1, z_2, \cdots, z_m)$ has the same fixed points ξ_1 and ξ_2 when treated as a Möbius transformation in any one of its variables. For the sake of simplicity, we choose the variable z_m. Let us denote $\zeta = \varphi(z_m) = \dfrac{z_m - \xi_1}{z_m - \xi_2}$ which carries $z_m = \xi_1$ into 0 and $z_m = \xi_2$ into ∞, then $M(\zeta) = \varphi \circ f_m \circ \varphi^{-1}(\zeta)$ (where f_m stands for $f_m(z_1, z_2, \cdots, z_m)$ in which all the variables except z_m are fixed) is a Möbius transformation having the fixed points 0 and ∞, hence $M(\zeta) = \mu\zeta$ for some complex number $\mu = \mu(z_1, z_2, \cdots, z_{m-1})$. This means

$$\begin{aligned} \mu \frac{z_m - \xi_1}{z_m - \xi_2} &= \varphi \circ f_m \circ \varphi^{-1}(\zeta) = \frac{f_m \circ \varphi^{-1}(\zeta) - \xi_1}{f_m \circ \varphi^{-1}(\zeta) - \xi_2} \\ &= \frac{f_m(z_1, z_2, \cdots, z_m) - \xi_1}{f_m(z_1, z_2, \cdots, z_m) - \xi_2} = \frac{w - \xi_1}{w - \xi_2} \end{aligned} \tag{31}$$

By writing repeatedly the recursive formula (3) we obtain f_m under the form

$$f_m(z_1, z_2, \cdots, z_m) = \frac{[\omega f_{m-1}(z_1, z_2, \cdots, z_{m-1}) - 1] z_m + 1 - f_{m-1}(z_1, z_2, \cdots, z_{m-1})}{[f_{m-1}(z_1, z_2, \cdots, z_{m-1}) - 1] z_m + \omega - f_{m-1}(z_1, z_2, \cdots, z_{m-1})} \tag{32}$$

It can be easily seen from here that

$$f_m\left(z_1,z_2,\cdots,z_{m-1},\infty\right)=\frac{\omega f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1}{f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1} \tag{33}$$

Thus, by (31) we have

$$\mu=\frac{\left[\omega f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1\right]-\left[f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1\right]\xi_1}{\left[\omega f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1\right]-\left[f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)-1\right]\xi_2} \tag{34}$$

This formula shows how the multiplier μ depends on $z_1,z_2,\cdots,z_{m-1}$. It can be written also under the form:

$$f_{m-1}\left(z_1,z_2,\cdots,z_{m-1}\right)=\frac{\left(\xi_2-1\right)\mu+\left(1-\xi_1\right)}{\left(\xi_2-\omega\right)\mu+\left(\omega-\xi_1\right)} \tag{35}$$

which agrees with (28) and (29). We have seen in Theorem 4 that when $\mu=\mathrm{e}^{i\theta}$ describes the unit circle, the right hand side in (35) describes a generalized circle. The pre-image by f_{m-1} of this circle is an object $\mathbf{C}_e$ in $\overline{\mathbb{C}}^{m-1}$. When $\left(z_1,z_2,\cdots,z_{m-1}\right)\in\mathbf{C}_e$, the function $f_m\left(z_1,z_2,\cdots,z_m\right)$ is an elliptic Möbius transformation in z_m. Similarly, when $\mu\in\mathbb{R}\setminus\{1\}$, the right hand side in (35) describes a circle or a straight line, the pre-image by f_{m-1} of which is an object $\mathbf{C}_h$ in $\overline{\mathbb{C}}^{m-1}$. When $\left(z_1,z_2,\cdots,z_{m-1}\right)\in\mathbf{C}_h$, the function $f_m\left(z_1,z_2,\cdots,z_m\right)$ is a hyperbolic Möbius transformation in z_m. In all the other cases this Möbius transformation is loxodromic. Due to the symmetry of f_m this is true for any other variable z_k instead of z_m.

