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Periodic Solutions of Integro-Differential Equations

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Abstract

The aim of this work is to study the existence of periodic solutions of integro-differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-L(x_t)\right]=A\left[x(t)-L(x_t)\right]+G(x_t)+\int_{-\infty}^{t}a(t-s)x(s)\mathrm{d}s+f(t)$, ($0\le t\le 2\pi$) with the periodic condition $x(0)=x(2\pi)$, where $a\in L^1(\mathbb{R}_+)$. Our approach is based on the M-boundedness of linear operators $B_{p,q}^{s}$-multipliers and some results in Besov space.

Keywords

Integro-Differential Equations, $B_{p,q}^{s}$-Multipliers, Besov Space

1. Introduction

The aim of this paper is to study the existence and solutions for some neutral functional integro-differential equations with delay by using methods of maximal regularity in spaces of vector-valued functions and Besov space. Motivated by the fact that neutral functional integro-differential equations with finite delay arise in many areas of applied mathematics, this type of equation has received much attention in recent years. In particular, the problem of the existence of periodic solutions has been considered by several authors. We refer the readers to papers [1] [2] [3] and the references listed therein for information on this subject. One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years, it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow treating certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the subjet, we refer to Lizama *et al.* [4], Hino [5], Hale [6] and Pazy [7].

We characterize the existence of periodic solutions for the following integro-differential equations in vector-valued spaces and Besov. In the case of vector-valued space, our results involve UMD spaces, the concept of R-boundedness and a condition on the resolvent operator. We remark that many of the most powerful modern theorems are valid in UMD spaces, *i.e.*, Banach space in which martingale is unconditional differences. The probabilistic definition of UMD spaces turns out to be equivalent to the L^p-boundedness of the Hilbert transform, a transformation, which is, in a sense, the typical representative example of a multiplier operator. On the other hand, the notion of R-boundedness has played an important role in the functional analytic approach to partial differential equations.

In the case of, Besov space, our results involve only boundedness of the resolvent.

In this work, we study the existence of periodic solutions for the following integro-differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)-L(x_t)\right]=A\left[x(t)-L(x_t)\right]+G(x_t)+\int_{-\infty}^{t}a(t-s)x(s)\mathrm{d}s+f(t) \quad (1)$$

where $A:D(A)\subseteq X\to X$ is a linear closed operator on Banach space $(X,\|.\|)$ and $f\in L^p(\mathbb{T},X)$ for all $p\geq 1$. For $r_{2\pi}:=2\pi N$ (some $N\in\mathbb{N}$) L and G are in $B\left(L^p\left([-r_{2\pi},0],X\right);X\right)$ is the space of all bounded linear operators and x_t is an element of $L^p\left([-r_{2\pi},0],X\right)$ which is defined as follows:

$$x_t(\theta)=x(t+\theta)\ \text{ for } \theta\in[-r_{2\pi},0].$$

For example:

$$\frac{\partial}{\partial t}\left(w(t,x)-w(t-1,x)\right)=\frac{\partial^2}{\partial t^2}\left(w(t,x)-w(t-1,x)\right)+w\left(t-\frac{\pi}{2},x\right)+g(t,x)$$

Put $y(t)(x)=w(t,x)$, $L(\varphi)=\varphi(-1)$, $G(\varphi)=\varphi\left(-\frac{\pi}{2}\right)$, $f(t)(x)=g(t,x)$ and $A\varphi=\varphi''$.

Then we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(y(t)-L(y_t)\right)=A\left(y(t)-L(y_t)\right)+G(y_t)+f(t)$$

In [8], the author investigated the existence of solutions of the following fractional integrodifferential equation:

$$\begin{aligned}\left(x(t)-L(x_t)\right)'&=A\left(x(t)-L(x_t)\right)+G(x_t)\\&\quad+\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha}\left(\int_{-\infty}^{s}b(s-\xi)x(\xi)\mathrm{d}\xi\right)\mathrm{d}s+f(t)\end{aligned} \quad (2)$$

In [9], S. Koumla, Kh. Ezzinbi and R. Bahloul., study the existence of mild solutions for some partial functional integrodifferential equations with finite delay in a Fréchet space for equation:

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)\mathrm{d}s + f(t,u_t) + h(t,u_t)$$

In [10], Ezzinbi *et al.* gave necessary and sufficient conditions for the existence of periodic solutions of Equation (1) for $a = 0$.

