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Representations of Ten-Dimensional Levi Decomposition Lie Algebras

Narayana Mudiyanselage Pushpamali Sarada Kumari Bandara[1], **Gerard Thompson**[2]

[1]Department of Mathematics, Florida A & M University, Tallahassee, USA
[2]Department of Mathematics, University of Toledo, Toledo, USA
Email: narayana.bandara@famu.edu, gerard.thompson@utoledo.edu

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Abstract

The authors have recently completed a partial classification of the ten-dimensional real Lie algebras that have the non-trivial Levi decomposition, namely, for such algebras whose semi-simple factor is $\mathfrak{so}(3)$. In the present paper, we obtain a matrix representation for each of these Lie algebras. We are able to find such representations by exploiting properties of the radical, principally, when it has a trivial center, in which case we can obtain such a representation by restricting the adjoint representation. Another important subclass of algebras is where the radical has a codimension one abelian nilradical and for which a representation can readily be found. In general, finding matrix representations for abstract Lie algebras is difficult and there is no algorithmic process, nor is it at all easy to program by computer, even for algebras of low dimension. The present paper represents another step in our efforts to find linear representations for all the low dimensional abstract Lie algebras.

Keywords

Levi Decomposition Lie Algebra, Lie Algebra Representation, *R*-Representation, Ten-Dimensions Representation

1. Introduction

In this article, we continue the ongoing project of documenting, up to isomorphism, the low-dimensional Lie algebras and giving each such algebra a faithful linear representation. Recently, we have classified a subclass of ten-dimensional indecomposable Lie algebras that admits a non-trivial Levi decomposition [1]. In fact, it can be shown that apart from four exceptions, where the radical is of dimension six, the radical is either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$. We have classified all such Lie algebras for which the radical is not $\mathfrak{sl}(2,\mathbb{R})$, although the algebras associated to $\mathfrak{sl}(2,\mathbb{R})$ comprise a much more extensive class. Our results have appeared in [1] and [2]. The present article is concerned solely with providing representations for this class of Lie algebras. We intend to revisit the cases involving $\mathfrak{sl}(2,\mathbb{R})$ in another venue.

The classification of the low-dimensional solvable Lie algebras, in general, has been well documented in [3] as far as dimension six. Among many names, we should mention [4] [5] [6] and [7]. Beyond [3], there is an encyclopaedic account of the seven-dimensional nilpotent [8]. On the non-solvable side, the classification of the semi-simple Lie algebras is well-known, see [9] [10] for a recent account, and Lie algebras that admit a non-trivial Levi decomposition have been studied by Turkowski in [11] [12] up to and including dimension nine. Thus, [1] comprises the first foray into Lie algebras that admits a non-trivial Levi decomposition in dimension ten.

Given a real Lie algebra L of dimension n, a well-known theorem due to Ado asserts that L has a faithful representation as a subalgebra of $\mathfrak{gl}(p,\mathbb{R})$ for some p. However, Ado's Theorem is essentially only an existence result and does not provide a practical method for constructing such a representation. In [13], representations were given for each of the nine-dimensional Lie algebras that have a non-trivial Levi decomposition. This class of Lie algebras was studied and classified by Turkowski [12]. Prior to [13], one of the current authors and others have investigated the problem of finding minimal dimensional representations of indecomposable Lie algebras of the dimension of eight and less [14] [15] [16] [17] [18]. In fact, minimal dimensional representations are known for all Lie algebras, indecomposable and decomposable alike, of the dimension of five and less [15] [16]. Furthermore, minimal dimensional representations are known for six-dimensional indecomposable nilpotent Lie algebras [17] [18] and also for Lie algebras of the dimensions of five, six, seven and eight that have a non-trivial Levi decomposition [14]. We refer the reader also to [19], where Turkowski's algebras were studied from the point of view of their invariants.

In general, to find minimal dimension representations for the ten-dimensional Levi decomposition algebras is difficult, therefore, we address the problem of finding just one faithful, linear representation for each of these 38 Lie algebras. Nonetheless, it would not be difficult to assert in many, indeed most cases, that the representations given here are minimal, since many of the underlying algebras are associated with irreducible representations. However, since we cannot be definitive in every case, we prefer to defer the issue of minimality for the present.

We shall supply a few words about terminology and notation. Much of the notation adopted here is based on [12]. The summation convention on repeated indices, one a subscript and one a superscript, is usually in operation and sometimes with two separate ranges of indices. We generally use S to denote a simple or occasionally semi-simple Lie algebra. We use N for a solvable Lie algebra and NR for its associated nilradical. Then a Levi decomposition is written as $S \ltimes N$. Turkowski classified nine-dimensional, indecomposable Lie algebras that have a non-trivial Levi decomposition. Actually, in [12], two algebras were omitted: one is denoted as $L_{9,7}^{*}$ in [19] and is a "real" form corresponding to $L_{9,52}$. The second shall be denoted here as $L_{9,11}^{*}$ and is a semi-direct product of $\mathfrak{so}(3)$ and $\mathbb{R}^6$ coming from the irreducible 6 × 6 representation of $\mathfrak{so}(3)$. See [13] for more details. Turkowski denotes by *R* the representation of the semi-simple factor *S* by automorphisms of the radical *N*. He discerns, up to isomorphism, 63 classes of such Lie algebras and they are denoted by $L_{9,i}$ where $1 \le i \le 63$. Following the lead given in [12], the Lie algebras for which representations are constructed, are listed below as $L_{10,i}$, where 10 pertains to the dimension of the algebra and *i* to the *i*th algebra in the list.

As regards the other classes of low-dimensional indecomposable Lie algebras, algebras of dimension less than or equal to five and the nilpotent algebras of dimension *i* are denoted by $A_{i,j}$ where $3 \le i \le 6$, and *j* signifies the *j*th algebra in the list, following the listing in [20]. The indecomposable solvable Lie algebras of dimension six that have a five-dimensional nilradical were classified by Mubarakzyanov [5] and are denoted by $g_{6,i}$ where $1 \le i \le 99$; see also [7] for an updated classification. The indecomposable Lie algebras of dimension six that have a four-dimensional nilradical classified by Turkowski [6] are denoted by $N_{6,i}$ where $1 \le i \le 40$. For much more information about low-dimensional Lie algebras in general, the reader may refer to [3]. There is also a memoir devoted to studying the invariants of the nine-dimensional, Levi decomposition algebras [19]. For abelian Lie algebras of dimension *n*, we usually say "Abelian" and refer to $\mathbb{R}^n$ rather than writing nA_1. The trivial representation of dimension *n* is denoted by nD_0. The irreducible representation of $\mathfrak{so}(3)$ of dimension *n* is denoted by R_n for $n > 3$ and for $n = 3$ by $\operatorname{ad}\mathfrak{so}(3)$. We shall refer to the *n*-dimensional Lie algebra with non-zero brackets given by:

$$[e_i, e_n] = e_i, \ (1 \le i \le n-1)$$

as the "Milnor algebra". The reason for so doing, is that it was introduced in [21] albeit in a rather different way; Milnor described the algebra as the unique up to isomorphism, non-abelian Lie with the property that the Lie bracket of any two elements is a linear combination of those same two elements. This algebra and small variants of it, occur repeatedly in the study of Lie algebras that have a non-trivial Levi decomposition.

The Lie algebra of dimension $2n+1$ that has brackets:

$$[e_i, e_{n+j}] = \delta_{ij} e_{2n+1}, \ (1 \le i \le j \le n)$$

is the Heisenberg Lie algebra and for $n = 3$ it is denoted by 17 in [8]. On the other hand, the Lie algebra of dimension $2n+1$ that has brackets:

$$[e_i, e_{2n+1}] = \delta_{ij} e_{n+j}, \ (1 \le i \le j \le n)$$

is the anti-Heisenberg Lie algebra and for $n = 3$ it is denoted by 37*A* in [8]. Unfortunately the Heisenberg and anti-Heisenberg coincide for $n = 1$ but in any case, that Lie algebra is denoted by *H*. Another indecomposable seven-dimensional nilpotent Lie algebra that is needed is denoted by 37*D*1 in [8].

Finally, we should like to say that it might be assumed that finding representations for low-dimensional Lie algebras is simply a matter of implementing a suitable computer program. However, in our opinion, such an assessment is overly optimistic. Such a routine can prove effective for nilpotent algebras of dimension up to about seven; even then, one is likely to obtain a representation that is wanting in elegance and that has to be simplified "by hand". An important reason for having to restrict attention to nilpotent algebras is that up to dimension six, such algebras are "atomic", by which we mean that the algebras do not depend on essential parameters. In dimension seven, there are some algebras that depend on a single parameter [8] and the situation is bound to be worse in dimension eight. Without intervention, a computer routine will assume generic values for the parameters and a given representation may fail for special values. In higher dimensions, the conditions seem to us to be too complicated to be amenable to a conventional routine. However, the reader is referred to a recent paper [22] which does claim success in finding representations for six and seven-dimensional nilpotent algebras. When it comes to Lie algebras that admit a non-trivial Levi decomposition, it seems to us that it is necessary to understand the role of the *R*-representation in constructing a representation for the entire algebra.