Now, let us project onto the (z_l)-plane sections of $\mathbf{C}_e$ and $\mathbf{C}_h$ obtained by keeping z_j constant for all $j\neq l$. These projections are in turn generalized circles $C_{e,l}$ respectively $C_{h,l}$ in the (z_l)-plane. Obviously, $z_l\in C_{e,l}$ for all $l\neq k$ if and only if $\left(z_1,z_2,\cdots,z_{k-1},z_{k+1},\cdots,z_m\right)\in\mathbf{C}_e$, therefore, a non parabolic m-Möbius transformation f_m is an elliptic Möbius transformation in z_k if and only if $z_l\in C_{e,l}$ for every $l\neq k$. A similar property is true for $C_{h,l}$. In all the other cases f_m is loxodromic.

5. Parabolic *m*-Möbius Transformations

The function f_2, considered as a Möbius transformation in any one of its variables, is parabolic if and only if it has a unique fixed point, *i.e.* the equation $z^2-\left(\omega+1\right)z+1=0$ has a double root. Since $\omega\neq 1$, this can happen if and only if $\omega=-3$ and then the fixed point is −1, *i.e.* for $\omega=-3$ we have $f_2\left(-1,z_2\right)=f_2\left(z_1,-1\right)=-1$.

For $\zeta=\varphi\left(z_1\right)=1/\left(z_1+1\right)$, let us define $M_1\left(\zeta\right)=\varphi\circ f_2\circ\varphi^{-1}\left(\zeta\right)$, where

$$f_2\left(z_1,z_2\right)=\frac{\left(3z_2+1\right)z_1+\left(z_2-1\right)}{\left(1-z_2\right)z_1+\left(z_2+3\right)} \tag{36}$$

Here f_2 is considered as a function of z_1 depending on the parameter z_2. The function $M_1\left(\zeta\right)$ is a Möbius transformation having the unique fixed point $\zeta=\infty$.

Then, necessarily,

$$M_1(\zeta)=\zeta+\mu_1 \tag{37}$$

for a complex number $\mu_1=\mu_1(z_2)$. This number can be determined knowing $M_1(0)$, *i.e.* $f_2(\infty,z_2)$. We have

$$f_2(\infty,z_2)=\frac{3z_2+1}{1-z_2} \tag{38}$$

and then

$$\begin{aligned}\mu_1(z_2)&=M_1(0)=\varphi(f_2(\infty,z_2))=\varphi\left(\frac{3z_2+1}{1-z_2}\right)\\&=1\Big/\left(\frac{3z_2+1}{1-z_2}+1\right)=\frac{1-z_2}{2(z_2+1)},z_2=\frac{1-2\mu_1}{1+2\mu_1}\end{aligned} \tag{39}$$

It can be easily checked that (39) implies $\frac{1}{f_2(z_1,z_2)+1}=\frac{1}{z_1+1}+\mu_1(z_2)$

We find analogously:

$$\mu_2(z_1)=\frac{1-z_1}{2(z_1+1)},z_1=\frac{1-2\mu_2}{1+2\mu_2},\frac{1}{f_2(z_1,z_2)+1}=\frac{1}{z_2+1}+\mu_2(z_1) \tag{40}$$

Having in view (37), every straight line parallel to the vector μ_1 is mapped by M_1 onto itself and every orthogonal line to μ_1 is mapped onto another orthogonal line. On the other hand, we have $z_1=\varphi^{-1}(\zeta)=1/\zeta-1$, which shows that φ^{-1} maps those orthogonal lines into two families of orthogonal circles passing through $z_1=-1$, for every $z_2\in\overline{\mathbb{C}}$. An analogous result is obtained if we switch z_1 and z_2. These nets are mapped by $f_2(z_1,z_2)$ into a similar net in the (w)-plane passing through $w=-1$. **Figure 3** below illustrates this phenomenon.

