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Continuity of the Solution Mappings for Parametric Generalized Strong Vector Equilibrium Problems

Xianzheng Dong, Chi Zhang, Lizhi Zhang

Jiangxi V & T College of Communications, Nanchang, China
Email: jijian@jxjtxy.edu.cn, 504775867@qq.com, 1065064716@qq.com

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Abstract

The stability analysis of the solution mappings for vector equilibrium problems is an important topic in optimization theory and its applications. In this paper, we focus on the continuity of the solution mapping for a parametric generalized strong vector equilibrium problem. By virtue of a nonlinear scalarization technique, a new density result of the solution mapping is obtained. Based on the density result, we give sufficient conditions for the lower semicontinuity and the Hausdorff upper semicontinuity of the solution mapping to the parametric generalized strong vector equilibrium problem. In addition, some examples were given to illustrate that our results improve ones in the literature.

Keywords

Parametric Generalized Strong Vector Equilibrium Problem, Lower Semicontinuity, Hausdorff Upper Semicontinuity, Nonlinear Scalarization

1. Introduction

The vector equilibrium problem is an interesting and important model in applied mathematics. It is a unified model, includes vector optimization problem, vector variational inequality, the vector complementarity problem and vector saddle point problem. Over the past two decades, this problem has received an increasing interest from many researchers. In the literature, existence and connectedness results for various types of vector equilibrium problems have been investigated intensively, e.g., [1]-[7] and the references therein.

The stability analysis of the solution mappings for vector equilibrium problems is another important topic in optimization theory and has recently received increasing attention from mathematicians with various approaches. The stability of solutions can be understood as lower or upper semicontinuity, continuity in the sense of Berge or Hausdorff, and Lipschitz/Hölder continuity of solution maps. There have been a large number of contributions to the semicontinuity, especially the lower semicontinuity, of solution maps to parametric vector equilibrium problems in the literature, such as [8]-[18]. In addition, many results on the Hölder/Lipschitz continuity of the solution maps are archived, e.g., [19] [20] [21] [22] and the references therein.

We observe that the scalarization technique is one of effective approaches to deal with the lower semicontinuity and the upper semicontinuity of solution mappings to parametric vector equilibrium problems. It is worth noting that the linear scalarization approach to the semicontinuity of solution mappings to parametric generalized vector equilibrium problems always requires (generalized) cone-convexity or strict cone-monotonicity of the objective functions, even the assumptions involve the information about the solution set. To avoid using these assumptions, nonlinear scalarization approaches have been applied for discussing the stability analysis in parametric generalized weak vector equilibrium problems. Recently, Sach [23] [24] has used some nonlinear scalarization functions (generalized versions of Gerstewizt's function) to investigative the semicontinuity of the solution mappings of parametric generalized weak and non-weak vector equilibrium problems, which are also called parametric generalized Ky Fan inequalities. However, to the best of our knowledge, there are few results in the literature on the continuity of the solution mappings to parametric strong vector equilibrium problems. In particular, so far, there is no work with contribution to the continuity of the solution mappings of parametric strong vector equilibrium problems by using nonlinear scalarization methods.

Hence, motivated by the works reported in [12] [14] [15] [16] [17] [23] [24], this paper aims to explore an application of the oriented distance function defined in [25] to discuss the lower semicontinuity and the Hausdorff continuity of the solution mapping of a parametric generalized strong vector equilibrium problem. To this end, we firstly establish a density theorem concerned with the solution set to parametric generalized vector equilibrium problem and the solution set of parametric generalized strong vector equilibrium problem by using the nonlinear scalarization method. Then by the density result, the lower semicontinuity and the Hausdorff upper semicontinuity of the solution mapping to the parametric generalized strong vector equilibrium problem are given. Our methods as well as results are different from ones in [23] [24], since the models discussed in [23] [24] are not parametric generalized strong vector equilibrium problems. Furthermore, some examples are given to illustrate that our results improve the corresponding ones in [12] [14] [15] [16] [17].

The outline of the paper is as follows. In Section 2, we introduce the parametric generalized strong vector equilibrium problem, and recall some basic concepts and their properties. In Section 3, based on the nonlinear scalarization method, we obtain a new density result. We also discuss the lower semicontinuity of the solution mapping to the parametric generalized strong vector equilibrium problem by means of the density result. In Section 4, we give the upper semicontinuity of the solution mapping to the parametric generalized strong vector equilibrium problem. In Section 5, we give the conclusions of the paper.