This work is organized as follows: after preliminaries in the second section, we are able to characterize in Section 3 the existence and uniqueness of the strong solution of the Equation (1) in Besov space, we obtain that the following assertion are equivalent If $\sup_k \left\| ik\left(ikD_k - AD_k - G_k - \tilde{a}(ik)\right)^{-1} \right\| < \infty$ and $\sup_k \left\| \left(ikD_k - AD_k - G_k - \tilde{a}(ik)\right)^{-1} \right\| < \infty$ then for every $f \in B_{p,q}^s(\mathbb{T}, X)$ there exist a unique strong $B_{p,q}^s$-solution of (**1**). In section 4, we give the conclusion.

2. Preliminaries

In this section we introduce some of the concepts to be used thereafter. We also review the classical results that provide material for a better understanding of the paper. We study the notion of M-boundedness. We present the notion of multipliers. Fourier multiplier theorems are of crucial importance in the study of maximal regularity of evolution equations. Let X be a Banach Space. Firstly, we denote By $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.

Given $1 \le p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of 2π-periodic locally p-integrable functions from $\mathbb{R}$ into X, with the norm:

$$\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p \mathrm{d}t\right)^{1/p}$$

For $f \in L^p(\mathbb{T}; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the k-th Fourier coefficient of f that is defined by:

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-ikt} f(t)\mathrm{d}t \text{ for } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For $1 \le p < \infty$, the periodic vector-valued space is defined by

$$H^{1,p}(\mathbb{T}; X) = \left\{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\right\} \quad (3)$$

Lemma 2.1 [7]:

Let $G : L^p(\mathbb{T}, X) \to X$ be a bounded linear operateur. Then:

$$\widehat{G(u_\cdot)}(k) = G(e_k\hat{u}(k)) := G_k\hat{u}(k) \text{ for all } k \in \mathbb{Z}$$

Next we give some preliminaries. Given $a \in L^1(\mathbb{R}^+)$ and $u : [0, 2\pi] \to X$ (extended by periodicity to $\mathbb{R}$), we define:

$$F(t) = \int_{-\infty}^t a(t-s)u(s)\mathrm{d}s.$$

Let $\tilde{a}(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} a(t)\mathrm{d}t$ be the Laplace transform of a. An easy computation shows that:

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), \text{ for all } k \in \mathbb{Z} \quad (4)$$

3. Periodic Strong Solutions in Besov Spaces

3.1. Preliminary

In this section, we consider the periodic solutions of Equation (1) in periodic Besov spaces $B^s_{p,q}(\mathbb{T};X)$. Firstly, we briefly recall the definition of periodic Besov spaces. Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of all rapidly decreasing smooth functions on $\mathbb{R}$. Let $D(\mathbb{T})$ be the space of all infinitely differentiable functions on $\mathbb{T}$ equipped with the locally convex topology given by the seminorms $\|f\|_n = \sup_{x\in\mathbb{T}}\left|f^{(n)}(x)\right|$ for $n \in \mathbb{N}$. Let $D'(\mathbb{T};X) = \mathcal{L}(D(\mathbb{T}),X)$. In order to define Besov spaces, we consider the dyadic-like subsets of $\mathbb{R}$:

$$I_0 = \{t \in \mathbb{R} : |t| \le 2\},\ I_k = \{t \in \mathbb{R}, 2^{k-1} < |t| \le 2^{k+1}\}$$

for $k \in \mathbb{N}$. Let $\phi(\mathbb{R})$ be the set of all systems $\phi = (\phi_k)_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ satisfying $supp(\phi_k) \subset \bar{I}_k$, for each $k \in \mathbb{N}$, $\sum_{k\in\mathbb{N}} \phi_k(x) = 1$.

Let $1 \le p,q \le \infty$, $s \in \mathbb{R}$ and $(\phi_j)_{j\ge 0} \in \phi(\mathbb{R})$ the X-valued periodic Besov space is defined by:

$$B^s_{p,q}(\mathbb{T};X) = \left\{ f \in D'(\mathbb{T};X) : \|f\|_{B^s_{p,q}} := \left(\sum_{j\ge 0} 2^{sjq}\left\|\sum_{k\in\mathbb{Z}} e_k\phi_j(k)\hat{f}(k)\right\|_p^q\right)^{1/q} < \infty \right\}.$$

For more information about the standard definitions and properties, see [7].

