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Regular Decimations Result in Irregular Distribution of Primes

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Abstract

Purpose: Primes are notorious for their irregular distribution in natural numbers. Such a lack of regularity makes primes elusive. Many NP-hard problems are related to the irregular occurrence of primes in natural numbers. **Methods:** To extract the underlying regularity of prime distribution, author started from the complementary relationship between composites and primes, through the regular occurrence of composites to infer the regularity underlying primes. **Results:** Previously random-appearing occurrence of primes resulted from the regular periodic decimations of various frequencies and cycles set by primes. **Conclusions:** Primes are the survivors of natural numbers after periodic decimations caused by primes. This leads to a novel concise representation of the set of all primes using sine function, suggestive of periodicity for both primes and composites.

Keywords

Primes, Composites, Complementary, Periodicity, Super Product, Fourier Transform

1. Introduction

Primes have been foci of intensive studies for many mathematicians over centuries, mainly because their occurrence in natural numbers appears elusive [1] [2] [3]. Such a random prime occurrence is closely hinged with various NP-hard problems in mathematics, for example, Rieman Conjecture [1] [2] and Goldbach Conjecture [2] [3] [4] [5] are all related to primes. Various special features of primes, including first digit [6], symmetrical distribution [7], periodicity [8], almost even distribution [9] as well as twin primes [10], have been revealed. Although primes are notorious for their irregular occurrence in natural numbers, there appears to be a lack of hypothesis accounting for such an irregular occurrence. Here the author revealed the periodicities underlying the irregular-appearing occurrence of primes, and showed they were the survivors of natural numbers after regular decimations by primes. Such regular decimations could be represented using sine function and Fourier transform. This new information and treatment shed new light on and paved a new way for related studies.

2. Methods

The analysis was carried out through the following steps. First, the author analyzed the complementary relationship between composites and primes. Then the author analyzed the regularity and periodicity of composites, using sine function to represent such regularity, and finally, represent the sets of composites and primes in a novel way, using sine function.

3. Results and Discussions

3.1. Periodicities of Composites and Primes

3.1.1. Complementary Relationship between Primes and Composites

As Guy [11] pointed out, natural numbers include numbers in the following three classes.

1) 1

2) Primes

3) Composites

Primes are famous for their lack of regularity. This is the origin of hard problems related to primes [1] [2] [3] [4] [11]-[15]. Therefore it is better for us to bypass this unnegotiable problem somehow. As seen above, primes constitute a subset of natural numbers. This subset can be obtained easily by subtracting 1 and all composites from the set of all natural numbers (N). Although primes are disobedient to rules, the occurrence of composites (the complementary of primes) is regular. Therefore we may study primes through studying composites.

Such a bypass is plausible as illustrated in **Figure 1**. Suppose each rectangle in **Figure 1** represents the set of all natural numbers greater than 1 (N_1). The two rectangles in **Figure 1** are exactly the same, but the separations between two

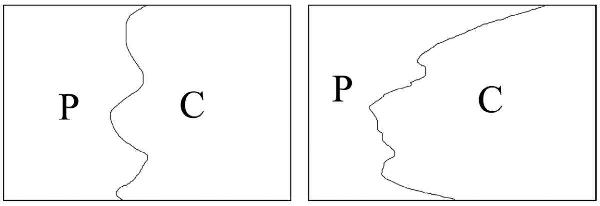

Figure 1. While both rectangles are equal, the separations between two regions, P and C, differ. Independent of the separation, the union of these two exclusive regions, P and C, remains constant. If the separating line characterizes the left region, so does it the right region.

regions differ. If the left region in the rectangle represents the set of primes (P) and the right the composites (C), it is easy to see that the union of left and right regions (P and C) always equals to N_1. This generalization is independent of the separation between P and C. Considering the complementary relationship between the left and right regions, to characterize of the set of primes P, we may bypass the difficulty by characterizing its complementary set, the set of composites C.