For the general case, we notice that for $\omega=-3$ we have $f_m(z_1,z_2,\cdots,z_{k-1},-1,z_{k+1},\cdots,z_m)=-1$, hence if $\zeta=\varphi(z_k)=1/(z_k+1)$, then $M_k(\zeta)=\varphi\circ f_m\circ\varphi^{-1}(\zeta)$, where the argument of f_m is z_k, is a Möbius transformation having the only fixed point $\zeta=\infty$. Therefore $M_k(\zeta)=\zeta+\mu_k$,

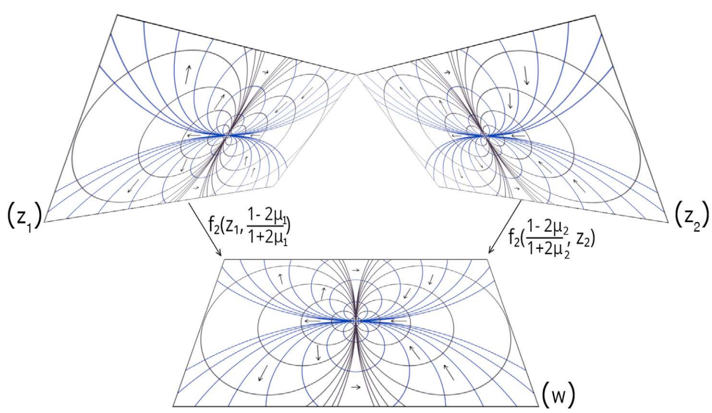

Figure 3. Moving parabolic Steiner nets by f_2 from the coordinate planes to the image plane.

where this time

$$\begin{aligned}\mu_k &= \varphi\left(f_m\left(z_1, z_2, \cdots, z_{k-1}, \infty, z_k, \cdots, z_m\right)\right) \\ &= \frac{1 - f_{m-1}\left(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_m\right)}{2\left[f_{m-1}\left(z_1, z_2, \cdots, z_{k-1}, z_{k+1}, \cdots, z_m\right) + 1\right]}.\end{aligned} \tag{41}$$

Again, every straight line parallel to the vector μ_k in the (ζ)-plane is mapped by M_k into itself and every straight line orthogonal to μ_k is mapped by M_k into another line orthogonal to μ_k. On the other hand, since $z_k = \frac{1}{\zeta} - 1$, the function $\varphi^{-1}(\zeta)$ maps those orthogonal lines into two families orthogonal circles passing through $z_k = -1$. The image by f_m of this net is a similar net in the (w)-plane. The pre-image by f_m of this last net is an object in $\overline{\mathbb{C}}^m$ whose every section obtained by keeping z_k fixed, $k \neq j$ is projected onto the (z_j)-plane into a similar net.

6. Groups of *m*-Möbius Transformations

Given $\omega_1, \omega_2 \in \mathbb{C} \setminus \{1\}$, $\omega_k = a_k + 1/a_k - 1$, let us define $M : \overline{\mathbb{C}}^2 \to \overline{\mathbb{C}}^2$ by

$$M\left(z_1, z_2\right) = \left(w_1, w_2\right) = \left(f_{\omega_1}\left(z_1, z_2\right), f_{\omega_2}\left(z_1, z_2\right)\right) \tag{42}$$

where

$$w_k = f_{\omega_k}\left(z_1, z_2\right) = \frac{\omega_k z_1 z_2 - z_1 - z_2 + 1}{z_1 z_2 - z_1 - z_2 + \omega_k}, k = 1, 2. \tag{43}$$

We will stick with this harmless change of notation in what follows since we need to specify the parameter on which every bi-Möbius transformation (42) depends.

We notice that $f_{\omega_k}\left(z_1, z_2\right) = f_{\omega_k}\left(z_2, z_1\right)$, hence $M\left(z_1, z_2\right) = M\left(z_2, z_1\right)$, which implies that M is not injective. However, we can choose a sub-domain of $\overline{\mathbb{C}}^2$ in which M is injective, as for example $G_1 \times G_2$ where $G_1 = \{z_1 \mid Rez_1 \geq 0$ and if $Rez_1 = 0$ then $Imz_1 > 0\}$ and $G_2 = \overline{\mathbb{C}} \setminus G_1$. Let us notice that $z \in G_1$ if and only if $1/z \in G_2$. In the following we will deal with the function $M : G_1 \times G_2 \to \overline{\mathbb{C}}^2$ defined by $M\left(z_1, z_2\right) = \left(w_1, w_2\right)$, where w_1 and w_2 are given by (43).