An outline of the paper is as follows. In Section 2, we consider the problem in general of constructing Lie algebras that have a Levi decomposition. In Section 3, we give a list of the possible *R*-representations of $\mathfrak{so}(3)$ that occur in the ten-dimensional Levi decomposition algebras. In Section 4, we briefly discuss the properties of the subalgebra of *R*-constants. In Section 5, we provide a few general comments about our classification of the ten-dimensional Levi decomposition algebras, which attempt to put the arguments given in [1] and [2] into context. In Section 6, we outline several techniques that we have used to construct representations and we refer the reader to [13] for more details. Finally, Section 7 gives the matrix representations of the 37 classes of Lie algebra that are concerned. We supply also the Lie brackets of the Lie algebra and the *R*-representation (see next paragraph) for each case where the semi-simple factor is just $\mathfrak{so}(3)$.

2. Constructing Algebras with Levi Decompositions in General

Let us consider the problem of constructing a Lie algebra that has a Levi decomposition $L = N \rtimes S$ in general. We have the following structure equations:

$$[e_a, e_b] = C_{ab}^c e_c,\ [e_a, e_i] = C_{ai}^k e_k,\ [e_i, e_j] = C_{ij}^k e_k \tag{1}$$

where $1 \le a,b,c,d \le r$ and $r+1 \le i,j,k,l \le n$ and $\{e_a\}$ is a basis for the semi-simple subalgebra *S* and $\{e_i\}$ is a basis for the radical *N*.

To find all possible Lie algebras of dimension *n* that have a Levi decompose-tion $L = N \rtimes S$ we can proceed as follows: choose a semi-simple algebra *S* of dimension *r*. Then pick any solvable algebra *N* of dimension $n-r$ and consider a representation of *S* in *N*, considered simply as a vector space of dimension

$n-r$, should one exist. All such representations are known and are completely reducible since S is semi-simple. Finally, it only remains to check that the matrices representing S act as derivations of the Lie algebra N. In the affirmative case, we have our sought after Lie algebra; in the negative case, there is no such algebra and we have to choose a different representation of S in N. If all such representations lead to a null result, then there can be no non-trivial Levi decomposition involving S and N.

3. *R*-Representations for $\mathfrak{so}(3)$

It can be shown [1] that in the enumeration of the ten-dimensional Levi decomposition algebras, either the semi-simple factor S is of dimension six, or else it is either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$. The case where $S=\mathfrak{sl}(2,\mathbb{R})$ leads to a very extensive list of Lie algebras and was not the concern of [1] or [2], nor will it be here.

Since we are interested from now on only with the case where the semi-simple part of the Levi decomposition algebra is $\mathfrak{so}(3)$, we shall need to find all possible representations of $\mathfrak{so}(3)$ in $\mathfrak{gl}(7,\mathbb{R})$. They are as follows:

- R_7;
- $R_5 \oplus 2D_0$;
- $R_4 \oplus 3D_0$;
- $\mathrm{ad}(\mathfrak{so}(3)) \oplus 4D_0$;
- $\mathrm{ad}(\mathfrak{so}(3)) \oplus R_4$;
- $2\mathrm{ad}(\mathfrak{so}(3)) \oplus D_0$.

We are following here the notation used in [12] so that $\mathrm{ad}(\mathfrak{so}(3))$ denotes the standard, or equivalently, adjoint representation of $\mathfrak{so}(3)$ and D_0 the one-dimensional trivial representation and kD_0 signifies k copies of it. Moreover, by R_q we mean the standard irreducible representation of $\mathfrak{so}(3)$ in $\mathfrak{gl}(q,\mathbb{R})$ where $1 \le q \le 7$. We shall not write down the matrices for these representations here. However, they can be gleaned from the representations supplied in Section 7 by taking the span of the s_1, s_2, s_3 matrices and deleting zero rows and matching columns. For example, algebras $L_{10.6}, L_{10.4}, L_{10.15}$ and $L_{10.36}$ engender $\mathrm{ad}(\mathfrak{so}(3)), R_4, R_5$ and R_7, respectfully. There is only one Levi decomposition Lie algebra for R_7 and a unique class depending on one parameter for R_5. Several "accidental" algebras are associated to R_4, as a result of the failure of Schur's Lemma. In [1] it was incorrectly stated that there is an irreducible representation R_6. However, R_6 is actually conjugate to the representation $2\mathrm{ad}(\mathfrak{so}(3))$. Accordingly algebra $L_{10.14}$ in [1] must be removed from the list.

4. *R*-Constants

Following Turkowski [12], we call an element in a Levi decomposition Lie algebra $S \ltimes N$, an *R*-constant if it commutes with the *R*-representation. The terminology arises from the fact that the *R*-representation also acts as a derivations on N, and the *R*-representation maps the *R*-constants to zero. We shall denote the space of R-constants by R and it readily follows that R forms a solvable subalgebra of N containing a complement to NR in N. In fact, R may even be nilpotent, see for example, algebra $L_{10.27}$ in Section 7. The nilradical of R contains $R \cap NR$, the latter being an ideal in R.

5. Supplementary Remarks about the Classification of Ten-Dimensional Levi Decomposition Lie Algebras

In this section, we shall make a few comments that are intended to elucidate and supplement the accounts in [1] and [2].

5.1. Dimension of NR Is Four

It can be shown that it is not possible to have the nilradical NR of dimension four if the semi-simple factor S contains just one copy of $\mathfrak{so}(3)$. However, if S is of dimension greater than three, then it can be one of $\mathfrak{so}(4) \approx \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, $\mathfrak{so}(2,2) \approx \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ or $\mathfrak{so}(3,1)$. In each of the first two cases we obtain just a single indecomposable Levi decomposition Lie algebra and in the case of $\mathfrak{so}(3,1)$, two up to isomorphism. In all of these cases NR is abelian.

5.2. Dimension of NR Is Five

The nilpotent Lie algebras of dimension five are
$A_{5.1}, A_{5.2}, A_{5.3}, A_{5.4}, A_{5.5}, A_{5.6}, A_{4.1} \oplus \mathbb{R}, H \oplus \mathbb{R}^2, \mathbb{R}^5$. Of these nine, only $A_{5.4}$ and $\mathbb{R}^5$ admit subalgebras of derivations that are isomorphic to $\mathfrak{so}(3)$. In the case of $A_{5.4}$, the R-representation comes in the form of R_4. In the case of $\mathbb{R}^5$, only the R-representations, R_5 $R_4 \oplus D_0$ and $\mathfrak{so}(3) \oplus 2D_0$ are possible. However, $\mathbb{R}^5$ leads to a decomposable ten-dimensional Levi decomposition Lie algebra in view of Schur's Lemma. For $R_4 \oplus D_0$, there is a unique class of Levi decomposition Lie algebras, but only because Schur's Lemma fails for R_4.

In the case where the R-representation is $\mathfrak{so}(3) \oplus 2D_0$, four classes of algebra appear. Assuming that the Jacobi identities arising from the R-representation have been satisfied, the remaining Jacobi identities come from the condition that R-constants R should be a subalgebra. It is found that in the various cases, R is one of $A_{2.1} \oplus A_{2.1}$, $A_{4.12}$, $A_{4.9(b=0)}$ or $A_{4.3}$ corresponding to algebras $(L_{10.6} - L_{10.9})$, respectively.

5.3. Dimension of NR Is Six

In dimension six the following nilpotent Lie algebras admit subalgebras of derivations that are isomorphic to $\mathfrak{so}(3)$: $A_{6.3}, A_{6.5}, A_{5.4} \oplus \mathbb{R}, H \oplus \mathbb{R}^3$ and $\mathbb{R}^6$. For $NR = A_{6.3}$ there is a unique ten-dimensional Levi decomposition Lie algebra for which the R-representation is $2\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ and for $A_{6.5}$ a class of algebras that depend on two parameters and R-representation $R_4 \oplus 3D_0$. For $NR = A_{5.4} \oplus \mathbb{R}$ there is one class of algebras that depend on two parameters and the R-representation is $R_4 \oplus 3D_0$. For $NR = H \oplus \mathbb{R}^3$ there is a unique Lie algebra for which the R-representation is $\mathrm{ad}(\mathfrak{so}(3)) \oplus 4D_0$.

There is a variety of cases for which *NR* is six-dimensional abelian. As we explained in Section 3, the algebra $L_{10.14}$ must be removed from the list. There is a unique classes of algebra arising from the *R*-representations $R_5 \oplus 2D_0$ $(L_{10.15})$. For the *R*-representation $R_4 \oplus 3D_0$, three cases arise $(L_{10.16}, L_{10.17}, L_{10.18})$, which may be distinguished by their algebra of *R*-constants $(A_{3.2}, A_{3.5}, A_{3.7})$, respectively. For the *R*-representation $\mathrm{ad}(\mathfrak{so}(3)) \oplus 4D_0$, there are five class of algebra $(L_{10.19} - L_{10.23})$ that may be distinguished by their algebra of *R*-constants $(A_{4.2} - A_{4.6})$. Finally, for the *R*-representation $2\mathrm{ad}(\mathfrak{so}(3)) \oplus D_0$, there are three classes of algebra $(L_{10.24}, L_{10.25}, L_{10.26})$ that as far as we can ascertain, are not mutually isomorphic.