3.1.2. Regularities of Composites

It is easy to envision that multiples of a prime (composites) are of regular occurrence, with fixed differences between adjacent ones, just as in any arithmetic progression. The regular occurrence of multiples of first few primes is shown in Figures 2-6.

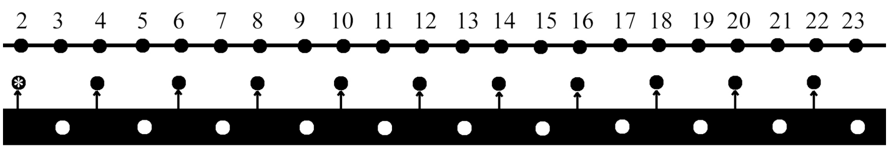

Figure 2. Regular occurrence of the multiples of 2. After removing these multiples, the left-over in natural numbers (odds) is also regular, with a fixed difference of 2 between adjacent ones.

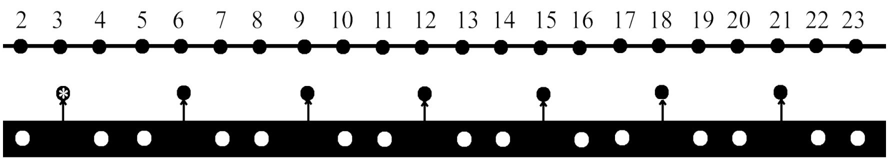

Figure 3. Regular occurrence of the multiples of 3. After removing these multiples, the left-over in natural numbers is also regular, with a fixed differences of 3 between adjacent sequences.

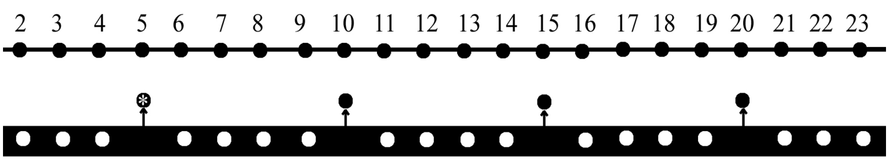

Figure 4. Regular occurrence of the multiples of 5. After removing these multiples, the left-over in natural numbers is also regular, with a fixed differences of 5 between adjacent sequences.

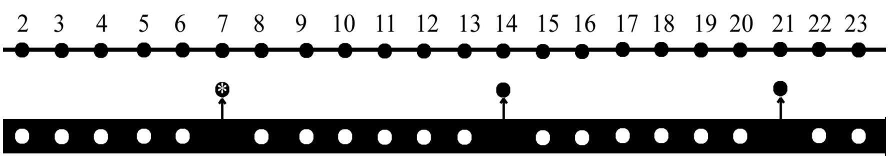

Figure 5. Regular occurrence of the multiples of 7. After removing these multiples, the left-over in natural numbers is also regular, with a fixed difference of 7 between adjacent sequences.

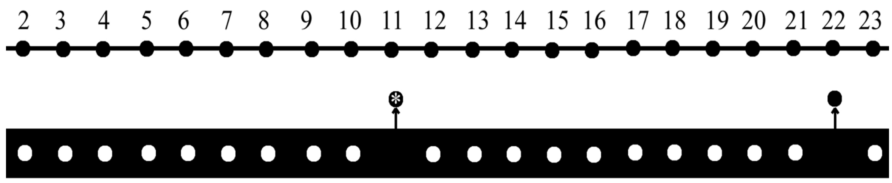

Figure 6. Regular occurrence of the multiples of 11. After removing these multiples, the left-over in natural numbers is also regular, with a fixed difference of 11 between adjacent sequences.