Theorem 6. The function M maps $G_1 \times G_2$ one to one and onto $\overline{\mathbb{C}}^2$.

Proof: Indeed, let $\left(w_1, w_2\right) \in \overline{\mathbb{C}}^2$. We are looking for $\left(z_1, z_2\right) \in G_1 \times G_2$ such that $\left(w_1, w_2\right) = M\left(z_1, z_2\right)$. For arbitrary $z_2 \in G_2 \setminus \{a_2, 1/a_2\}$, solving the first Equation (43) for z_1 we get $z_1 = \frac{\left(w_1 - 1\right) z_2 - \omega_1 w_1 + 1}{\left(w_1 - \omega_1\right) z_2 - w_1 + 1}$ and dividing both the denominator and numerator by $-z_2$ we obtain

$$z_1 = \frac{\omega_1 w_1 / z_2 - w_1 - 1/z_2 + 1}{w_1 / z_2 - w_1 - 1/z_2 + \omega_1} = f_{\omega_1}\left(1/z_2, w_1\right) \tag{44}$$

Similarly, solving the second Equation (43) for z_2, with z_1 already found, we get

$$z_2 = f_{\omega_2}\left(1/z_1, w_2\right) \tag{45}$$

and both Equation (43) are satisfied with these values of z_1 and z_2. Hence we have found a couple $(z_1, z_2) \in G_1 \times G_2$ such that $M(z_1, z_2) = (w_1, w_2)$, which means that M maps $G_1 \times G_2$ onto $\overline{\mathbb{C}}^2$. Moreover, both z_1 and z_2 have been uniquely determined since the first Equation (43) is a Möbius transformation in z_1 for every $z_2 \in G_2 \setminus \{a_2, 1/a_2\}$ and the second Equation (43) is a Möbius transformation in z_2 for every $z_1 \in G_1 \setminus \{a_1, 1/a_1\}$, therefore M is injective, which completely proves the theorem.

The mapping $(w_1, w_2) \to (z_1, z_2)$ given by (44) and (45) is the inverse mapping M^{-1} of M. We notice that although $f_{\omega_1}(1/z_2, w_1)$ is a Möbius transformation in w_1 for every $z_2 \in G_2 \setminus \{a_1, 1/a_1\}$ and $f_{\omega_2}(1/z_1, w_2)$ is a Möbius transformation in w_2 for every $z_1 \in G_1 \setminus \{a_2, 1/a_2\}$, the mapping M^{-1} is not of the same nature as M. To avoid this inconvenience, let us redefine M in the following way. With ω_1 and ω_2, as previously given, we choose two other parameters $\zeta_1 \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$ and $\zeta_2 \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$ and set:

$$w_1 = f_{\omega_1}(z_1, \zeta_1) = \frac{\omega_1 \zeta_1 z_1 - \zeta_1 - z_1 + 1}{\zeta_1 z_1 - \zeta_1 - z_1 + \omega_1} \tag{46}$$

$$w_2 = f_{\omega_2}(z_2, \zeta_2) = \frac{\omega_2 \zeta_2 z_2 - \zeta_2 - z_2 + 1}{\zeta_2 z_2 - \zeta_2 - z_2 + \omega_2} \tag{47}$$

The functions f_{ω_1} and f_{ω_2} are Möbius transformations in z_1 and respectively z_2, hence we can solve (46) and (47) for these variables and we get:

$$z_1 = f_{\omega_1}(w_1, 1/\zeta_1) \tag{48}$$

$$z_2 = f_{w_2}(w_2, 1/\zeta_2) \tag{49}$$

This time

$$(w_1, w_2) = M(z_1, z_2) = \left(f_{\omega_1}(z_1, \zeta_1), f_{\omega_2}(z_2, \zeta_2)\right) : \overline{\mathbb{C}}^2 \to \overline{\mathbb{C}}^2 \tag{50}$$

is a bijective function for every $\zeta_1 \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$ and $\zeta_2 \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$ and

$$M^{-1}(w_1, w_2) = \left(f_{\omega_1}(w_1, 1/\zeta_1), f_{\omega_2}(w_2, 1/\zeta_2)\right). \tag{51}$$

Thus, M and M^{-1} are functions of the same nature depending on the parameters $\zeta_1 \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$, $\zeta_2 \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$ and respectively $1/\zeta_1 \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$, $1/\zeta_2 \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$.