Proposition 3.1 1) $B^s_{p,q}((0,2\pi);X)$ is a Banach space;

2) Let $s > 0$. Then $f \in B^{s+1}_{p,q}((0,2\pi);X)$ in and only if *f* is differentiale and $f' \in B^s_{p,q}((0,2\pi);X)$

Definition 3.1 For $1 \le p < \infty$, a sequence $\{M_k\}_{k\in Z} \subset B(X,Y)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}(\mathbb{T},X)$, there exists $u \in B^s_{p,q}(\mathbb{T},Y)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

Remark 3.1 1) When $s > 0$, then $B^s_{p,q}(\mathbb{T},X) \subset L^p(\mathbb{T},X)$.

2) when $u \in B^{s+1}_{p,q}(\mathbb{T},X)$ is a strong $B^s_{p,q}$-solution of (1), then $u \in H^{1,p}(\mathbb{T},X)$, therefore u is twicely differentiable a.e. and $u(0) = u(2\pi)$.

Definition 3.2 We say that $\{M_k\}_{k\in Z} \subseteq B(X,Y)$ is M-bounded if:

$$\sup_k \|M_k\| < \infty,\ \sup_k \|k(M_{k+1} - M_k)\| < \infty \quad (5)$$

$$\sup_k \left\|k^2(M_{k+1} - 2M_k + M_{k-1})\right\| < \infty \quad (6)$$

We recall the following operator-valued Fourier multiplier theorem on Besov spaces.

Theorem 3.1 [7]

Let *X*, *Y* be Banach spaces and let $\{M_k\}_{k\in Z} \subseteq B(X,Y)$ be a M-bounded sequence. Then for $1 \le p,q < \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in Z}$ is an $B^s_{p,q}$-multiplier:

3.2. Main Result

For convenience, we introduce the following notations:

$$B_k = kA\left(L_{k+1} - L_k\right), P_k = k\left(\tilde{a}\left(i\left(k+1\right)\right) - \tilde{a}\left(ik\right)\right), Q_k = k\left(L_{k+1} - L_k\right),$$
$$R_k = k\left(G_{k+1} - G_k\right), I_k = k^2\left(C_k - 2C_{k+1} + C_{k-1}\right), J_k = k^2\left(L_k - 2L_{k+1} + L_{k-1}\right).$$

In order to give our result, the following hypotheses are fundamental.

$$\sup_k \left\|\tilde{a}\left(ik\right)\right\| := a < \infty, \sup_{k\in\mathbb{Z}} \left\|B_k\right\| := b < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\|F_k\right\| := f < \infty, \sup_{k\in\mathbb{Z}} \left\|P_k\right\| := p < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\|Q_k\right\| := q < \infty, \sup_{k\in\mathbb{Z}} \left\|R_k\right\| := r < \infty,$$
$$\sup_{k\in\mathbb{Z}} \left\|I_k\right\| := i < \infty, \sup_{k\in\mathbb{Z}} \left\|J_k\right\| := j < \infty.$$

Definition 3.3: Let $1 \le p, q < \infty$ and $s > 0$. We say that a function $x \in B^s_{p,q}\left(\mathbb{T}; X\right)$ is a strong $B^s_{p,q}$-solution of (1) if $\left(x(t) - L\left(x_t\right)\right) \in D(A)$, $Dx_t \in B^{s+1}_{p,q}\left(\mathbb{T}; X\right)$ and Equation (1) holds for all $t \in \mathbb{T}$.

We prove the following result.

Theorem 3.2: Let A be a linear closed operator. Suppose that $\left(ikD_k - AD_k - G_k - \tilde{a}\left(ik\right)\right)$ is invertible for all $k \in \mathbb{Z}$. If $\sup_k \left\|ik\left(ikD_k - AD_k - G_k - \tilde{a}\left(ik\right)\right)^{-1}\right\| < \infty$ and $\sup_k \left\|\left(ikD_k - AD_k - G_k - \tilde{a}\left(ik\right)\right)^{-1}\right\| < \infty$ then $\left\{ik\left(ikD_k - AD_k - G_k - \tilde{a}\left(ik\right)\right)^{-1}\right\}_{k\in Z}$ is an $B^s_{p,q}$-multiplier for $1 \le p, q < \infty$ and $s > 0$.