Figure 7 shows the regular occurring patterns of composites. The pattern in the black rectangles repeats in the figure. This pattern repetition is constant and will last to the infinite. However, patterns related to larger primes occur more sparsely and in larger extent, so they cannot be demonstrated graphically here since they have longer periods and lower frequencies. For example, the pattern in the gray rectangle repeats in [36, 40], that in the broken black rectangle repeats in [216, 224], while that in the broken gray rectangle repeats in [2316, 2332].

As seen in **Figure 7**, the repeated patterns for multiples of primes (composites) have various constant periods and frequencies. The differences in period and frequency cause the chaotic occurrence of composites, which actually repeat their own patterns in larger scales, although the patterns do not surface in smaller scales, implying regularities underlying the irregular-appearing occurrence of composites. Since the set of primes is complementary to that of composites in natural numbers, it is logical to expect periodicities for the set of primes, at least theoretically (**Figure 8**).

3.1.3. Inferring Regularities Underlying Primes

The above reasoning suggests that there are certain regularities, namely, repeated patterns, underlying primes. This inference is line with Dirichlet's theorem, which states that, given an arithmetic progression (A.P.) terms of $a \cdot n + b$, $n \in \mathrm{N}$, the series contains an infinite number of primes if a and b are relatively prime.

Dirichlet's theorem implies that there is a periodicity of length a for certain primes. An easy way to find suitable a and b required for the theorem is to find them in primes and their products. a can be either a prime or a product of randomly selected primes, while b can be a prime or a product of primes randomly selected from the primes not selected for a.

Such random selecting will generate infinite number of as and bs. Correspondingly, it will also generate infinite number of A.P. of primes with various periods. This will end up in chaos that is characteristic of primes. To avoid such an unpleasant scenario, we prefer to do it in an orderly manner.

An existing concept, super product of primes X_n [8], can help us on this. X_n is defined as the product of all primes smaller than a specified prime P_n (n_{th} prime) [8], namely,

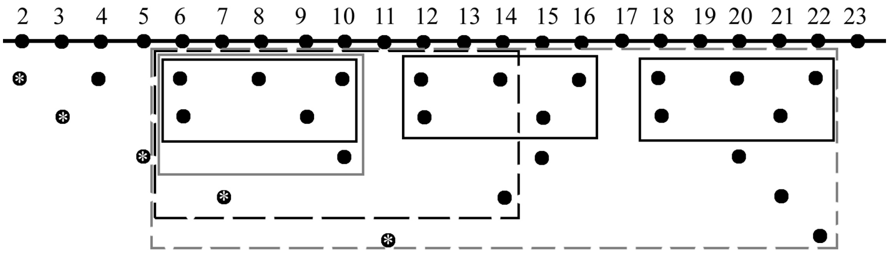

Figure 7. Irregular occurrence of first few primes and their multiples (composites). This irregularity is a sum of regular occurrences of multiples and irregular occurrences of primes.

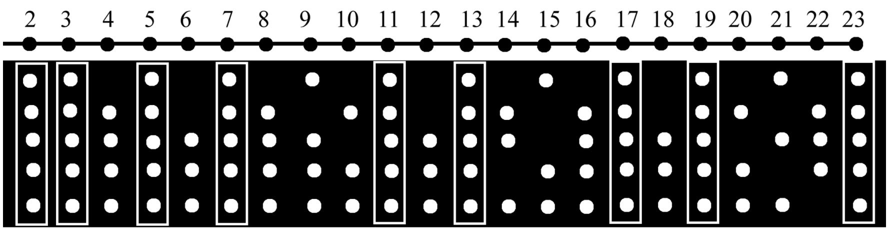

Figure 8. Sum of the survivors after decimations of primes. A number escaping the decimations of all primes is a prime (rectangles).

$$X_n = \prod_{i=1}^{n-1} P_i$$

The first a few super products of primes are shown in Table 1.

You may say, "I cannot see patterns in primes".