5.4. Dimension of NR Is Seven

In dimension seven, the following seven-dimensional nilpotent Lie algebras admit subalgebras of derivations that are isomorphic to $\mathfrak{so}(3)$, using the listing given in [8]: 17, 37*A*, 37*D*1. For 17, the seven-dimensional Heisenberg algebra, the *R*-representations $R_4 \oplus 3D_0$ and $2\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ both occur and lead to the unique algebras $L_{10.27}$) and $L_{10.28}$). In the case of 37*A*, the *R*-representation is $2\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ and gives the unique algebra $L_{10.29}$). For 37*D*1 each of the *R*-representations, $R_4 \oplus \mathrm{ad}(\mathfrak{so}(3))$, $R_4 \oplus 3D_0$ and $2\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ is possible. The first two of these representations lead to the unique algebras $L_{10.30}$ and $L_{10.31}$, respectively. The case of the *R*-representation $2\mathrm{ad}\,\mathfrak{so}(3) \oplus D_0$ produce four algebras $(L_{10.32}, L_{10.33}, L_{10.34}, L_{10.35})$, which, although very similar, appear to us not to be mutually isomorphic.

Taking into account the various *R*-representations and the decomposable seven-dimensional nilpotent Lie algebras that can occur, the only case for which there can be an indecomposable Levi decomposition Lie algebra, is for $NR = R^7$ with *R*-representation either R_7 or $R_4 \oplus \mathrm{ad}(\mathfrak{so}(3))$. Each of these cases gives a unique algebra $(L_{10.36}, L_{10.37})$.

6. Summary of Main Techniques for Finding Representations

6.1. Radical Has Trivial Center

If the radical has trivial center then the whole Lie algebra has trivial center. Instead of taking the full adjoint representation, we take the adjoint representation restricted to the radical *N*.

6.2. Abelian Codimension One Nilradical

If the radical *N* has a codimension one abelian nilradical then we have a representation of *N* in $\mathfrak{gl}(n, \mathbb{R})$ where *n* is the dimension of *N*. After finding a representation of the radical it remains to add the semi-simple factor, if possible.

6.3. Radical Is Abelian

If the radical *N* is Abelian then $L = S \rtimes N$ has a representation in $\mathfrak{gl}(n+1, \mathbb{R})$ where the dimension of *N* is *n*. We have the following result.

Theorem 6.1. Let a simple Lie algebra S have a faithful representation in End(N) for some vector space N. Then there is a Lie algebra $S \rtimes N$ that has a Levi decomposition with N being an abelian radical. Conversely, every Lie algebra that has a Levi decomposition with abelian radical arises in this way. Such a Lie algebra is decomposable if and only if the representation of S, being completely reducible, contains a trivial subrepresentation.

The representation is obtained by augmenting the R-representation by an extra row and column, the row consisting of zeroes and the column consisting of n arbitrary entries and $(n+1,n+1)$-entry zero.

6.4. Entire Lie Algebra Has Trivial Center

If the Lie algebra as a whole has trivial center, whereas the radical has non-trivial center, then we can take full the adjoint representation.

6.5. Using the *R*-Representation

An interesting question is to what extent the R-representation helps to determine a representation of the Lie algebra $L = S \rtimes N$. It may be helpful, for example, to consider the invariant subspaces coming from the R-Representation when a representation for the radical has been found and one wishes to extend it to the full Levi decomposition algebra by adding a representation of the semi-simple factor.

7. Representations of Indecomposable Lie Algebras of Dimension Ten that Have Non-Trivial Levi Decomposition and Whose Semi-Simple Part Is Not $\mathfrak{sl}(2,\mathbb{R})$

7.1. NR Four-Dimensional Abelian

7.1.1. $S = \mathfrak{so}(3,1)$

$$\begin{aligned} L_{10.1}:&[e_1,e_2]=e_3,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_1,e_8]=e_9,\\ &[e_1,e_9]=-e_8,[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_2,e_7]=-e_9,\\ &[e_2,e_9]=e_7,[e_3,e_4]=e_5,[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_3,e_8]=-e_7,\\ &[e_4,e_5]=-e_3,[e_4,e_6]=e_2,[e_4,e_7]=e_{10},[e_4,e_{10}]=e_7,[e_5,e_6]=-e_1,\\ &[e_5,e_8]=e_{10},[e_5,e_{10}]=e_8,[e_6,e_9]=e_{10},[e_6,e_{10}]=e_9 \end{aligned}$$

$$\begin{bmatrix} 0 & s_3 & -s_2 & s_4 & s_7 \\ -s_3 & 0 & s_1 & s_5 & s_8 \\ s_2 & -s_1 & 0 & s_6 & s_9 \\ s_4 & s_5 & s_6 & 0 & s_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.1'}:&[e_1,e_2]=2e_2,[e_1,e_3]=-2e_3,[e_1,e_7]=e_7,[e_1,e_8]=-e_8,[e_1,e_9]=e_9,\\ &[e_1,e_{10}]=-e_{10},[e_2,e_3]=e_1,[e_2,e_8]=e_7,[e_2,e_{10}]=e_9,[e_3,e_7]=e_8,\\ &[e_3,e_9]=e_{10},[e_4,e_5]=2e_5,[e_4,e_6]=-2e_6,[e_4,e_7]=e_7,[e_4,e_8]=e_8,\\ &[e_4,e_9]=-e_9,[e_4,e_{10}]=-e_{10},[e_5,e_6]=e_4,[e_5,e_9]=e_7,[e_5,e_{10}]=e_8,\\ &[e_6,e_7]=e_9,[e_6,e_8]=e_{10} \end{aligned}$$

$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_7 \\ s_5 & -s_1 & s_4 & s_6 & s_8 \\ s_6 & -s_4 & -s_1 & -s_5 & s_9 \\ -s_4 & s_3 & -s_2 & s_1 & s_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.1.2. $S = \mathfrak{so}(4) \approx \mathfrak{so}(3) \oplus \mathfrak{so}(3)$

$$L_{10.2}: [e_1,e_2]=e_3,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_1,e_8]=e_9,$$
$$[e_1,e_9]=-e_8,[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_2,e_7]=-e_9,$$
$$[e_2,e_9]=e_7,[e_3,e_4]=e_5,[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_3,e_8]=-e_7,$$
$$[e_4,e_5]=-e_3,[e_4,e_6]=e_2,[e_4,e_7]=e_{10},[e_4,e_{10}]=e_7,[e_5,e_6]=-e_1,$$
$$[e_5,e_8]=e_{10},[e_5,e_{10}]=e_8,[e_6,e_9]=e_{10},[e_6,e_{10}]=-e_9$$

$$\begin{bmatrix} 0 & s_3 & -s_2 & s_4 & s_7 \\ -s_3 & 0 & s_1 & s_5 & s_8 \\ s_2 & -s_1 & 0 & s_6 & s_9 \\ -s_4 & -s_5 & -s_6 & 0 & s_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.1.3. $S = \mathfrak{so}(2,2) \approx \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

$$L_{10.3}: [e_1,e_2]=e_6,[e_1,e_3]=e_5,[e_1,e_5]=e_3,[e_1,e_6]=e_2,[e_1,e_7]=e_9,$$
$$[e_1,e_9]=e_7,[e_2,e_4]=e_5,[e_2,e_5]=e_4,[e_2,e_6]=-e_1,[e_2,e_7]=e_{10},$$
$$[e_2,e_{10}]=e_7,[e_3,e_4]=e_6,[e_3,e_5]=-e_1,[e_3,e_6]=e_4,[e_3,e_8]=e_9,$$
$$[e_3,e_9]=e_8,[e_4,e_5]=-e_2,[e_4,e_6]=-e_3,[e_4,e_8]=e_{10},[e_4,e_{10}]=e_8,$$
$$[e_5,e_7]=-e_8,[e_5,e_8]=e_7,[e_6,e_9]=-e_{10},[e_6,e_{10}]=e_9$$

$$\begin{bmatrix} 0 & s_5 & s_1 & s_2 & s_7 \\ -s_5 & 0 & s_3 & s_4 & s_8 \\ s_1 & s_3 & 0 & s_6 & s_9 \\ s_2 & s_4 & -s_6 & 0 & s_{10} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.2. $S = \mathfrak{so}(3)$, *NR* Five-Dimensional Heisenberg $= A_{5.4}$, *R*-Representation $R_4 \oplus 3D_0$

$$L_{10.4}: [e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_5,[e_1,e_5]=-\frac{1}{2}e_4,[e_1,e_6]=\frac{1}{2}e_7,$$
$$[e_1,e_7]=-\frac{1}{2}e_6,[e_2,e_3]=e_1,[e_2,e_4]=\frac{1}{2}e_6,[e_2,e_5]=-\frac{1}{2}e_7,$$
$$[e_2,e_6]=-\frac{1}{2}e_4,[e_2,e_7]=\frac{1}{2}e_5,[e_3,e_4]=\frac{1}{2}e_7,[e_3,e_5]=\frac{1}{2}e_6,$$
$$[e_3,e_6]=-\frac{1}{2}e_5,[e_3,e_7]=-\frac{1}{2}e_4,[e_4,e_6]=e_8,[e_4,e_9]=-e_6,[e_4,e_{10}]=e_4,$$
$$[e_5,e_7]=e_8,[e_5,e_9]=-e_7,[e_5,e_{10}]=e_5,[e_6,e_9]=e_4,[e_6,e_{10}]=e_6,$$
$$[e_7,e_9]=e_5,[e_7,e_{10}]=e_7,[e_8,e_{10}]=2e_8,[e_9,e_{10}]=ae_8,(a=-1,0,1)$$