Proof. Let $S_k = ikN_k$, $N_k = \left(C_k - AD_k\right)^{-1}$, $T_k = G_k N_k$, $F_k = \tilde{a}\left(ik\right)N_k$ and $C_k = ikD_k - G_k - \tilde{a}\left(ik\right)$.

For convenience, we introduce the following result:

$$\begin{aligned} C_k - C_{k+1} &= \left[ikD_k - G_k - \tilde{a}\left(ik\right)\right] - \left[i\left(k+1\right)D_{k+1} - G_{k+1} - \tilde{a}\left(i\left(k+1\right)\right)\right] \\ &= ikI - ikL_k - G_k - \tilde{a}\left(ik\right) - \left(ik+i\right)\left(I - L_{k+1}\right) + G_{k+1} + \tilde{a}\left(i\left(k+1\right)\right) \\ &= -iI + iL_{k+1} + ik\left(L_{k+1} - L_k\right) + \left(G_{k+1} - G_k\right) + \left(\tilde{a}\left(i\left(k+1\right)\right) - \tilde{a}\left(ik\right)\right) \end{aligned}$$

Then we have:

$$\begin{aligned} k\left(C_k - C_{k+1}\right) &= -ikI + ikL_{k+1} + ik\left(k\left(L_{k+1} - L_k\right)\right) + k\left(G_{k+1} - G_k\right) \\ &\quad + k\left(\tilde{a}\left(i\left(k+1\right)\right) - \tilde{a}\left(ik\right)\right) \\ &= -ikI + ikL_{k+1} + ikQ_k + R_k + P_k \end{aligned}$$

Now, we are going to show that:

$$\begin{cases} \sup_k \left\|k\left(N_{k+1} - N_k\right)\right\| < \infty, \\ \sup_k \left\|k\left(S_{k+1} - S_k\right)\right\| < \infty, \\ \sup_k \left\|k\left(T_{k+1} - T_k\right)\right\| < \infty \end{cases} \tag{7}$$

and:

$$\begin{cases} \sup_k \left\| k^2 \left(N_{k+1} - 2N_k + N_{k-1} \right) \right\| < \infty, \\ \sup_k \left\| k^2 \left(S_{k+1} - 2S_k + S_{k-1} \right) \right\| < \infty, \\ \sup_k \left\| k^2 \left(T_{k+1} - 2T_k + T_{k-1} \right) \right\| < \infty \end{cases} \tag{8}$$

Put $\sup_{k\in\mathbb{Z}} \|N_k\| = a_1$, $\sup_{k\in\mathbb{Z}} \|S_k\| = a_2$ and $\sup_{k\in\mathbb{Z}} \|T_k\| = a_3$. We have:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \left\| k\left(N_{k+1} - N_k\right) \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| k\left[\left(C_{k+1} - AD_{k+1}\right)^{-1} - \left(C_k - AD_k\right)^{-1} \right] \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| kN_{k+1} \left[C_k - AD_k - C_{k+1} + AD_{k+1} \right] N_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| kN_{k+1} \left[\left(C_k - C_{k+1}\right) - A\left(L_{k+1} - L_k\right) \right] N_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| kN_{k+1} \left[k\left(C_k - C_{k+1}\right) - kA\left(L_{k+1} - L_k\right) \right] N_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| N_{k+1} \left[-ikI + ikL_{k+1} + ikQ_k + R_k + P_k - B_k \right] N_k \right\| \\
&\le \sup_{k\in\mathbb{Z}} \left\| \left[-N_{k+1} + N_{k+1}L_{k+1} + N_{k+1}Q_k \right] S_k + N_{k+1}R_k N_k \right\| \\
&\quad + \sup_{k\in\mathbb{Z}} \left\| N_{k+1}P_k N_k - N_{k+1}B_k N_k \right\| \\
&\le \left(a_1 + a_1 \left(2r_{2\pi}\right)^{1/p} \|L\| + a_1 q \right) a_2 + a_1^2 r + a_1^2 p + a_1^2 b.
\end{aligned}$$