You can say that again. But if you check out Figures 2-6, you may have noticed that each prime periodically removes its multiples from the pool of natural numbers, and, in the meantime, leaves periodical sequences of natural numbers as candidates for primes. Such left-overs all constitute periodical sequences, although various periods and frequencies are set by the concerned primes (Figure 8). The greater a prime is, the longer is the period, and the lower is the frequency. The number of primes involved is directly correlated with the diversity of the periodicities as well as the complexity of irregularities of primes. Since numerous periodicities effect in tandem, it is rather expected that irregularity of primes can be represented using Fourier transform in future studies.

Following are a few of examples of periodicity of primes.

One A. P. of primes with a period of 2:

3, 5, 7

This A.P. of primes ends because of the decimation of 3, as the next element would be 9 (= 3 × 3).

Two A. P. of primes with a period of 6 (1 exception, underlined):

5, 11, 17, 23, 29

7, 13, 19, <u>25</u>, 31

The exception in the second A. P. (25) is due to the decimation of 5, as 25 = 5 × 5.

Table 1. The first ten super products of primes.

Super Product	Expression	Value
X_2	2	2
X_3	2×3	6
X_4	$2 \times 3 \times 5$	30
X_5	$2 \times 3 \times 5 \times 7$	210
X_6	$2 \times 3 \times 5 \times 7 \times 11$	2310
X_7	$2 \times 3 \times 5 \times 7 \times 11 \times 13$	30,030
X_8	$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$	510,510
X_9	$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19$	9,699,690
X_{10}	$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23$	223,092,870
X_{11}	$2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 29$	6,469,693,230

Eight A. P. of primes with a period of 30 (12 exceptions, underlined, due to decimations from primes greater than 5):

7, 37, 67, 97, 127, 157, $\underline{187}$
11, 41, 71, 101, 131, $\underline{161}$, 191
13, 43, 73, 103, $\underline{133}$, 163, 193
17, 47, $\underline{77}$, 107, 137, 167, 197
19, $\underline{49}$, 79, 109, 139, $\underline{169}$, 199
23, 53, 83, 113, $\underline{143}$, 173, $\underline{203}$
29, 59, 89, $\underline{119}$, 149, 179, $\underline{209}$
31, 61, $\underline{91}$, $\underline{121}$, 151, 181, 211

In spite of the obvious periodicities of 2, 6, and 30 in the above A. P. of primes, the occurrences of exceptions appear bothering. However, it is rather enlightening that all the exceptions (composites) have factors greater than the factors of the periods. For example, 143 has factors 11 and 13, which are all greater than 2, 3, and 5 (factors of period 30). Apparently, the periodicity-breaking is due to the decimations of greater primes. This is rather expected as increasingly greater ranges and greater primes are taken into consideration.

3.1.4. Reason Underlying Periodic Occurrence of Primes

As shown above, primes should have certain periodicities theoretically, since primes are complementary to composites that demonstrate clear periodicities. We have seen the periods of 2, 6, 30 for primes, now we try to figure out the underlying reason for such periodicities, using period of 30 as an example.

$2 \times 3 \times 5 + 1 = 31$, prime
$\mathbf{2} \times 3 \times 5 + \mathbf{2} = 32$, composite
$2 \times \mathbf{3} \times 5 + \mathbf{3} = 33$, composite
$2 \times 3 \times 5 + 4 = 34$, composite
$2 \times 3 \times \mathbf{5} + \mathbf{5} = 35$, composite
$2 \times 3 \times 5 + 6 = 36$, composite

$2 \times 3 \times 5 + 7 = 37$, prime

$2 \times 3 \times 5 + 11 = 41$, prime

$2 \times 3 \times 5 + 13 = 43$, prime

2, 3, 4, 5, and 6 all have at least one of 2, 3, 5 (factors of $X_4 = 30$) as their factors, so each of their sums with X_4 can be divided exactly by such factors, naturally, their corresponding sums must be composites. 31 is a prime due to 1 is not an element of {2, 3, 5}. When primes greater than 5, such as 7, 11, 13, 17…… involve, the sums are primes, simply because these primes have no factor in {2, 3, 5}.