$$\begin{bmatrix} -s_{10} & -\frac{1}{2}s_1 & -\frac{1}{2}s_2-s_9 & -\frac{1}{2}s_3 & 0 & s_4 \\ \frac{1}{2}s_1 & -s_{10} & -\frac{1}{2}s_3 & \frac{1}{2}s_2-s_9 & 0 & s_5 \\ \frac{1}{2}s_2+s_9 & \frac{1}{2}s_3 & -s_{10} & -\frac{1}{2}s_1 & 0 & s_6 \\ \frac{1}{2}s_3 & -\frac{1}{2}s_2+s_9 & \frac{1}{2}s_1 & -s_{10} & 0 & s_7 \\ -s_6 & -s_7 & s_4 & s_5 & -2s_{10} & as_9+2s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.3. $S = \mathfrak{so}(3)$, *NR* Five-Dimensional Abelian, *R*-Representation $R_4 \oplus 3D_0$

$$\begin{aligned} L_{10.5}: &[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_7,[e_1,e_5]=\frac{1}{2}e_6,[e_1,e_6]=-\frac{1}{2}e_5, \\ &[e_1,e_7]=-\frac{1}{2}e_4,[e_2,e_3]=e_1,[e_2,e_4]=\frac{1}{2}e_5,[e_2,e_5]=-\frac{1}{2}e_4, \\ &[e_2,e_6]=\frac{1}{2}e_7,[e_2,e_7]=-\frac{1}{2}e_6,[e_3,e_4]=\frac{1}{2}e_6,[e_3,e_5]=-\frac{1}{2}e_7, \\ &[e_3,e_6]=-\frac{1}{2}e_4,[e_3,e_7]=\frac{1}{2}e_5,[e_4,e_9]=-e_6,[e_4,e_{10}]=e_4, \\ &[e_5,e_9]=-e_7,[e_5,e_{10}]=e_5,[e_6,e_9]=e_4,[e_6,e_{10}]=e_6, \\ &[e_7,e_9]=e_5,[e_7,e_{10}]=e_7,[e_8,e_9]=ae_8,[e_8,e_{10}]=be_8, \\ &[e_9,e_{10}]=ce_8,\left(c=0 \text{ or } 1, a^2+b^2+c^2\neq 0\right) \end{aligned}$$

$$\begin{bmatrix} -s_{10} & -\frac{1}{2}s_2 & -\frac{1}{2}s_3-s_9 & -\frac{1}{2}s_1 & 0 & s_4 \\ \frac{1}{2}s_2 & -s_{10} & -\frac{1}{2}s_1 & \frac{1}{2}s_3-s_9 & 0 & s_5 \\ \frac{1}{2}s_3+s_9 & \frac{1}{2}s_1 & -s_{10} & -\frac{1}{2}s_2 & 0 & s_6 \\ \frac{1}{2}s_1 & -\frac{1}{2}s_3+s_9 & \frac{1}{2}s_2 & -s_{10} & 0 & s_7 \\ 0 & 0 & 0 & 0 & -as_9-bs_{10} & bs_8+cs_9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.4. $S = \mathfrak{so}(3)$, *NR* Five-Dimensional Abelian, *R*-Representation $\mathrm{ad}(\mathfrak{so}(3)) \oplus 4D_0$

$$\begin{aligned} L_{10.6}: &[e_1,e_2]=e_3,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5, \\ &[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_3,e_4]=e_5, \\ &[e_3,e_5]=-e_4,[e_4,e_9]=ae_4,[e_5,e_9]=ae_5,[e_6,e_9]=ae_6, \\ &[e_7,e_9]=e_7,[e_4,e_{10}]=be_4,[e_5,e_{10}]=be_5, \\ &[e_6,e_{10}]=be_6,[e_8,e_{10}]=e_8,(ab\neq 0) \end{aligned}$$

$$\begin{bmatrix} -as_9-bs_{10} & -s_3 & s_2 & 0 & 0 & s_4 \\ s_3 & -as_9-sbs_{10} & -s_1 & 0 & 0 & s_5 \\ -s_2 & s_1 & -as_9-bs_{10} & 0 & 0 & s_6 \\ 0 & 0 & 0 & -s_9 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -s_{10} & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.7}: & [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2, [e_1,e_5]=e_6, \\ & [e_2,e_4]=-e_6, [e_3,e_4]=e_5, [e_1,e_6]=-e_5, [e_2,e_6]=e_4, \\ & [e_3,e_5]=-e_4, [e_4,e_9]=ae_4, [e_5,e_9]=ae_5, [e_6,e_9]=ae_6, \\ & [e_7,e_9]=e_7, [e_8,e_9]=e_8, [e_4,e_{10}]=be_4, [e_5,e_{10}]=be_5, \\ & [e_6,e_{10}]=be_6, [e_7,e_{10}]=-e_8, [e_8,e_{10}]=e_7, \left(a^2+b^2\neq 0\right) \end{aligned}$$

$$\begin{bmatrix} -as_9-bs_{10} & -s_3 & s_2 & 0 & 0 & s_4 \\ s_3 & -as_9bs_{10} & -s_1 & 0 & 0 & s_5 \\ -s_2 & s_1 & -as_9-bs_{10} & 0 & 0 & s_6 \\ 0 & 0 & 0 & -s_9 & -s_{10} & s_7 \\ 0 & 0 & 0 & s_{10} & -s_9 & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.8}: & [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2, [e_1,e_5]=e_6, \\ & [e_2,e_4]=-e_6, [e_3,e_4]=e_5, [e_1,e_6]=-e_5, [e_2,e_6]=e_4, \\ & [e_3,e_5]=-e_4, [e_4,e_9]=ae_4, [e_5,e_9]=ae_5, [e_6,e_9]=ae_6, \\ & [e_7,e_9]=e_7, [e_8,e_9]=e_8, [e_4,e_{10}]=e_4, [e_5,e_{10}]=e_5, \\ & [e_6,e_{10}]=e_6, [e_8,e_{10}]=e_7 \end{aligned}$$

$$\begin{bmatrix} -as_9-s_{10} & -s_3 & s_2 & 0 & 0 & s_4 \\ s_3 & -as_9-s_{10} & -s_1 & 0 & 0 & s_5 \\ -s_2 & s_1 & -as_9-s_{10} & 0 & 0 & s_6 \\ 0 & 0 & 0 & -s_9 & -s_{10} & s_7 \\ 0 & 0 & 0 & 0 & -s_9 & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.9}: & [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2, [e_1,e_5]=e_6, \\ & [e_2,e_4]=-e_6, [e_3,e_4]=e_5, [e_1,e_6]=-e_5, [e_2,e_6]=e_4, \\ & [e_3,e_5]=-e_4, [e_4,e_9]=e_4, [e_5,e_9]=e_5, [e_6,e_9]=e_6, \\ & [e_7,e_{10}]=e_7, [e_9,e_{10}]=e_8 \end{aligned}$$

$$\begin{bmatrix} -s_9 & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_9 & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_9 & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -s_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & 0 & -s_{10} & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.5. $S=\mathfrak{so}(3)$, $NR=A_{6.3}$, R-Representation $2\mathrm{ad}(\mathfrak{so}(3))\oplus D_0$

$$\begin{aligned}L_{10.10}:&[e_1,e_2]=e_3,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_1,e_8]=e_9,\\&[e_1,e_9]=-e_8,[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_2,e_7]=-e_9,\\&[e_2,e_9]=e_7,[e_3,e_4]=e_5,[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_3,e_8]=-e_7,\\&[e_4,e_5]=e_9,[e_4,e_6]=-e_8,[e_4,e_{10}]=e_4,[e_5,e_6]=e_7,[e_5,e_{10}]=e_5,\\&[e_6,e_{10}]=e_6,[e_7,e_{10}]=2e_7,[e_8,e_{10}]=2e_8,[e_9,e_{10}]=2e_9\end{aligned}$$

$$\begin{bmatrix}-s_{10}&s_3&-s_2&0&0&0&s_4\\-s_3&-s_{10}&s_1&0&0&0&s_5\\s_2&-s_1&-s_{10}&0&0&0&s_6\\0&-s_6&s_5&-2s_{10}&-s_3&s_2&2s_7\\s_6&0&-s_4&s_3&-2s_{10}&-s_1&2s_8\\-s_5&s_4&0&-s_2&s_1&-2s_{10}&2s_9\\0&0&0&0&0&0&0\end{bmatrix}$$