We obtain:

$$\sup_{k\in\mathbb{Z}} \left\| k\left(N_{k+1} - N_k\right) \right\| < \infty \tag{9}$$

On the other hand, we have:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}} \left\| k\left(S_{k+1} - S_k\right) \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| k\left[i(k+1)\left(C_{k+1} - AD_{k+1}\right)^{-1} - ik\left(C_k - AD_k\right)^{-1} \right] \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| kN_{k+1} \left[i(k+1)\left(C_k - A\left(I - L_k\right)\right) - ik\left(C_{k+1} - A\left(I - L_{k+1}\right)\right) \right] N_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| kN_{k+1} \left[ik\left(C_k - C_{k+1}\right) + i\left(C_k - A\left(I - L_k\right)\right) - ikA\left(L_{k+1} - L_k\right) \right] N_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| N_{k+1} \left[k\left(C_k - C_{k+1}\right) \right] ikN_k + ikN_{k+1} - N_{k+1}kA\left(L_{k+1} - L_k\right) ikN_k \right\| \\
&= \sup_{k\in\mathbb{Z}} \left\| N_{k+1} \left[k\left(C_k - C_{k+1}\right) \right] ikN_k + ikN_{k+1} - N_{k+1}B_k ikN_k \right\| \\
&\le \sup_{k\in\mathbb{Z}} \left\| N_{k+1} \left[-ikI + ikL_{k+1} + ikQ_k + R_k + P_k \right] S_k \right\| + \sup_{k\in\mathbb{Z}} \left\| \frac{k}{k+1} S_{k+1} - N_{k+1}B_k S_k \right\| \\
&\le \sup_{k\in\mathbb{Z}} \left\| \frac{k}{k+1} S_{k+1} \left[-I + L_{k+1} + Q_k \right] S_k + N_{k+1} \left[R_k + P_k \right] S_k \right\| \\
&\quad + \sup_{k\in\mathbb{Z}} \left\| \frac{k}{k+1} S_{k+1} - N_{k+1}B_k S_k \right\| \\
&\le a_2^2 \left(1 + \left(2r_{2\pi}\right)^{1/p} \|L\| + q \right) + a_1 a_2 \left(r + p\right) + a_2 + b a_1 a_2.
\end{aligned}$$

Then:

$$\sup_{k\in\mathbb{Z}} \left\| k\left(S_{k+1} - S_k\right) \right\| < \infty$$

Finally we have:

$$\begin{aligned}\sup_{k\in\mathbb{Z}}\left\|k\left(T_{k+1}-T_k\right)\right\| &= \sup_{k\in\mathbb{Z}}\left\|k\left[G_{k+1}N_{k+1}-G_kN_k\right]\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|k\left(G_{k+1}-G_k\right)N_{k+1}+G_kk\left(N_{k+1}-N_k\right)\right\| \\ &\le \sup_{k\in\mathbb{Z}}\left\|k\left(G_{k+1}-G_k\right)N_{k+1}\right\|+\sup_{k\in\mathbb{Z}}\left\|G_kk\left(N_{k+1}-N_k\right)\right\| \\ &\le \sup_{k\in\mathbb{Z}}\left\|R_kN_{k+1}\right\|+\sup_{k\in\mathbb{Z}}\left\|G_k\right\|\sup_{k\in\mathbb{Z}}\left\|k\left(N_{k+1}-N_k\right)\right\| \\ &\le ra_1+\left\|G\right\|\left(2r_{2\pi}\right)^{1/p}\sup_{k\in\mathbb{Z}}\left\|k\left(N_{k+1}-N_k\right)\right\|\end{aligned}$$

Then by (9) we have:

$$\sup_{k\in\mathbb{Z}}\left\|k\left(T_{k+1}-T_k\right)\right\|<\infty$$

and:

$$\begin{aligned}&\sup_{k\in\mathbb{Z}}\left\|k\left(F_{k+1}-F_k\right)\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|k\left[\tilde{a}\left(i\left(k+1\right)\right)N_{k+1}-\tilde{a}\left(ik\right)N_k\right]\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|k\left(N_{k+1}-N_k\right)\tilde{a}\left(i\left(k+1\right)\right)+k\left(\tilde{a}\left(i\left(k+1\right)\right)-\tilde{a}\left(ik\right)\right)N_k\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|k\left(N_{k+1}-N_k\right)\tilde{a}\left(i\left(k+1\right)\right)+F_kN_k\right\| \\ &\le \sup_{k\in\mathbb{Z}}\left\|k\left(N_{k+1}-N_k\right)\right\|a+fa_1\end{aligned}$$