2. Preliminaries

Throughout this paper, without special statements, let *X*, *Y* and *Z* be normed vector spaces. We denote by $\mathrm{cl}A$, A^c, $\mathrm{ri}A$, $\mathrm{int}A$ and ∂A the closure, the complement, the relative interior, the interior and the boundary of a set $A \subseteq Y$, respectively. We also assume that C be a pointed closed convex cone in Y with nonempty interior. Let Y^* be the topological dual space of Y and let $C^* := \{ f \in Y^* : f(x) \geq 0, \forall x \in C \}$.

Let Λ be a subset of *Z*. Let $A : \Lambda \rightrightarrows X$ be a set-valued mapping. Let $F : X \times X \rightrightarrows Y$ be a set-valued mapping. In this paper, we consider the following parametric generalized strong vector equilibrium problem:

(PGSVEP) find $x \in A(\mu)$ such that $F(x, y, \mu) \cap (-C \setminus \{0_Y\}) = \emptyset, \forall y \in A(\mu)$.

For each $\mu \in \Lambda$, let $S(\mu)$ denote the solution set of (PGSVEP), *i.e.*,

$$S(\mu) = \{ x \in A(\mu) : F(x, y, \mu) \cap (-C \setminus \{0_Y\}) = \emptyset, \forall y \in A(\mu) \}.$$

For each $\mu \in \Lambda$, let $S_w(\mu)$ denote the solution set of the following parametric generalized weak vector equilibrium problem, *i.e.*,

$$S_w(\mu) = \{ x \in A(\mu) : F(x, y, \mu) \subset Y \setminus -\mathrm{int}\, C, \forall y \in A(\mu) \}.$$

For each $\mu \in \Lambda$, let $\tilde{S}(\mu)$ denote the solution set of the following parametric generalized vector equilibrium problem, *i.e.*,

$$\tilde{S}(\mu) = \{ x \in A(\mu) : F(x, y, \mu) \subset Y \setminus -C, \forall y \in A(\mu) \}.$$

For each $f \in C^* \setminus \{0\}$ and for each $\mu \in \Lambda$, let $V_f(\mu)$ denote the set of *f*-efficient solutions to (PGSVEP), *i.e.*,

$$V_f(\mu) = \left\{ x \in A(\mu) : \inf_{z \in F(x,y,\mu)} f(z) \geq 0, \forall y \in A(\mu) \right\}.$$

It is easy to observe that $\tilde{S}(\mu) \subseteq S(\mu) \subseteq S_w(\mu)$. Throughout this paper, we assume that $\tilde{S}(\mu) \neq \emptyset$ for all μ in Λ. In this paper, by using the nonlinear scalarization method, we will discuss the lower semicontinuity and the Hausdorff upper semicontinuity of the solution mapping $S(\cdot)$ as a set-valued mapping from the set Λ to *X*. Now we recall some basic definitions and their properties.

Definition 2.1. [26] A set-valued mapping $\Phi : X \rightrightarrows Y$ is said to be properly quasiconcave on X If for any $x_1, x_2 \in X$ and for any $\alpha \in [0,1]$, one has

either $\Phi(\alpha x_1 + (1-\alpha) x_2) \subseteq \Phi(x_1) + C$ or $\Phi(\alpha x_1 + (1-\alpha) x_2) \subseteq \Phi(x_2) + C$.

Definition 2.2. [27] [28] Let T_1 and T_2 be two topological vector spaces. A set-valued mapping $G : T_1 \rightrightarrows T_2$ is said to be

1) Lower semicontinuous (l.s.c.) at $\bar{t} \in T_1$ iff, for every open set $V \subseteq T_2$ with $G(\bar{t}) \cap V \neq \varnothing$, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$, for every $t \in N(\bar{t})$, $G(t) \cap V \neq \varnothing$.

2) Upper semicontinuous (u.s.c.) at $\bar{t} \in T_1$ iff, for every open set $V \subseteq T_2$ with $G(\bar{t}) \subseteq V$, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$, for every $t \in N(\bar{t})$, $G(t) \subseteq V$.