7.6. $S=\mathfrak{so}(3)$, $NR=A_{6.5}$, R-Representation $R_4\oplus 3D_0$

$$\begin{aligned}L_{10.11}:&[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_7,[e_1,e_5]=\frac{1}{2}e_6,[e_1,e_6]=-\frac{1}{2}e_5,\\&[e_1,e_7]=-\frac{1}{2}e_4,[e_2,e_3]=e_1,[e_2,e_4]=\frac{1}{2}e_5,[e_2,e_5]=-\frac{1}{2}e_4,\\&[e_2,e_6]=\frac{1}{2}e_7,[e_2,e_7]=-\frac{1}{2}e_6,[e_3,e_4]=\frac{1}{2}e_6,[e_3,e_5]=-\frac{1}{2}e_7,\\&[e_3,e_6]=-\frac{1}{2}e_4,[e_3,e_7]=\frac{1}{2}e_5,[e_4,e_6]=e_8,[e_4,e_7]=e_9,\\&[e_4,e_{10}]=ae_4-be_5,[e_5,e_6]=-e_9,[e_5,e_7]=e_8,[e_5,e_{10}]=ae_5+be_4,\\&[e_6,e_{10}]=ae_6+be_7,[e_7,e_{10}]=ae_7-be_6,[e_8,e_{10}]=2ae_8+2be_9,\\&[e_9,e_{10}]=2ae_9-2be_8,a^2+b^2\neq 0\end{aligned}$$

$$\begin{bmatrix}-as_{10}&-\frac{1}{2}s_2-bs_{10}&-\frac{1}{2}s_3&-\frac{1}{2}s_1&0&0&s_4\\\frac{1}{2}s_2+bs_{10}&-as_{10}&-\frac{1}{2}s_1&\frac{1}{2}s_3&0&0&s_5\\\frac{1}{2}s_3&\frac{1}{2}s_1&-as_{10}&-\frac{1}{2}s_2+bs_{10}&0&0&s_6\\\frac{1}{2}s_1&-\frac{1}{2}s_3&\frac{1}{2}s_2-bs_{10}&-as_{10}&0&0&s_7\\-s_6&-s_7&s_4&s_5&-2as_{10}&2bs_{10}&s_8\\-s_7&s_6&-s_5&s_4&-2bs_{10}&-2as_{10}&s_9\\0&0&0&0&0&0&0\end{bmatrix}$$

7.7. $S=\mathfrak{so}(3)$, $NR=A_{5.4}\oplus\mathbb{R}$, R-Representation $R_4\oplus 3D_0$

$$\begin{aligned}L_{10.12}:&[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_5]=\frac{1}{2}e_8,[e_1,e_6]=\frac{1}{2}e_7,[e_1,e_7]=-\frac{1}{2}e_6,\\&[e_1,e_8]=-\frac{1}{2}e_5,[e_2,e_3]=e_1,[e_2,e_5]=\frac{1}{2}e_7,[e_2,e_6]=-\frac{1}{2}e_8,\end{aligned}$$

$$[e_2,e_7]=-\frac{1}{2}e_5,[e_2,e_8]=\frac{1}{2}e_6,[e_3,e_5]=-\frac{1}{2}e_6,[e_3,e_6]=\frac{1}{2}e_5,$$
$$[e_3,e_7]=-\frac{1}{2}e_8,[e_3,e_8]=\frac{1}{2}e_7,[e_5,e_7]=e_4,[e_6,e_8]=e_4,$$
$$[e_4,e_{10}]=2e_4,[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=e_7,[e_8,e_{10}]=e_8,$$
$$[e_9,e_{10}]=ae_4+be_9,(b\neq 0;b\neq 2,a=0;b=2,a=0,1)$$

$$\begin{bmatrix} -2s_{10} & -s_7 & -s_8 & s_5 & s_6 & -as_{10} & as_9+2s_4 \\ 0 & -s_{10} & \frac{1}{2}s_3 & -\frac{1}{2}s_2 & -\frac{1}{2}s_1 & 0 & s_5 \\ 0 & -\frac{1}{2}s_3 & -s_{10} & -\frac{1}{2}s_1 & \frac{1}{2}s_2 & 0 & s_6 \\ 0 & \frac{1}{2}s_2 & \frac{1}{2}s_1 & -s_{10} & \frac{1}{2}s_3 & 0 & s_7 \\ 0 & \frac{1}{2}s_1 & -\frac{1}{2}s_2 & -\frac{1}{2}s_3 & -s_{10} & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & -bs_{10} & bs_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.8. $S=\mathfrak{so}(3)$, $NR=H\oplus\mathbb{R}^3$, R-Representation $\mathrm{ad}(\mathfrak{so}(3))\oplus 4D_0$

$$L_{10.13}:[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_8]=e_9,[e_1,e_9]=-e_8,[e_2,e_3]=e_1,$$
$$[e_2,e_7]=-e_9,[e_2,e_9]=e_7,[e_3,e_7]=e_8,[e_3,e_8]=-e_7,[e_4,e_{10}]=2e_4,$$
$$[e_5,e_6]=e_4,[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=e_7,[e_8,e_{10}]=e_8,$$
$$[e_9,e_{10}]=e_9$$

$$\begin{bmatrix} -2s_{10} & 0 & 0 & s_4 & 0 & 0 & 0 & 0 \\ 0 & -s_{10} & 0 & s_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -s_{10} & s_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -s_{10} & -s_3 & s_2 & s_7 \\ 0 & 0 & 0 & 0 & s_3 & -s_{10} & -s_1 & s_8 \\ 0 & 0 & 0 & 0 & -s_2 & s_1 & -s_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.9. $S=\mathfrak{so}(3)$, NR Six-Dimensional Abelian, R-Representation $R_5\oplus 2D_0$

$$L_{10.15}:[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_7,[e_1,e_5]=-\frac{1}{2}e_6,$$
$$[e_1,e_6]=2e_5-e_8,[e_1,e_7]=-2e_4,[e_1,e_8]=3e_6,[e_2,e_3]=e_1,$$
$$[e_2,e_4]=\frac{1}{2}e_6,[e_2,e_5]=\frac{1}{2}e_7,[e_2,e_6]=-2e_4,[e_2,e_7]=-2e_5-e_8,$$
$$[e_2,e_8]=3e_7,[e_3,e_4]=2e_5,[e_3,e_5]=-2e_4,[e_3,e_6]=e_7,$$
$$[e_3,e_7]=-e_6,[e_4,e_{10}]=e_4,[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,$$
$$[e_7,e_{10}]=e_7,[e_8,e_{10}]=e_8,[e_9,e_{10}]=ae_9,(a\neq 0)$$

$$\begin{bmatrix} -s_{10} & -2s_3 & -2s_2 & -2s_1 & 0 & 0 & s_4 \\ 2s_3 & -s_{10} & 2s_1 & -2s_2 & 0 & 0 & s_5 \\ \frac{1}{2}s_2 & -\frac{1}{2}s_1 & -s_{10} & -s_3 & 3s_1 & 0 & s_6 \\ \frac{1}{2}s_1 & \frac{1}{2}s_2 & s_3 & -s_{10} & 3s_2 & 0 & s_7 \\ 0 & 0 & -s_1 & -s_2 & -s_{10} & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & -as_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.10. $S = \mathfrak{so}(3)$, *NR* Six-Dimensional Abelian, *R*-Representation $R_4 \oplus 3D_0$

$$L_{10.16}: [e_1,e_2]=e_3, [e_1,e_3]=-e_2, [e_1,e_4]=\frac{1}{2}e_7, [e_1,e_5]=\frac{1}{2}e_6, [e_1,e_6]=-\frac{1}{2}e_5,$$
$$[e_1,e_7]=-\frac{1}{2}e_4, [e_2,e_3]=e_1, [e_2,e_4]=\frac{1}{2}e_5, [e_2,e_5]=-\frac{1}{2}e_4,$$
$$[e_2,e_6]=\frac{1}{2}e_7, [e_2,e_7]=-\frac{1}{2}e_6, [e_3,e_4]=\frac{1}{2}e_6, [e_3,e_5]=-\frac{1}{2}e_7,$$
$$[e_3,e_6]=-\frac{1}{2}e_4, [e_3,e_7]=\frac{1}{2}e_5, [e_4,e_{10}]=ae_4-be_6,$$
$$[e_5,e_{10}]=ae_5-be_7, [e_6,e_{10}]=be_4+ae_6, [e_7,e_{10}]=be_5+ae_7,$$
$$[e_8,e_{10}]=ce_8, [e_9,e_{10}]=ce_9+e_8 \left(a=1 \text{ or } b=1\right)$$

$$\begin{bmatrix} -as_{10} & -\frac{1}{2}s_2 & -\frac{1}{2}s_3-bs_{10} & -\frac{1}{2}s_1 & 0 & 0 & s_4 \\ \frac{1}{2}s_2 & -as_{10} & -\frac{1}{2}s_1 & \frac{1}{2}s_3-bs_{10} & 0 & 0 & s_5 \\ \frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_1 & -as_{10} & -\frac{1}{2}s_2 & 0 & 0 & s_6 \\ \frac{1}{2}s_1 & -\frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_2 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -cs_{10} & -s_{10} & s_8 \\ 0 & 0 & 0 & 0 & 0 & -cs_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.17}: [e_1,e_2]=e_3, [e_1,e_3]=-e_2, [e_1,e_4]=\frac{1}{2}e_7, [e_1,e_5]=\frac{1}{2}e_6, [e_1,e_6]=-\frac{1}{2}e_5,$$
$$[e_1,e_7]=-\frac{1}{2}e_4, [e_2,e_3]=e_1, [e_2,e_4]=\frac{1}{2}e_5, [e_2,e_5]=-\frac{1}{2}e_4,$$
$$[e_2,e_6]=\frac{1}{2}e_7, [e_2,e_7]=-\frac{1}{2}e_6, [e_3,e_4]=\frac{1}{2}e_6, [e_3,e_5]=-\frac{1}{2}e_7,$$
$$[e_3,e_6]=-\frac{1}{2}e_4, [e_3,e_7]=\frac{1}{2}e_5, [e_4,e_{10}]=ae_4-be_6,$$
$$[e_5,e_{10}]=ae_5-be_7, [e_6,e_{10}]=be_4+ae_6, [e_7,e_{10}]=be_5+ae_7,$$
$$[e_8,e_{10}]=ce_8, [e_9,e_{10}]=de_9, \left(a=1 \text{ or } b=1, cd \neq 0\right)$$