Then by (9) we have:

$$\sup_{k\in\mathbb{Z}}\left\|k\left(F_{k+1}-F_k\right)\right\|<\infty$$

proving (7). To verify (8), put $b_k=C_k-\alpha AD_k$,

$$\begin{aligned}&\sup_{k\in\mathbb{Z}}\left\|k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|k^2\left[\left(C_{k+1}-AD_{k+1}\right)^{-1}-2\left(C_k-AD_k\right)^{-1}+N_{k-1}\right]\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|N_{k+1}I_kN_k+N_{k+1}J_kN_k+\left(b_k-b_{k-1}\right)\left(b_{k+1}+b_{k-1}\right)k^2N_{k-1}N_k\right\| \\ &\quad+\sup_{k\in\mathbb{Z}}\left\|b_{k-1}\left(b_{k+1}-b_k\right)k^2N_{k-1}N_k\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|N_{k+1}I_kN_k+N_{k+1}J_kN_k-\left(b_k-b_{k-1}\right)\left(b_{k+1}+b_{k-1}\right)S_{k-1}S_k\right\| \\ &\quad+\sup_{k\in\mathbb{Z}}\left\|b_{k-1}\left(b_{k+1}-b_k\right)S_{k-1}S_k\right\|\end{aligned}$$

we conclude that,

$$\sup_{k\in\mathbb{Z}}\left\|k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right\|<\infty \tag{10}$$

So, $\left(N_k\right)_{k\in\mathbb{Z}}$ is M-bounded and therefore, by Theorem 3.1 is an $B^s_{p,q}$-multiplier. Furthermore:

$$\begin{aligned}&\sup_{k\in\mathbb{Z}}\left\|k^3\left(N_{k+1}-2N_k+N_{k-1}\right)\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|N_{k+1}I_kkN_k+\alpha N_{k+1}J_kkN_k-k\left(b_k-b_{k-1}\right)\left(b_{k+1}+b_{k-1}\right)S_{k-1}S_k\right\| \\ &\quad+\sup_{k\in\mathbb{Z}}\left\|b_{k-1}k\left(b_{k+1}-b_k\right)S_{k-1}S_k\right\| \\ &= \sup_{k\in\mathbb{Z}}\left\|-N_{k+1}I_kS_k-\alpha N_{k+1}J_kS_k-k\left(b_k-b_{k-1}\right)\left(b_{k+1}+b_{k-1}\right)S_{k-1}S_k\right\| \\ &\quad+\sup_{k\in\mathbb{Z}}\left\|b_{k-1}k\left(b_{k+1}-b_k\right)S_{k-1}S_k\right\|\end{aligned}$$

Then:

$$\sup_{k\in\mathbb{Z}}\left\|k^3\left(N_{k+1}-2N_k+N_{k-1}\right)\right\|<\infty \tag{11}$$

On the other hand, we have:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left\|k^2\left(S_{k+1}-2S_k+S_{k-1}\right)\right\|\\
&=\sup_{k\in\mathbb{Z}}\left\|k^2\left[i(k+1)N_{k+1}-2ikN_k+i(k-1)N_{k-1}\right]\right\|\\
&=\sup_{k\in\mathbb{Z}}\left\|ik^3\left[N_{k+1}-2N_k+N_{k-1}\right]+ik^2\left(N_{k+1}-N_{k-1}\right)\right\|
\end{aligned}$$

Then by (10) and (11), we have:

$$\sup_{k\in\mathbb{Z}}\left\|k^2\left(S_{k+1}-2S_k+S_{k-1}\right)\right\|<\infty$$