3) Hausdorff upper semicontinuous (H-u.s.c.) at $\bar{t} \in T_1$ iff, for each neighbourhood U of 0_{T_1}, there is a neighbourhood $N(\bar{t})$ of $\bar{t}$ such that for any $t \in N(\bar{t})$, $G(t) \subseteq G(\bar{t}) + U$.

4) G is called closed at $\bar{x}$ iff for each sequence $(x_\alpha, y_\alpha) \in \operatorname{graph} G := \{(x, y) : y \in G(x)\}$, $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y})$, it follows that $(\bar{x}, \bar{y}) \in \operatorname{graph} G$.

We say that G is l.s.c. (resp. u.s.c.) on T_1, if it is l.s.c. (resp. u.s.c.) at each $t \in T_1$. G is said to be continuous on T_1 if it is both l.s.c. and u.s.c. on T_1.

In the following Proposition 2.1, let T_1 and T_2 be two normed vector spaces.

Proposition 2.1. [28] [29]

1) G is l.s.c. at $\bar{t}$ if and only if for any sequence $\{t_n\} \subset T_1$ with $t_n \to \bar{t}$ and for any $\bar{x} \in G(\bar{t})$, there exists $x_n \in G(t_n)$ such that $x_n \to \bar{x}$.

2) If G has compact values at $\bar{t}$, then G is u.s.c. at $\bar{t}$ if and only if for any sequence $\{t_n\} \subset T_1$ with $t_n \to \bar{t}$ and any $x_n \in G(t_n)$, there exist $\bar{x} \in G(\bar{t})$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$.

Definition 2.3. [25] For a set $A \subseteq Y$, the oriented distance function $\Delta_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\Delta_A(y) := d_A(y) - d_{Y \setminus A}(y),$$

where $d_A(y) := \inf_{z \in A} \|y - z\|$, $d_\varnothing(y) := +\infty$, and $\|y\|$ denotes the norm of y in Y.

Proposition 2.2. (See Proposition 3.2 in [30]) *If the set A is nonempty and $A \neq Y$. Then*

1) Δ_A is real valued;

2) Δ_A is 1-Lipschitzian;

3) $\Delta_A(y) < 0 \iff y \in \operatorname{int} A$;

4) $\Delta_A(y) = 0 \iff y \in \partial A$;

5) $\Delta_A(y) > 0 \iff y \in \operatorname{int} A^c$;

6) If A is a closed and convex cone, then Δ_A is nonincreasing with respect to the ordering relation induced on Y, *i.e.*, the following is true: if $y_1, y_2 \in Y$, then

$$y_1 - y_2 \in A \Rightarrow \Delta_A(y_1) \leq \Delta_A(y_2);$$

if A has a nonempty interior, then

$$y_1 - y_2 \in \operatorname{int} A \Rightarrow \Delta_A(y_1) < \Delta_A(y_2).$$

3. Lower Semicontinuity

In this section, we discuss the lower semicontinuity of the solution mapping

$S(\cdot)$ to (PGSVEP).

Firstly, we define the function $\zeta : X \times \Lambda \to \mathbb{R} \cup \{\pm\infty\}$ as follows.

$$\zeta(x,\mu) = \inf_{y \in A(\mu)} \inf_{z \in F(x,y,\mu)} \Delta_{-C}(z). \tag{1}$$

Proposition 3.1. *Assume that the following conditions hold.*

1) $A(\cdot)$ is continuous with nonempty compact values on Λ;

2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X \times X \times \Lambda$.

Then, $\zeta(\cdot,\cdot)$ is continuous on $X \times \Lambda$.

Proof. We define $\psi : X \times X \times \Lambda \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\psi(x,y,\mu) = \inf_{z \in F(x,y,\mu)} \Delta_{-C}(z) = -\sup_{z \in F(x,y,\mu)} -\Delta_{-C}(z).$$

Then

$$\zeta(x,\mu) = \inf_{y \in A(\mu)} \psi(x,y,\mu). \tag{2}$$

It follows from assumption (2), the continuity of $\Delta_{-C}(\cdot)$, Proposition 19 and Proposition 21 in [28] that $\psi(\cdot,\cdot,\cdot)$ is continuous on $X \times X \times \Lambda$. Furthermore, combining this with the assumption (1), (2), Proposition 19 and Proposition 21 in [28], we see that $\zeta(\cdot,\cdot)$ is continuous on $X \times \Lambda$. □

Remark 3.1. It is worth of being observed that if $A(\cdot)$ is l.s.c. on Λ and $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X \times X \times \Lambda$, then we can get that $\zeta(\cdot,\cdot)$ is lower semicontinuous on $X \times \Lambda$ by Proposition 19 in [28].