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$$\begin{bmatrix} -as_{10} & -\frac{1}{2}s_2 & -\frac{1}{2}s_3-bs_{10} & -\frac{1}{2}s_1 & 0 & 0 & s_4 \\ \frac{1}{2}s_2 & -as_{10} & -\frac{1}{2}s_1 & \frac{1}{2}s_3-bs_{10} & 0 & 0 & s_5 \\ \frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_1 & -as_{10} & -\frac{1}{2}s_2 & 0 & 0 & s_6 \\ \frac{1}{2}s_1 & -\frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_2 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -cs_{10} & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & -ds_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.18}: & [e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_7,[e_1,e_5]=\frac{1}{2}e_6,[e_1,e_6]=-\frac{1}{2}e_5, \\ & [e_1,e_7]=-\frac{1}{2}e_4,[e_2,e_3]=e_1,[e_2,e_4]=\frac{1}{2}e_5,[e_2,e_5]=-\frac{1}{2}e_4,[e_2,e_6]=\frac{1}{2}e_7, \\ & [e_2,e_7]=-\frac{1}{2}e_6,[e_3,e_4]=\frac{1}{2}e_6,[e_3,e_5]=-\frac{1}{2}e_7,[e_3,e_6]=-\frac{1}{2}e_4,[e_3,e_7]=\frac{1}{2}e_5, \\ & [e_4,e_{10}]=ae_4-be_6,[e_5,e_{10}]=ae_5-be_7,[e_6,e_{10}]=be_4+ae_6,[e_7,e_{10}]=be_5+ae_7, \\ & [e_8,e_{10}]=ce_8+de_9,[e_9,e_{10}]=-de_8+ce_9,\ (a=1 \text{ or } b=1, d\neq 0) \end{aligned}$$

$$\begin{bmatrix} -as_{10} & -\frac{1}{2}s_2 & -\frac{1}{2}s_3-bs_{10} & -\frac{1}{2}s_1 & 0 & 0 & s_4 \\ \frac{1}{2}s_2 & -as_{10} & -\frac{1}{2}s_1 & \frac{1}{2}s_3-bs_{10} & 0 & 0 & s_5 \\ \frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_1 & -as_{10} & -\frac{1}{2}s_2 & 0 & 0 & s_6 \\ \frac{1}{2}s_1 & -\frac{1}{2}s_3+bs_{10} & \frac{1}{2}s_2 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -cs_{10} & ds_{10} & s_8 \\ 0 & 0 & 0 & 0 & -ds_{10} & -cs_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.11. $S=\mathfrak{so}(3)$, *NR* Six-Dimensional Abelian, *R*-Representation $\mathrm{ad}(\mathfrak{so}(3))\oplus 4D_0$

$$\begin{aligned} L_{10.19}: & [e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6, \\ & [e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_4,e_{10}]=e_4, \\ & [e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=ae_7,[e_8,e_{10}]=e_7+ae_8, \\ & [e_9,e_{10}]=be_9,\ (b\neq 0) \end{aligned}$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & -s_{10} & 0 & s_7 \\ 0 & 0 & 0 & 0 & -as_{10} & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & -bs_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.20}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_4,e_{10}]=e_4,$$
$$[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=ae_7,[e_9,e_{10}]=e_8,\ (a\neq 0)$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & 0 & -s_{10} & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.21}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_4,e_{10}]=e_4,$$
$$[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=ae_7,[e_8,e_{10}]=e_7+ae_8,$$
$$[e_9,e_{10}]=e_8+ae_9$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & -s_{10} & 0 & s_7 \\ 0 & 0 & 0 & 0 & -as_{10} & -s_{10} & s_8 \\ 0 & 0 & 0 & 0 & 0 & -as_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.22}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,$$
$$[e_7,e_{10}]=ae_4,[e_8,e_{10}]=be_5,[e_9,e_{10}]=ce_9,\ (abc\neq 0)$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -bs_{10} & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & -cs_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.23}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_4,e_{10}]=e_4,$$
$$[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=ae_7,[e_8,e_{10}]=be_8-ce_9,$$
$$[e_9,e_{10}]=be_8+ce_9,\ (ac\neq 0)$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & -bs_{10} & -cs_{10} & s_8 \\ 0 & 0 & 0 & 0 & cs_{10} & -bs_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.12. $S=\mathfrak{so}(3)$, *NR* Six-Dimensional Abelian, *R*-Representation $2\mathrm{ad}(\mathfrak{so}(3))\oplus D_0$

$$L_{10.24}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_1,e_8]=e_9,$$
$$[e_2,e_7]=-e_9,[e_3,e_7]=e_8,[e_1,e_9]=-e_8,[e_2,e_9]=e_7,[e_3,e_8]=-e_7,$$
$$[e_4,e_{10}]=e_4,[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=ae_7,$$
$$[e_8,e_{10}]=ae_8,[e_9,e_{10}]=ae_9,\ (a\neq 0)$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & 0 & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & 0 & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & -as_{10} & -s_3 & s_2 & s_7 \\ 0 & 0 & 0 & s_3 & -as_{10} & -s_1 & s_8 \\ 0 & 0 & 0 & -s_2 & s_1 & -as_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.25}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_1,e_8]=e_9,$$
$$[e_2,e_7]=-e_9,[e_3,e_7]=e_8,[e_1,e_9]=-e_8,[e_2,e_9]=e_7,[e_3,e_8]=-e_7,$$
$$[e_4,e_{10}]=e_4,[e_5,e_{10}]=e_5,[e_6,e_{10}]=e_6,[e_7,e_{10}]=e_4+e_7,$$
$$[e_8,e_{10}]=e_5+e_8,[e_9,e_{10}]=e_6+e_9$$

$$\begin{bmatrix} -s_{10} & -s_3 & s_2 & -s_{10} & 0 & 0 & s_4 \\ s_3 & -s_{10} & -s_1 & 0 & -s_{10} & 0 & s_5 \\ -s_2 & s_1 & -s_{10} & 0 & 0 & -s_{10} & s_6 \\ 0 & 0 & 0 & -s_{10} & -s_3 & s_2 & s_7 \\ 0 & 0 & 0 & s_3 & -s_{10} & -s_1 & s_8 \\ 0 & 0 & 0 & -s_2 & s_1 & -s_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.26}:[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_5]=e_6,[e_2,e_4]=-e_6,$$
$$[e_3,e_4]=e_5,[e_1,e_6]=-e_5,[e_2,e_6]=e_4,[e_3,e_5]=-e_4,[e_1,e_8]=e_9,$$
$$[e_2,e_7]=-e_9,[e_3,e_7]=e_8,[e_1,e_9]=-e_8,[e_2,e_9]=e_7,[e_3,e_8]=-e_7,$$
$$[e_4,e_{10}]=ae_4-e_7,[e_5,e_{10}]=ae_5-e_8,[e_6,e_{10}]=ae_6-e_9,$$
$$[e_7,e_{10}]=e_4+ae_7,[e_8,e_{10}]=e_5+ae_8,[e_9,e_{10}]=e_6+ae_9$$

$$\begin{bmatrix} -as_{10} & -s_3 & s_2 & -s_{10} & 0 & 0 & s_4 \\ s_3 & -as_{10} & -s_1 & 0 & -s_{10} & 0 & s_5 \\ -s_2 & s_1 & -as_{10} & 0 & 0 & -s_{10} & s_6 \\ s_{10} & 0 & 0 & -as_{10} & -s_3 & s_2 & s_7 \\ 0 & s_{10} & 0 & s_3 & -as_{10} & -s_1 & s_8 \\ 0 & 0 & s_{10} & -s_2 & s_1 & -as_{10} & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.13. $S = \mathfrak{so}(3)$, *NR* = Seven-Dimensional Heisenberg = 17, *R*-Representation $R_4 \oplus 3D_0$

$$L_{10.27}: [e_1,e_2]=e_3, [e_1,e_3]=-e_2, [e_1,e_4]=\frac{1}{2}e_5, [e_1,e_5]=-\frac{1}{2}e_4, [e_1,e_6]=-\frac{1}{2}e_7,$$
$$[e_1,e_7]=\frac{1}{2}e_6, [e_2,e_3]=e_1, [e_2,e_4]=-\frac{1}{2}e_7, [e_2,e_5]=\frac{1}{2}e_6,$$
$$[e_2,e_6]=-\frac{1}{2}e_5, [e_2,e_7]=\frac{1}{2}e_4, [e_3,e_4]=-\frac{1}{2}e_6, [e_3,e_5]=-\frac{1}{2}e_7,$$
$$[e_3,e_6]=\frac{1}{2}e_4, [e_3,e_7]=\frac{1}{2}e_5, [e_4,e_5]=e_{10}, [e_6,e_7]=e_{10}, [e_8,e_9]=e_{10}$$