Finally we have:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left\|k^2\left(T_{k+1}-2T_k+T_{k-1}\right)\right\|\\
&=\sup_{k\in\mathbb{Z}}\left\|k^2\left[G_{k+1}N_{k+1}-2G_kN_k+G_{k-1}N_{k-1}\right]\right\|\\
&\le\sup_{k\in\mathbb{Z}}\left\|k^2\left[G_{k+1}\left(N_{k+1}-2N_k+N_{k-1}\right)-2\left(G_{k+1}-G_k\right)N_k\right\|\\
&\quad+\sup_{k\in\mathbb{Z}}\left\|\left(G_{k-1}-G_{k+1}\right)N_{k-1}\right]\right\|\\
&\le\sup_{k\in\mathbb{Z}}\left\|G_{k+1}\left[k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right]-2\left(G_{k+1}-G_k\right)k^2N_k\right\|\\
&\quad+\sup_{k\in\mathbb{Z}}\left\|\left(G_{k-1}-G_{k+1}\right)k^2N_{k-1}\right\|\\
&\le\sup_{k\in\mathbb{Z}}\left\|G_{k+1}\left[k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right]-2k\left(G_{k+1}-G_k\right)S_k\right\|\\
&\quad+\sup_{k\in\mathbb{Z}}\left\|k\left(G_{k-1}-G_{k+1}\right)\frac{k}{k-1}S_{k-1}\right\|\\
&=\sup_{k\in\mathbb{Z}}\left\|G_{k+1}\left[k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right]-2R_kS_k+R_k\frac{k}{k-1}S_{k-1}\right\|\\
&\le\left(2r_{2\pi}\right)^{1/p}\|G\|\sup_{k\in\mathbb{Z}}\left\|k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right\|+3ra_2
\end{aligned}$$

Then by (10) we have:

$$\sup_{k\in\mathbb{Z}}\left\|k^2\left(T_{k+1}-2T_k+T_{k-1}\right)\right\|<\infty$$

and:

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}\left\|k^2\left(F_{k+1}-2F_k+F_{k-1}\right)\right\|\\
&=\sup_{k\in\mathbb{Z}}\left\|k^2\left[\tilde{a}(k+1)N_{k+1}-2\tilde{a}(k)N_k+\tilde{a}(k-1)N_{k-1}\right]\right\|\\
&\le\sup_{k\in\mathbb{Z}}\left\|k^2\left[\tilde{a}(k+1)\left(N_{k+1}-2N_k+N_{k-1}\right)-2\left(\tilde{a}(k+1)-\tilde{a}(k)\right)N_k\right]\right\|\\
&\quad+\sup_{k\in\mathbb{Z}}\left\|k^2\left[\left(\tilde{a}(k-1)-\tilde{a}(k+1)\right)N_{k-1}\right]\right\|\\
&\le\sup_{k\in\mathbb{Z}}\left\|\tilde{a}(k+1)\left(k^2\left(N_{k+1}-2N_k+N_{k-1}\right)\right)+2ik\left(\tilde{a}(k+1)-\tilde{a}(k)\right)S_k\right\|\\
&\quad+\sup_{k\in\mathbb{Z}}\left\|\frac{k}{k-1}k\left(\tilde{a}(k-1)-\tilde{a}(k+1)\right)S_{k-1}\right\|
\end{aligned}$$

$$\leq a \sup_{k\in\mathbb{Z}} \left\| k^2 \left(N_{k+1} - 2N_k + N_{k-1} \right) \right\|$$
$$+ 2\sup_{k\in\mathbb{Z}} \left\| k \left(\tilde{a}(k+1) - \tilde{a}(k) \right) \right\| a_2 + \sup_{k\in\mathbb{Z}} \left\| k \left(\tilde{a}(k-1) - \tilde{a}(k+1) \right) \right\| a_2$$

Then by hypotheses and (10) we have

$$\sup_{k\in\mathbb{Z}} \left| k^2 \left(F_{k+1} - 2F_k + F_{k-1} \right) \right| < \infty$$

So, $(T_k)_{k\in\mathbb{Z}}$ and $(F_k)_{k\in\mathbb{Z}}$ are M-bounded and therefore, by **Theorem 3.1** are an $B^s_{p,q}$-multiplier.

Theorem 3.3 Let $1 \leq p, q < \infty$ and $s > 0$. Let X be a Banach space. Suppose that $(ik - AD_k - G_k - \tilde{a}(ik))$ is invertible for all $k \in \mathbb{Z}$.