Proposition 3.2. *Assume that* $F(\cdot,y,\mu)$ *is properly C-quasiconcave on X for each* $y \in X$ *and for each* $\mu \in \Lambda$. *Then* $\zeta(\cdot,\mu)$ *is quasiconcave on X for each* $\mu \in \Lambda$.

Proof. For any given $y \in X$ and $\mu \in \Lambda$, since $F(\cdot,y,\mu)$ is properly *C*-quasiconcave on Λ, for any $x_1, x_2 \in X$, we have either

$$F(\alpha x_1 + (1-\alpha)x_2, y, \mu) \subseteq F(x_1,y,\mu) + C, \quad \forall x_1, x_2 \in X,$$

or

$$F(\alpha x_1 + (1-\alpha)x_2, y, \mu) \subseteq F(x_2,y,\mu) + C, \quad \forall x_1, x_2 \in X.$$

Thus, in terms of Proposition 2.2 (vi), for any $u \in F(\alpha x_1 + (1-\alpha)x_2, y, \mu)$, one has

$$\text{either } \Delta_{-C}(u) \geq \inf_{z \in F(x_1,y,\mu)} \Delta_{-C}(z) \text{ or } \Delta_{-C}(u) \geq \inf_{z \in F(x_2,y,\mu)} \Delta_{-C}(z)$$

This together with the arbitrariness of $u \in F(\alpha x_1 + (1-\alpha)x_2, y, \mu)$ shows that

$$\text{either } \inf_{z \in F(\alpha x_1 + (1-\alpha)x_2, y, \mu)} \Delta_{-C}(z) \geq \inf_{z \in F(x_1,y,\mu)} \Delta_{-C}(z), \quad \forall x_1, x_2 \in X,$$

or

$$\inf_{z \in F(\alpha x_1 + (1-\alpha)x_2, y, \mu)} \Delta_{-C}(z) \geq \inf_{z \in F(x_2,y,\mu)} \Delta_{-C}(z), \quad \forall x_1, x_2 \in X.$$

For any given $x_1, x_2 \in X$, the above inequalities give us that either

$$\zeta(\alpha x_1+(1-\alpha)x_2,\mu)=\inf_{y\in A(\mu)}\inf_{z\in F(\alpha x_1+(1-\alpha)x_2,y,\mu)}\Delta_{-C}(z)$$
$$\geq \inf_{y\in A(\mu)}\inf_{z\in F(x_1,y,\mu)}\Delta_{-C}(z)=\zeta(x_1,\mu)$$

or

$$\zeta(\alpha x_1+(1-\alpha)x_2,\mu)=\inf_{y\in A(\mu)}\inf_{z\in F(\alpha x_1+(1-\alpha)x_2,y,\mu)}\Delta_{-C}(z)$$
$$\geq \inf_{y\in A(\mu)}\inf_{z\in F(x_2,y,\mu)}\Delta_{-C}(z)=\zeta(x_2,\mu),\ \forall\mu\in\Lambda.$$

Therefore, $\zeta(\cdot,\mu)$ is quasiconcave on X for each $\mu\in\Lambda$. □

Lemma 3.1. *Assume that the following conditions are satisfied.*

1) $A(\cdot)$ is l.s.c. on Λ;

2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X\times X\times\Lambda$.

Then $\tilde{S}(\cdot)$ is l.s.c. on Λ.

Proof. By Proposition 2.2 (v), it is not hard to see that

$$\tilde{S}(\mu):=\{x\in A(\mu):\zeta(x,\mu)>0\}.$$

To prove the result by contradiction, suppose that there exists $\mu_0\in\Lambda$ such that $\tilde{S}(\cdot)$ is not l.s.c. at μ_0. Then by Proposition 2.1 (1), there exist a sequence $\{\mu_n\}$ with $\mu_n\to\mu_0$ and $x_0\in\tilde{S}(\mu_0)$ such that for any $x_n\in\tilde{S}(\mu_n)$, we have $x_n\nrightarrow x_0$.