$$\begin{bmatrix} 0 & -s_5 & s_4 & -s_7 & s_6 & -s_9 & s_8 & 2s_{10} \\ 0 & 0 & -\frac{1}{2}s_1 & \frac{1}{2}s_3 & \frac{1}{2}s_2 ss & 0 & 0 & s_4 \\ 0 & \frac{1}{2}s_1 & 0 & -\frac{1}{2}s_2 & \frac{1}{2}s_3 & 0 & 0 & s_5 \\ 0 & -\frac{1}{2}s_3 & \frac{1}{2}s_2 & 0 & \frac{1}{2}s_1 & 0 & 0 & s_6 \\ 0 & -\frac{1}{2}s_2 & -\frac{1}{2}s_3 & -\frac{1}{2}s_1 & 0 & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.14. $S = \mathfrak{so}(3)$, *NR* = Seven-Dimensional Heisenberg = 17, *R*-Representation $2\mathrm{ad}(\mathfrak{so}(3)) \oplus D_0$

$$L_{10.28}: [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2, [e_1,e_5]=e_6, [e_1,e_6]=-e_5,$$
$$[e_1,e_8]=e_9, [e_1,e_9]=-e_8, [e_2,e_4]=-e_6, [e_2,e_6]=e_4, [e_2,e_7]=-e_9,$$
$$[e_2,e_9]=e_7, [e_3,e_4]=e_5, [e_3,e_5]=-e_4, [e_3,e_7]=e_8, [e_3,e_8]=-e_7,$$
$$[e_4,e_7]=e_{10}, [e_5,e_8]=e_{10}, [e_6,e_9]=e_{10}$$

$$\begin{bmatrix} 0 & s_4 & s_5 & s_6 & s_{10} \\ 0 & 0 & s_3 & -s_2 & s_7 \\ 0 & -s_3 & 0 & s_1 & s_8 \\ 0 & s_2 & -s_1 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.15. $S = \mathfrak{so}(3)$, *NR* Seven-Dimensional Anti-Heisenberg = 37*A*, *R*-Representation $2\mathrm{ad}(\mathfrak{so}(3)) \oplus D_0$

$$L_{10.29}: [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2, [e_1,e_5]=e_6, [e_1,e_6]=-e_5,$$
$$[e_1,e_8]=e_9, [e_1,e_9]=-e_8, [e_2,e_4]=-e_6, [e_2,e_6]=e_4, [e_2,e_7]=-e_9,$$
$$[e_2,e_9]=e_7, [e_3,e_4]=e_5, [e_3,e_5]=-e_4, [e_3,e_7]=e_8, [e_3,e_8]=-e_7,$$
$$[e_4,e_{10}]=e_7, [e_5,e_{10}]=e_8, [e_6,e_{10}]=e_9$$

$$\begin{bmatrix} 0 & -s_3 & s_2 & -s_{10} & 0 & 0 & s_7 \\ s_3 & 0 & -s_1 & 0 & -s_{10} & 0 & s_8 \\ -s_2 & s_1 & 0 & 0 & 0 & -s_{10} & s_9 \\ 0 & 0 & 0 & 0 & -s_3 & s_2 & s_4 \\ 0 & 0 & 0 & s_3 & 0 & -s_1 & s_5 \\ 0 & 0 & 0 & -s_2 & s_1 & 0 & s_6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.16. $S = \mathfrak{so}(3)$, $NR = 37D1$, R-Representation $R_4 \oplus \text{ad}(\mathfrak{so}(3))$

$$\begin{aligned} L_{10.30}: &[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_4]=-\frac{1}{2}e_5,[e_1,e_5]=\frac{1}{2}e_4, \\ &[e_1,e_6]=\frac{1}{2}e_7,[e_1,e_7]=-\frac{1}{2}e_6,[e_1,e_9]=e_{10},[e_1,e_{10}]=-e_9, \\ &[e_2,e_4]=-\frac{1}{2}e_6,[e_2,e_5]=-\frac{1}{2}e_7,[e_2,e_6]=\frac{1}{2}e_4,[e_2,e_7]=\frac{1}{2}e_5, \\ &[e_2,e_8]=-e_{10},[e_2,e_{10}]=e_8,[e_3,e_4]=-\frac{1}{2}e_7,[e_3,e_5]=\frac{1}{2}e_6, \\ &[e_3,e_6]=-\frac{1}{2}e_5,[e_3,e_7]=\frac{1}{2}e_4,[e_3,e_8]=e_9,[e_3,e_9]=-e_8,[e_4,e_5]=e_8, \\ &[e_4,e_6]=e_9,[e_4,e_7]=e_{10},[e_5,e_6]=-e_{10},[e_5,e_7]=e_9,[e_6,e_7]=-e_8 \end{aligned}$$

$$\begin{bmatrix} 0 & -s_3 & s_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ s_3 & 0 & -s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s_2 & s_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}s_5 & -\frac{1}{2}s_6 & -\frac{1}{2}s_7 & 0 & \frac{1}{2}s_1 & \frac{1}{2}s_2 & \frac{1}{2}s_3 & 0 & 0 & 0 \\ \frac{1}{2}s_4 & -\frac{1}{2}s_7 & \frac{1}{2}s_6 & -\frac{1}{2}s_1 & 0 & -\frac{1}{2}s_3 & \frac{1}{2}s_2 & 0 & 0 & 0 \\ \frac{1}{2}s_7 & \frac{1}{2}s_4 & -\frac{1}{2}s_5 & -\frac{1}{2}s_2 & \frac{1}{2}s_3 & 0 & -\frac{1}{2}s_1 & 0 & 0 & 0 \\ -\frac{1}{2}s_6 & \frac{1}{2}s_5 & \frac{1}{2}s_4 & -\frac{1}{2}s_3 & -\frac{1}{2}s_2 & \frac{1}{2}s_1 & 0 & 0 & 0 & 0 \\ 0 & -s_{10} & s_9 & -s_5 & s_4 & s_7 & -s_6 & 0 & -s_3 & s_2 \\ s_{10} & 0 & -s_8 & -s_6 & -s_7 & s_4 & s_5 & s_3 & 0 & -s_1 \\ -s_9 & s_8 & 0 & -s_7 & s_6 & -s_5 & s_4 & -s_2 & s_1 & 0 \end{bmatrix}$$

7.17. $S = \mathfrak{so}(3)$, $NR = 37D1$, R-Representation $R_4 \oplus 3D_0$

$$\begin{aligned} L_{10.31}: &[e_1,e_2]=e_3,[e_2,e_3]=e_1,[e_3,e_1]=e_2,[e_1,e_4]=-\frac{1}{2}e_5,[e_1,e_5]=\frac{1}{2}e_4, \\ &[e_1,e_6]=-\frac{1}{2}e_7,[e_1,e_7]=\frac{1}{2}e_6,[e_2,e_4]=-\frac{1}{2}e_6,[e_2,e_5]=\frac{1}{2}e_7, \\ &[e_2,e_6]=\frac{1}{2}e_4,[e_2,e_7]=-\frac{1}{2}e_5,[e_3,e_4]=\frac{1}{2}e_7,[e_3,e_5]=\frac{1}{2}e_6, \\ &[e_3,e_6]=-\frac{1}{2}e_5,[e_3,e_7]=-\frac{1}{2}e_4,[e_4,e_5]=e_8,[e_4,e_6]=e_9, \\ &[e_4,e_7]=e_{10},[e_5,e_6]=-e_{10},[e_5,e_7]=e_9,[e_6,e_7]=-e_8 \end{aligned}$$

$$\begin{bmatrix} 0 & s_4 & s_5 & s_6 & s_7 & 2s_8 & 2s_9 & 2s_{10} \\ 0 & 0 & -\frac{1}{2}s_1 & -\frac{1}{2}s_2 & -\frac{1}{2}s_3 & s_5 & s_6 & s_7 \\ 0 & \frac{1}{2}s_1 & 0 & -\frac{1}{2}s_3 & \frac{1}{2}s_2 & -s_4 & s_7 & -s_6 \\ 0 & \frac{1}{2}s_2 & \frac{1}{2}s_3 & 0 & -\frac{1}{2}s_1 & -s_7 & -s_4 & s_5 \\ 0 & \frac{1}{2}s_3 & -\frac{1}{2}s_2 & \frac{1}{2}s_1 & 0 & s_6 & -s_5 & -s_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.18. $S=\mathfrak{so}(3)$, $NR=37D1$, R-Representation $2\mathrm{ad}(\mathfrak{so}(3))\oplus D_0$

$$L_{10.32}:[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_1,e_8]=e_9,[e_1,e_9]=-e_8,$$
$$[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_2,e_7]=-e_9,[e_2,e_9]=e_7,[e_3,e_4]=e_5,$$
$$[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_3,e_8]=-e_7,[e_7,e_8]=e_6,[e_7,e_9]=-e_5,$$
$$[e_7,e_{10}]=e_4,[e_8,e_9]=e_4,[e_8,e_{10}]=e_5,[e_9,e_{10}]=e_6$$