If $\sup_k \left\| ik \left(ikD_k - AD_k - G_k - \tilde{a}(ik) \right)^{-1} \right\| < \infty$ and $\sup_k \left\| \left(ikD_k - AD_k - G_k - \tilde{a}(ik) \right)^{-1} \right\| < \infty$ then for every $f \in B^s_{p,q}(\mathbb{T}, X)$ there exist a unique strong $B^s_{p,q}$-solution of (**1**).

Proof. Define $S_k = ikN_k$, $N_k = \left(ikD_k - AD_k - G_k - \tilde{a}(ik) \right)^{-1}$, $F_k = \tilde{a}(ik) N_k$ and $T_k = G_k N_k$ for $k \in \mathbb{Z}$. Since by (7) and (8), $(S_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}}, (F_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are M-bounded, we have by Theorem 3.1 that $(S_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}}, (F_k)_{k\in\mathbb{Z}}$ and $(T_k)_{k\in\mathbb{Z}}$ are an $B^s_{p,q}$-multipliers. Since $D_k S_k - AD_k N_k - T_k - \tilde{a}(ik) N_k = I$ (because ($\left(ikD_k - AD_k - G_k - F_k \right) N_k = I$), we deduce $AD_k N_k$ is also an $B^s_{p,q}$-multiplicateur.

Now let $f \in B^s_{p,q}(\mathbb{T}, X)$. Then there exist $u, v, w, q, x \in B^s_{p,q}(\mathbb{T}, X)$, such that $\hat{u}(k) = N_k \hat{f}(k)$, $\hat{v}(k) = D_k S_k \hat{f}(k)$, $\hat{w}(k) = T_k \hat{f}(k)$, $\hat{x}(k) = F_k \hat{f}(k)$ and $\hat{q}(k) = AD_k N_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. So, We have $\left(\hat{u}(k) - L_k \hat{u}(k) \right) \in D(A)$ and $A\left(\hat{u}(k) - L_k \hat{u}(k) \right) = \hat{q}(k)$ for all $k \in \mathbb{Z}$, we deduce that $\left(u(t) - L(u_t) \right) \in D(A)$. On the other hand $\exists v \in B^s_{p,q}(\mathbb{T}, X)$ such that $\hat{v}(k) = D_k S_k \hat{f}(k) = ikD_k N_k \hat{f}(k) = ikD_k \hat{u}(k)$. By **Lemma 2.2** we obtain $(Du_t)' = v(t)$ a.e. Since $Du_t \in B^{s+1}_{p,q}(\mathbb{T}, X)$.

We have $\widehat{(Du_t)'}(k) = ikD_k \hat{u}(k)$, $\left(A\left(u(.) - L(u_.) \right) \right)^\wedge (k) = AD_k \hat{u}(k)$, $\widehat{Gu_.}(k) = G_k \hat{u}(k)$ and

$\widehat{\int_{-\infty}^t a(t-s) u(s) \mathrm{d}s}(k) = \tilde{a}(ik) \hat{u}(k)$ for all $k \in \mathbb{Z}$, It follows from the identity

$$ikD_k N_k - AD_k N_k - G_k N_k - \tilde{a}(ik) N_k = I$$

that:

$$\left(u(t) - L(u_t) \right)' = A\left(u(t) - L(u_t) \right) + G(u_t) + \int_{-\infty}^t a(t-s) u(s) \mathrm{d}s + f(t)$$

For the uniqueness we suppose two solutions u_1 and u_2, then $u = u_1 - u_2$ is strong L^p-solution of equation (1) corresponding to the function $f = 0$, taking Fourier transform, we get $\left(ikD_k - AD_k - G_k - \tilde{a}(ik) \right) \hat{u}(k) = 0$, which implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and $u(t) = 0$. Then $u_1 = u_2$.

On the other hand, we have $u \in H^{1,p}(\mathbb{T}; X)$ and by Remark 3.1 we deduce $Mx(0) = Mx(2\pi)$. The proof completed.

4. Conclusion

We are obtained necessary and sufficient conditions to guarantee the existence and uniqueness of periodic solutions to the equation $\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \int_{-\infty}^{t} a(t-s)x(s)\mathrm{d}s + f(t)$ in terms of either the R-boundedness of the modified resolvent operator determined by the equation. Our results are obtained in the vector-valued space and Besov space.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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