From $x_0\in\tilde{S}(\mu_0)$, we have $x_0\in A(\mu_0)$. As $A(\cdot)$ is l.s.c. at μ_0, there exists $\bar{x}_n\in A(\mu_n)$ such that $\bar{x}_n\to x_0$. By the above contradiction assumption, there must exist subsequence $\{\bar{x}_{n_k}\}\subset\{\bar{x}_n\}$ such that, $\forall k$ with $\bar{x}_{n_k}\notin\tilde{S}(\mu_{n_k})$, *i.e.*,

$$\zeta(\bar{x}_{n_k},\mu_{n_k})\leq 0. \tag{3}$$

It follows from Remark 3.1 that ζ is lower semicontinuous at (x_0,μ_0). This together with (3) implies that

$$\zeta(x_0,\mu_0)\leq\liminf_{\beta\to+\infty}\zeta(x_\beta,\mu_\beta)\leq 0,$$

which contradicts $x_0\in\tilde{S}(\mu_0)$. Thus $\tilde{S}(\cdot)$ is l.s.c. on Λ. □

Lemma 3.2. (See Theorem 1.1.2 in [31]) *Let* $A\subseteq Y$ *be a convex set. If* $\operatorname{int}A\neq\varnothing$, *then* $\operatorname{cl}(\operatorname{int}A)=\operatorname{cl}A$.

Lemma 3.3. *For each* $\mu\in\Lambda$, *suppose the following conditions are satisfied.*

1) $A(\cdot)$ is continuous with nonempty compact values on Λ;

2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X\times X\times\Lambda$;

3) $F(\cdot,y,\mu)$ is properly C-quasiconcave on X for each $y\in X$.

Then, we have

$$\tilde{S}(\mu)\subseteq S(\mu)\subseteq S_w(\mu)\subseteq\operatorname{cl}\tilde{S}(\mu).$$

Proof. Taking into consideration that $\tilde{S}(\mu)\subseteq S(\mu)\subseteq S_w(\mu)$, we only need to show that

$$S_w(\mu)\subseteq\operatorname{cl}\tilde{S}(\mu),\ \forall\mu\in\Lambda. \tag{4}$$

In fact, it follows from Proposition 2.2 (3) that $S_w(\mu)=\{x\in A(\mu): f(x,\mu)\geq 0,\forall y\in A(\mu)\}$ for each $\mu\in\Lambda$. In addition, by means of the assumption (2) and Proposition 3.1, we know that $f(\cdot,\cdot)$ is continuous on $X\times\Lambda$. Hence, it is easy to see that $S_w(\mu)$ is closed for each $\mu\in\Lambda$.

Next, we need to prove that the set $S_w(\mu)$ is convex for each $\mu\in\Lambda$. Indeed, it follows from the assumption (3) and Proposition 3.2 that $\zeta(\cdot,\mu)$ is properly quasi-concave on X for each $\mu\in\Lambda$. Hence, for any $x_1,x_2\in S_w(\mu)$, and for any $\alpha\in[0,1]$, we have either

$$\zeta(\alpha x_1+(1-\alpha)x_2,\mu)\geq\zeta(x_1,\mu)\geq 0,\ \ \forall\mu\in\Lambda.$$

or

$$\zeta(\alpha x_1+(1-\alpha)x_2,\mu)\geq\zeta(x_2,\mu)\geq 0,\ \ \forall\mu\in\Lambda.$$

This shows that for each $\mu\in\Lambda$, $\alpha x_1+(1-\alpha)x_2\in S_w(\mu)$ for any $\alpha\in[0,1]$ and so $S_w(\mu)$ is convex for each $\mu\in\Lambda$. Taking into account $\operatorname{int}S_w(\mu)=\tilde{S}(\mu)\neq\varnothing$ for each $\mu\in\Lambda$ and with the help of Lemma 3.2, we see that (4) is valid. □

Remark 3.2. It is worth mentioning that Lemma 3.2 shows that the solution set $\tilde{S}(\mu)$ is dense in the solution set $S(\mu)$ for each $\mu\in\Lambda$. The density result is obtained with the help of nonlinear scalarization method, which is better than ones derived in [12] [14] [15] [16] [17] by using linear scalarization methods.