Let us denote by $\mathcal{L}_2 = \mathcal{L}_2(\omega_1, \omega_2)$ the class of these functions, where ω_1 and ω_2 are fixed and notice that $M \in \mathcal{L}_2$ if and only if $M^{-1} \in \mathcal{L}_2$. Different values of the parameters ζ_1 and ζ_2 define different functions $M \in \mathcal{L}_2$. Two of them can be composed following the usual rule of function composition. We will show next that the result is an element of $\mathcal{L}_2$.

Theorem 7. If $M, M' \in \mathcal{L}_2$ then $M' \circ M \in \mathcal{L}_2$.

Proof: Let $\zeta_1, \zeta_1' \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$ and $\zeta_2, \zeta_2' \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$ and let $(w_1, w_2) = M(z_1, z_2) = \left(f_{\omega_1}(z_1, \zeta_1), f_{\omega_2}(z_2, \zeta_2)\right)$, $(\eta_1, \eta_2) = M'(w_1, w_2) = \left(f_{\omega_1}(w_1, \zeta_1'), f_{\omega_2}(w_2, \zeta_2')\right)$. Then

$$M' \circ M(z_1,z_2) = \left(f_{\omega_1}\left(f_{\omega_1}(z_1,\zeta_1),\zeta_1'\right), f_{\omega_2}\left(f_{\omega_2}(z_2,\zeta_2),\zeta_2'\right)\right)$$
$$= \left(f_{\omega_1}\left(z_1, f_{\omega_1}(\zeta_1,\zeta_1')\right), f_{\omega_2}\left(z_2, f_{\omega_2}(\zeta_2,\zeta_2')\right)\right)$$
$$= \left(f_{\omega_1}(z_1,\zeta_1''), f_{\omega_2}(z_2,\zeta_2'')\right)$$

where $\zeta_1'' = f_{\omega_1}(\zeta_1,\zeta_1') \in \overline{\mathbb{C}} \setminus \{a_1, 1/a_1\}$ and $\zeta_2'' = f_{\omega_2}(\zeta_2,\zeta_2') \in \overline{\mathbb{C}} \setminus \{a_2, 1/a_2\}$, which shows that indeed $M' \circ M \in \mathcal{L}_2$.

When $M' = M^{-1}$ then $\zeta_1' = 1/\zeta_1$ and $\zeta_2' = 1/\zeta_2$, thus $\zeta_1'' = f_{\omega_1}(\zeta_1, 1/\zeta_1) = 1$ and $\zeta_2'' = f_{\omega_2}(\zeta_2, 1/\zeta_2) = 1$ (see [2]), which means that $M^{-1} \circ M(z_1,z_2) = \left(f_{\omega_1}(z_1,1), f_{\omega_2}(z_2,1)\right) = (z_1,z_2)$ (see [2]), hence the unit element of $\mathcal{L}_2$ is $M_0 : \overline{\mathbb{C}}^2 \to \overline{\mathbb{C}}^2$, defined by $M_0(z_1,z_2) = (z_1,z_2)$ for every $z_1, z_2 \in \overline{\mathbb{C}}$.

Since $f_{\omega_k}(\zeta_k,\zeta_k') = f_{\omega_k}(\zeta_k',\zeta_k)$, this composition law in $\mathcal{L}_2$ is commutative. Finally, with the proper notations, we have