This explains the cases related to periodicity of 30. Similarly, the periodicities of other super products effect in a similar way.

3.2. Divisibility, Periodicity, and Sine Function

For each prime P_i, multiples of the prime regularly alternate with numbers that are not multiples of the prime. Each period includes 1 multiple of the prime and $P_i - 1$ natural numbers that cannot be divided exactly by P_i. This period can be repeated infinitely. Such a periodic phenomenon can be represented by trigonometric functions, for example, sine. The period of 7, as an example, is shown in **Figure 9**. The periods of other primes can be shown in a similar way, although period length varies.

3.3. Potential Representing Primes Using Discrete Fourier Transform

Since multiples of a prime P_i are periodic with a period of P_i between adjacent ones, all of them (denoted as x) satisfy the equation $\sin(x \cdot \pi / P_i) = 0$. Thus all multiples of all primes constitute a set of composites C, which can be represented as

$$\mathrm{C} = \left\{ n \mid \prod \sin\left(n \cdot \pi / P_i\right) = 0, n \in \mathrm{N}, n > 1, i \in \mathrm{N} \right\} \quad \left(P_i^2 < n\right)$$

While the set of primes, P, can be similarly represented as

$$\mathrm{P} = \left\{ n \mid \prod \sin\left(n \cdot \pi / P_i\right) \neq 0, n \in \mathrm{N}, n > 1, i \in \mathrm{N} \right\} \quad \left(P_i^2 < n\right)$$

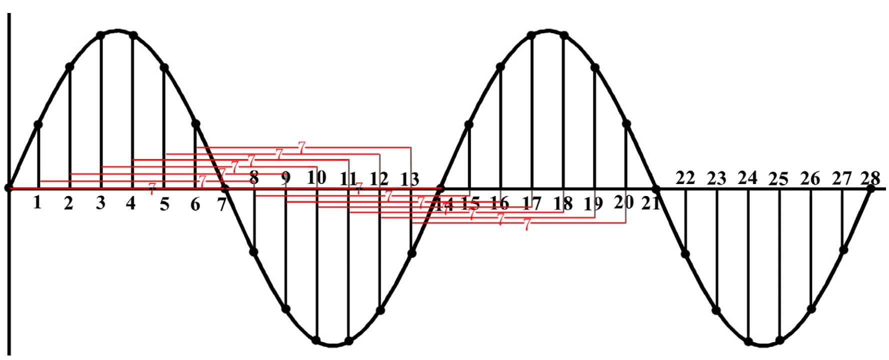

Figure 9. The constant difference of 7 between multiples of 7 as well as between non-multiples of 7. The value of $\sin(x \cdot \pi / 7)$ ($x \in \mathrm{N}$, $x > 1$) is a proxy of divisibility of x by 7: whenever $\sin(x \cdot \pi / 7) = 0$, x is a multiple of 7; whenever $\sin(x \cdot \pi / 7) \neq 0$, x is not a multiple of 7.

This representation of set P implies periodicity of primes.

It is noteworthy that the set of natural numbers, N, can be represented as

$$\mathrm{N} = \{1\} \cup \mathrm{P} \cup \mathrm{C}$$

This returns to the classification of natural numbers given by Guy (2007), at the beginning of this paper.

4. Conclusion

Irregular primes are the survivors of natural numbers after regular decimations by primes. Periods equal to super products of primes, X_n, effect in tandem, leading to the irregular occurrence of prime distribution. As super products of primes, $X_n = \prod_{i=1}^{n-1} P_i$, increase as more primes join, periodicities with increasingly longer periods and lower frequencies effect in sculpturing the occurrence of primes. The occurrence of primes may be represented and studied using sine function and Fourier transform.

Conflicts of Interest

The author declares no conflicts of interests regarding the publication of this paper.

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