Theorem 3.1. *Suppose the following conditions are satisfied*:

1) $A(\cdot)$ is continuous with nonempty compact values on Λ;
2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X\times X\times\Lambda$;
3) $F(\cdot,y,\mu)$ is properly *C*-quasiconcave on *X* for each $y\in X$.

Then $S(\cdot)$ is l.s.c. at Λ.

Proof. By virtue of Lemma 3.3, we have

$$\tilde{S}(\mu)\subset S(\mu)\subset\operatorname{cl}\tilde{S}(\mu).$$

For any given $\mu_0\in\Lambda$, we claim that $S(\cdot)$ is l.s.c. at μ_0. Indeed, for any $x\in S(\mu_0)$ and any neighborhood $x+U(0_X)$ of *x*, where $U(0_X)$ is a neighborhood of 0_X in *X*. Since

$$x\in S(\mu_0)\subset\operatorname{cl}\tilde{S}(\mu_0),$$

we have

$$(x+U(0_X))\cap\tilde{S}(\mu_0)\neq\varnothing.$$

By Lemma 3.1, we have $\tilde{S}(\cdot)$ is l.s.c at μ_0. For the above $U(0_X)$, there exists a neighborhood $U(\mu_0)$ of μ_0 such that

$$\tilde{S}(\mu)\cap(x+U(0_X))\neq\varnothing,\ \ \forall\mu\in U(\mu_0).$$

Taking into account $\tilde{S}(\mu)\subset S(\mu)$ for each $\mu\in\Lambda$, we arrive at

$$S(\mu)\cap(x+U(0_X))\neq\varnothing,\ \ \forall\mu\in U(\mu_0).$$

This states that $S(\cdot)$ is l.s.c. at μ_0. □

The following example illustrates that Theorem 3.1 can be applied, while the corresponding lower semicontinuity results in [12] [14] [15] [16] [17] are not applicable.

Example 3.1. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $\Lambda = [0,1]$. Let $A(\mu) = [0,1]$ for each $\mu \in \Lambda$ and

$$F(x,y,\mu) = \begin{cases} \{y-x-2\} \times [1+x,2], & \text{if } \mu = 0, \\ [-4, x+1] \times [1+x+\mu, +\infty), & \text{if } \mu \in (0,1]. \end{cases}$$

It follows form a direct computation that $\zeta(x,\mu) = 1+x+\mu$ and so $S(\mu) = \tilde{S}(\mu) = [0,1]$ for all $\mu \in [0,1]$. It is not hard to check that the assumptions (1)-(3) in Theorem 3.1 are satisfied. However, $F(\cdot,\cdot,\mu_0)$ ($\mu_0 = 0$) is not strictly C-monotone on $X \times X$, that is, the assumption (2) in Theorem 2.1 of [12] and the assumption (3) in Theorem 3.2 of [14] are violated at μ_0. Indeed, there exist $x = 0$ and $y = 1$ in $A(\mu_0) = [0,1]$ such that

$$F(x,y,\mu_0) + F(y,x,\mu_0) \subseteq Y \setminus -\operatorname{int} C.$$

The assumption (5) in Theorem 3.1 of [15] is not valid at $\mu_0 = 0$ for $f_0 = \left(1, \frac{5}{3}\right) \in C^* \setminus \left\{0_{\mathbb{R}^2_+}\right\}$. Indeed, for each $x \in A(\mu_0) \setminus V_{f_0}(\mu_0) = \left(\frac{1}{2}, 1\right]$, $y \in V_{f_0}(\mu_0) = \left[0, \frac{1}{2}\right]$, we have

$$F(x,y,\mu_0) + F(y,x,\mu_0) + B(0, d(x,y)) \not\subseteq -C.$$

The assumption (4) (*i.e.*, *f*-property) in Theorem 3.1 of [16] is also violated at $\mu_0 = 0$ for $f_0 = \left(1, \frac{5}{3}\right)$. Indeed, for any $x, y \in [0,1]$ and $\{\mu_n\}$ with $\mu_n \to \mu_0$, we have $\inf_{z \in F(x,y,\mu_0)} f_0(z) = y + \frac{2}{3}x - \frac{1}{3} \geq 0$, but $\inf_{z \in F(x,y,\mu_n)} f_0(z) = -4 + 1 + x + \mu_n < 0$, $\forall n$.