$$M'' \circ (M' \circ M) = M'' \circ \left(f_{\omega_1}\left(z_1, f_{\omega_1}(\zeta_1,\zeta_1')\right), f_{\omega_2}\left(z_2, f_{\omega_2}(\zeta_2,\zeta_2')\right)\right)$$
$$= \left(f_{\omega_1}\left(z_1, f_{\omega_1}\left(f_{\omega_1}(\zeta_1,\zeta_1'),\zeta_1''\right)\right), f_{\omega_2}\left(z_2, f_{\omega_2}\left(f_{\omega_2}(\zeta_2,\zeta_2'),\zeta_2''\right)\right)\right)$$
$$= \left(f_{\omega_1}\left(z_1, \left(f_{\omega_1}\left(\zeta_1, f_{\omega_1}(\zeta_1',\zeta_1'')\right)\right)\right), f_{\omega_2}\left(z_2, \left(f_{\omega_2}\left(\zeta_2, f_{\omega_2}(\zeta_2',\zeta_2'')\right)\right)\right)\right)$$
$$= (M'' \circ M') \circ M$$

hence the composition law in $\mathcal{L}_2$ is associative.

Corollary 1. The function composition law in $\mathcal{L}_2$ defines a structure of Abelian group on $\mathcal{L}_2$.

This result is in contrast with the case of ordinary Möbius transformations in the plane for which the composition law is not commutative.

The generalization of this theory to the dimension m is straightforward. Let $a_k \in \mathbb{C} \setminus \{0,1\}$, $k = 1,2,\cdots,m$ be arbitrary complex numbers and let $\omega_k = a_k + 1/a_k - 1$. For every k and a parameter $\zeta_k \in \overline{\mathbb{C}} \setminus \{a_k, 1/a_k\}$ we define the Möbius transformation in z_k depending on the parameter ζ_k, $w_k = f_{\omega_k}(z_k,\zeta_k) = \frac{\omega_k \zeta_k z_k - z_k - \zeta_k + 1}{\zeta_k z_k - z_k - \zeta_k + \omega_k}$. These transformations define a bijective mapping $M : \overline{\mathbb{C}}^m \to \overline{\mathbb{C}}^m$ $(w_1, w_2, \cdots, w_m) = \left(f_{\omega_1}(z_1,\zeta_1), f_{\omega_2}(z_2,\zeta_2), \cdots, f_{\omega_m}(z_m,\zeta_m)\right)$. Let $\mathcal{L}_m$ be the set of these functions endowed with the usual function composition law. Proceeding as for $\mathcal{L}_2$, it can be easily proved that $\mathcal{L}_m$ is an Abelian group.

7. Conclusions

The m-Möbius transformations have been introduced in connection with Lie groups' actions on complex manifolds (see [3] and [4]). They represent an interesting mathematical topic in itself and we dedicated ourselves to performing in this paper a study of these transformations parallel to that of classical Möbius transformations of the complex plane. The geometric properties of m-Möbius transformations revealed in [1] have been expanded in this paper by using the tool of multipliers. This became possible after proving that regarded as an ordinary Möbius transformation in any one of its variables, a m-Möbius transformation has the same fixed points. This is the main result and it was instrumental in the classification of these transformations. We ended the study with group properties of m-Möbius transformations by showing that they form Abelian groups.

The topic we dealt with here is a new one and it has been studied just in [1] [2] [3] [4]. No other reference was needed. In [5] one can find everything about ordinary Möbius transformations.

Acknowledgements

We thank Aneta Costin for her support with technical matters.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Ghisa, D. (2022) Some Geometric Properties of the m-Möbius Transformations. *Advances in Pure Mathematics*, **12**, 1-6.

[2] Ghisa, D. (2021) A Note on m-Möbius Transformations. *Advances in Pure Mathematics*, **11**, 883-890. https://doi.org/10.4236/apm.2021.1111057

[3] Cao-Huu, T. and Ghisa, D. (2021) Lie Groups Actions on Non Orientable n-Dimensional Complex Manifolds. *Advances in Pure Mathematics*, **11**, 604-610. https://doi.org/10.4236/apm.2021.116039

[4] Barza, I. and Ghisa, D. (2020) Lie Groups Actions on Non Orientable Klein Surfaces. In: Dobrev, V., Ed., *Lie Theory and Its Applications in Physics*, Springer, Singapore, 421-428. https://doi.org/10.1007/978-981-15-7775-8_33

[5] Needham, T. (1997) Visual Complex Analysis. Clarendon Press, Oxford.