$$\begin{bmatrix} 0 & -s_3 & s_2 & -2s_{10} & -s_9 & s_8 & 2s_4 \\ s_3 & 0 & -s_1 & s_9 & -2s_{10} & -s_7 & 2s_5 \\ -s_2 & s_1 & 0 & -s_8 & s_7 & -2s_{10} & 2s_6 \\ 0 & 0 & 0 & 0 & -s_3 & s_2 & s_7 \\ 0 & 0 & 0 & s_3 & 0 & -s_1 & s_8 \\ 0 & 0 & 0 & -s_2 & s_1 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.33}:[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_5]=e_6,[e_1,e_6]=-e_5,[e_1,e_8]=e_9,$$
$$[e_1,e_9]=-e_8,[e_2,e_3]=e_1,[e_2,e_4]=e_6,[e_2,e_6]=-e_4,[e_2,e_7]=e_9,$$
$$[e_2,e_9]=-e_7,[e_3,e_4]=-e_5,[e_3,e_5]=e_4,[e_3,e_7]=-e_8,[e_3,e_8]=e_7,$$
$$[e_7,e_8]=e_6,[e_7,e_9]=-e_5,[e_7,e_{10}]=e_4,[e_8,e_9]=e_4,[e_8,e_{10}]=e_5,$$
$$[e_9,e_{10}]=e_6$$

$$\begin{bmatrix} 0 & s_3 & -s_2 & -2s_{10} & -s_9 & s_8 & 2s_4 \\ -s_3 & 0 & -s_1 & s_9 & -2s_{10} & -s_7 & 2s_5 \\ s_2 & s_1 & 0 & -s_8 & s_7 & -2s_{10} & 2s_6 \\ 0 & 0 & 0 & 0 & s_3 & -s_2 & s_7 \\ 0 & 0 & 0 & -s_3 & 0 & -s_1 & s_8 \\ 0 & 0 & 0 & s_2 & s_1 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L_{10.34}:[e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_5]=-e_6,[e_1,e_6]=e_5,[e_1,e_8]=-e_9,[e_1,e_9]=e_8,$$
$$[e_2,e_3]=e_1,[e_2,e_4]=-e_6,[e_2,e_6]=e_4,[e_2,e_7]=-e_9,[e_2,e_9]=e_7,[e_3,e_4]=-e_5,$$
$$[e_3,e_5]=e_4,[e_3,e_7]=-e_8,[e_3,e_8]=e_7,[e_7,e_8]=e_6,[e_7,e_9]=-e_5,[e_7,e_{10}]=e_4,$$
$$[e_8,e_9]=e_4,[e_8,e_{10}]=e_5,[e_9,e_{10}]=e_6$$

$$\begin{bmatrix} 0 & s_3 & s_2 & -2s_{10} & -s_9 & s_8 & 2s_4 \\ -s_3 & 0 & s_1 & s_9 & -2s_{10} & -s_7 & 2s_5 \\ -s_2 & -s_1 & 0 & -s_8 & s_7 & -2s_{10} & 2s_6 \\ 0 & 0 & 0 & 0 & s_3 & s_2 & s_7 \\ 0 & 0 & 0 & -s_3 & 0 & s_1 & s_8 \\ 0 & 0 & 0 & -s_2 & -s_1 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} L_{10.35}: & [e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_5]=-e_6,[e_1,e_6]=e_5,[e_1,e_8]=-e_9, \\ & [e_1,e_9]=e_8,[e_2,e_3]=e_1,[e_2,e_4]=e_6,[e_2,e_6]=-e_4,[e_2,e_7]=e_9, \\ & [e_2,e_9]=-e_7,[e_3,e_4]=e_5,[e_3,e_5]=-e_4,[e_3,e_7]=e_8,[e_3,e_8]=-e_7, \\ & [e_7,e_8]=e_6,[e_7,e_9]=-e_5,[e_7,e_{10}]=e_4,[e_8,e_9]=e_4,[e_8,e_{10}]=e_5, \\ & [e_9,e_{10}]=e_6 \end{aligned}$$

$$\begin{bmatrix} 0 & -s_3 & -s_2 & -2s_{10} & -s_9 & s_8 & 2s_4 \\ s_3 & 0 & s_1 & s_9 & -2s_{10} & -s_7 & 2s_5 \\ s_2 & -s_1 & 0 & -s_8 & s_7 & -2s_{10} & 2s_6 \\ 0 & 0 & 0 & 0 & -s_3 & -s_2 & s_7 \\ 0 & 0 & 0 & s_3 & 0 & s_1 & s_8 \\ 0 & 0 & 0 & s_2 & -s_1 & 0 & s_9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.19. $S=\mathfrak{so}(3)$, *NR* Seven-Dimensional Abelian, *R*-Representation R_7

$$\begin{aligned} L_{10.36}: & [e_1,e_2]=e_3,[e_1,e_3]=-e_2,[e_1,e_4]=\frac{1}{2}e_7,[e_1,e_5]=-\frac{1}{2}e_6, \\ & [e_1,e_6]=e_9+3e_5,[e_1,e_7]=-e_8-3e_4,[e_1,e_8]=3e_{10}+\frac{5}{2}e_7, \\ & [e_1,e_9]=-\frac{5}{2}e_6,[e_1,e_{10}]=-2e_8,[e_2,e_3]=e_1,[e_2,e_4]=\frac{1}{2}e_6, \\ & [e_2,e_5]=\frac{1}{2}e_7,[e_2,e_6]=e_8-3e_4,[e_2,e_7]=e_9-3e_5,[e_2,e_8]=-\frac{5}{2}e_6, \\ & [e_2,e_9]=3e_{10}-\frac{5}{2}e_7,[e_2,e_{10}]=-2e_9,[e_3,e_4]=3e_5,[e_3,e_5]=-3e_4, \\ & [e_3,e_6]=2e_7,[e_3,e_7]=-2e_6,[e_3,e_8]=e_9,[e_3,e_9]=-e_8 \end{aligned}$$

$$\begin{bmatrix} 0 & -3s_3 & -3s_2 & -3s_1 & 0 & 0 & 0 & s_4 \\ 3s_3 & 0 & 3s_1 & -3s_2 & 0 & 0 & 0 & s_5 \\ \frac{1}{2}s_2 & -\frac{1}{2}s_1 & 0 & -2s_3 & -\frac{5}{2}s_2 & -\frac{5}{2}s_1 & 0 & s_6 \\ \frac{1}{2}s_1 & \frac{1}{2}s_2 & 2s_3 & 0 & \frac{5}{2}s_1 & -\frac{5}{2}s_2 & 0 & s_7 \\ 0 & 0 & s_2 & -s_1 & 0 & -s_3 & -2s_1 & s_8 \\ 0 & 0 & s_1 & s_2 & s_3 & 0 & -2s_2 & s_9 \\ 0 & 0 & 0 & 0 & 3s_1 & 3s_2 & 0 & s_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7.20. $S = \mathfrak{so}(3)$, *NR* Seven-Dimensional Abelian, *R*-Representation $R_4 \oplus \mathrm{ad}(\mathfrak{so}(3))$

$$L_{10.37}: [e_1,e_2]=e_3, [e_1,e_3]=-e_2, [e_1,e_4]=\frac{1}{2}e_7, [e_1,e_5]=\frac{1}{2}e_6, [e_1,e_6]=-\frac{1}{2}e_5,$$
$$[e_1,e_7]=-\frac{1}{2}e_4, [e_1,e_9]=e_{10}, [e_1,e_{10}]=-e_9, [e_2,e_3]=e_1, [e_2,e_4]=\frac{1}{2}e_5,$$
$$[e_2,e_5]=-\frac{1}{2}e_4, [e_2,e_6]=\frac{1}{2}e_7, [e_2,e_7]=-\frac{1}{2}e_6, [e_2,e_8]=-e_{10},$$
$$[e_2,e_{10}]=e_8, [e_3,e_4]=\frac{1}{2}e_6, [e_3,e_5]=-\frac{1}{2}e_7, [e_3,e_6]=-\frac{1}{2}e_4,$$
$$[e_3,e_7]=\frac{1}{2}e_5, [e_3,e_8]=e_9, [e_3,e_9]=-e_8$$

$$\begin{bmatrix} 0 & -\frac{1}{2}s_2 & -\frac{1}{2}s_3 & -\frac{1}{2}s_1 & 0 & 0 & 0 & s_4 \\ \frac{1}{2}s_2 & 0 & -\frac{1}{2}s_1 & \frac{1}{2}s_3 & 0 & 0 & 0 & s_5 \\ \frac{1}{2}s_3 & \frac{1}{2}s_1 & 0 & -\frac{1}{2}s_2 & 0 & 0 & 0 & s_6 \\ \frac{1}{2}s_1 & -\frac{1}{2}s_3 & \frac{1}{2}s_2 & 0 & 0 & 0 & 0 & s_7 \\ 0 & 0 & 0 & 0 & 0 & -s_3 & s_2 & s_8 \\ 0 & 0 & 0 & 0 & s_3 & 0 & -s_1 & s_9 \\ 0 & 0 & 0 & 0 & -s_2 & s_1 & 0 & s_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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