The assumption (5) in Theorem 3.1 of [17] does not hold at $\mu_0 = 0$ for $\tilde{f} = \left(1, \frac{3}{2}\right)$. Indeed, $\inf_{z \in F(x,y,\mu_0)} \tilde{f}(z) = y + \frac{1}{2}x - \frac{1}{2} = 0 \nRightarrow y = x$.

As a consequence, Theorem 2.1 of [12], Theorem 3.2 of [14], Theorem 3.1 of [15] [16] [17] are not applicable.

4. Hausdorff Upper Semicontinuity

In the section, we give the Hausdorff upper semicontinuity of the solution mapping $S(\cdot)$ to (PGSVEP).

Lemma 4.1. *Assume that*

1) $A(\cdot)$ is continuous with nonempty compact values on Λ;

2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X \times X \times \Lambda$.

Then $S_w(\cdot)$ is u.s.c. with compact values on Λ.

Proof. By Proposition 2.2 (3), it is not hard to see that

$$S_w(\mu) := \{x \in A(\mu) : \zeta(x,\mu) \geq 0\}, \tag{5}$$

where ζ is given in (1). Suppose to the contrary that there exists μ_0 such that $S_w(\cdot)$ is not u.s.c. at μ_0. Then there exist an open set W_0 with $S_w(\mu_0)\subseteq W_0$ and a sequence $\{\mu_n\}$ with $\mu_n \to \mu_0$ such that $S_w(\mu_n)\not\subseteq W_0, \forall n\in\mathbb{N}$. This implies that there is $x_n \in S_w(\mu_n)$ such that

$$x_n \notin W_0, \quad \forall n \in \mathbb{N}. \tag{6}$$

As $A(\cdot)$ is u.s.c. with compact values at μ_0 and $x_n \in A(\mu_n)$, by Proposition 2.1 (2), there exist $x_0 \in A(\mu_0)$ and a subsequence $\{x_{n_k}\}$ of x_n such that $x_{n_k} \to x_0$. Without loss of generality, we assume that $x_n \to x_0$.

Now, we claim that $x_0 \in S_w(\mu_0)$. Indeed, suppose that $x_0 \notin S_w(\mu_0)$. Then it follows from (5) that

$$\zeta(x_0,\mu_0)<0. \tag{7}$$

With the help of Proposition 3.1, we see that $f(\cdot,\cdot)$ is continuous on $X\times\Lambda$. As a result, it follows from (7) that

$$\zeta(x_n,\mu_n)<0, \text{ for } n \text{ large enough},$$

which contradicts $x_n \in S_w(\mu_n)$ because of (5). Thus, $x_0 \in S_w(\mu_0)$. This together with $x_n \to x_0$ and $S_w(\mu_0)\subseteq W_0$ shows that $x_n \in W_0$ for n large enough, which contradicts (6). Therefore, $S_w(\cdot)$ is u.s.c. on Λ.

Next, we show that $S_w(\mu)$ is compact for each $\mu\in\Lambda$. Indeed, by (5) and Proposition 3.1, we can see that $S_w(\mu)$ is closed. This together with the compactness of $A(\mu)$ and $S_w(\mu)\subseteq A(\mu)$ for each $\mu\in\Lambda$ gives us desired result. □

Theorem 4.1. *Assume that the following conditions hold.*

1) $A(\cdot)$ is continuous with nonempty compact values on Λ;

2) $F(\cdot,\cdot,\cdot)$ is continuous with nonempty compact values on $X\times X\times\Lambda$;

3) $F(\cdot,y,\mu)$ is properly *C*-quasiconcave on *X* for each $y\in X$.

Then $S(\cdot)$ is H-u.s.c. on Λ.

Proof. To prove the result by contradiction, suppose that there exists $\mu_0\in\Lambda$ such that $S(\cdot)$ is not H-u.s.c. at μ_0. Then there exists an open set W_0 of 0_X such that for any neighborhood U_0 of μ_0 and there exists $\mu'\in U_0$ with $S(\mu')\not\subseteq S(\mu_0)+W_0$. Hence, there exists a sequence $\{\mu_n\}$ with $\mu_n\to\mu_0$ such that $S(\mu_n)\not\subseteq S(\mu_0)+W_0, \forall n\in\mathbb{N}$. This implies that there is $x_n\in S(\mu_n)$ such that

$$x_n \notin S(\mu_0)+W_0, \quad \forall n\in\mathbb{N}. \tag{8}$$

Taking into consideration that $x_n\in S(\mu_n)$ and $S(\mu_n)\subseteq S_w(\mu_n)$, we have $x_n\in S_w(\mu_n), \ \forall n\in\mathbb{N}$. Due to Lemma 4.1, we know that $S_w(\cdot)$ is u.s.c. with compact values at μ_0. Then by Proposition 2.1 (2), there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x_0\in S_w(\mu_0)$ such that $x_{n_k}\to x_0$. It follows from the closedness of $S_w(\mu_0)$ and Lemma 3.3 that

$$\mathrm{cl}\, S(\mu_0)=S_w(\mu_0).$$

Consequently,

$$x_0\in \mathrm{cl}\, S(\mu_0)\subseteq S(\mu_0)+W_0.$$

This together with $x_{n_k} \to x_0$ indicates that $x_{n_k} \in S(\mu_0) + W_0$ for k large enough, which contradicts (8). Therefore, $S(\cdot)$ is H-u.s.c. at μ_0. □

The example is given to show that Theorem 4.1 is applicable, but Theorem 4.1 of [15] and Theorem 4.1 of [17] are not applicable.

Example 4.1. Let $X = Z = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_+^2$ and $\Lambda = [0,1]$. Let $A(\mu) = [0,1]$ and

$$F(x, y, \mu) = \{y - x - 1\} \times \left[1 + \frac{x}{2} + \mu, 3\right].$$

It follows form a direct computation that $\zeta(x, \mu) = 1 + \frac{x}{2} + \mu$ and so $S(\mu) = [0,1]$ for all $\mu \in [0,1]$. Hence $S(\cdot)$ is H-u.s.c. on Λ. It is not hard to see that the assumptions (1)-(3) in Theorem 4.1 are satisfied. However, the assumption (5) in Theorem 4.1 of [15] is violated at $\mu_0 = 0$ for $\bar{f} = (1,1) \in C^* \setminus \{0_{\mathbb{R}_+^2}\}$. Indeed, for each $x \in A(\mu_0) \setminus V_{\bar{f}}(\mu_0) = (0,1]$, $y \in V_{\bar{f}}(\mu_0) = \{0\}$, we have

$$F(x, y, \mu_0) + F(y, x, \mu_0) + B(0, d(x, y)) \not\subseteq -C.$$

The assumption (5) in Theorem 4.1 of [17] is also violated at $\mu_0 = 0$ for $\bar{f} = (1,1)$. Indeed, $\inf_{z \in F(x,y,\mu_0)} \bar{f}(z) = y - x - 1 + 1 + \frac{1}{2}x = y - \frac{1}{2}x = 0$, $\not\Rightarrow$ $y = x$. Therefore, Theorem 4.1 of [15] and Theorem 4.1 of [17] are not applicable.

5. Conclusions

In this paper, we established a density result in regard to the solution set to parametric generalized vector equilibrium problem and the solution set of parametric generalized strong vector equilibrium problem by using the nonlinear scalarization method. Then by using the density result, we obtained the lower semicontinuity and the Hausdorff upper semicontinuity of the solution mapping to the parametric generalized strong vector equilibrium problem. Additionally, some examples were given to illustrate that our results improve ones in [12] [14] [15] [16] [17].

The multi-criteria traffic network equilibrium model is used to evaluate the traffic flow pattern and the travel costs, and it has played an important role in the traffic network programming and the traffic control. The topic has received increasing interest from many researchers, e.g., [32] [33] [34] [35]. It is worth noting that the multi-criteria traffic network equilibrium model can be shifted to a vector equilibrium problem. Therefore, it would be interesting to discuss the stability of the multi-criteria traffic network equilibrium model.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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