

On M -Asymmetric Semi-Open Sets and Semicontinuous Multifunctions in Bitopological Spaces

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How to cite this paper: Matindih, L.K. and Moyo, E. (2021) On M -Asymmetric Semi-Open Sets and Semicontinuous Multifunctions in Bitopological Spaces. *Advances in Pure Mathematics*, 11, 218-236.
<https://doi.org/10.4236/apm.2021.114016>

Received: January 10, 2021

Accepted: April 5, 2021

Published: April 8, 2021

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Abstract

The purpose of this paper is to introduce the notions of m -asymmetric semiopen sets and M -asymmetric semicontinuous multifunctions defined between asymmetric sets satisfying certain minimal conditions in the framework of bitopological spaces. Some new characterizations of m -asymmetric semiopen sets and M -asymmetric semicontinuous multifunctions will be investigated and several fundamental properties will be obtained.

Keywords

m -Spaces, m -Semiasymmetric Open and Closed Sets, Upper and Lower M -Asymmetric Semicontinuous Multifunction

1. Introduction

The concept of topology is an important tool that has received considerable attention from many scholars in many fields of applied sciences and other branches of pure mathematics. Continuity and multifunctions on the other hand, which are basic notions in the theory of classical point set topology also play a vital role not only in the field of pure mathematics but also in applied mathematics. In this regard, several scholars have generalizations the notion of continuity in topological spaces using the weaker forms of open and closed sets the semiopen and semiclosed sets [1] [2] [3].

The concept of semiopen sets and semi-continuity of function first appeared in a paper by Levine [1] in the realm of topological spaces and this idea was then generalized and extended to the frameworks of bitopological spaces by Maheshwari and Prasad [2], and also by Bose [4]. In 1963, Berge [5] introduced the concepts of upper and lower continuous multifunctions and lately, Whyburn [6]

also studied the general continuity of multifunctions in the realm of topological spaces. Smithson [7] and Popa [8] respectively then generalized these notions of multifunction continuity to the setting of bitopological spaces. In 2000, Popa and Noiri [9], introduced and studied the concepts of m -structures and M -continuous function as a function defined between topologies satisfying certain minimal conditions. They showed that the M -continuous functions have properties similar to those of continuous functions between topological spaces. With these ideas of multifunction continuity and M -continuous functions, Noiri and Popa [10] then introduced and studied upper and lower M -continuous multifunctions and showed how these functions have properties similar to those of upper and lower continuous functions and continuous multifunctions between topological spaces.

In this present paper, we introduce and investigate some basic characterizations and properties of m -asymmetric semiopen sets, m -asymmetric semiclosed sets and, upper and lower M -asymmetric semicontinuous multifunctions in the realm of bitopological spaces satisfying a minimal structure, which is a generalization of results of Noiri and Popa [10] and also Berge [5].

This paper is organized as follows. Section 2 presents some necessary preliminaries concerning semiopen sets, continuity and multifunctions on sets satisfying certain minimal structures. In Section 3, we investigate and study the concept of m -asymmetric semiopen and semiclosed sets in the realm of bitopological spaces, a generalization of semiopen and semiclosed sets in [1] [2] and [4]. Section 4 presents and discusses some results of upper and lower M -asymmetric semicontinuous multifunctions as a generalized idea of upper and lower upper M -continuous multifunctions first introduced in [8] and then [10]. Section 5 gives some concluding remarks.

2. Preliminaries and Basic Properties

Throughout the section, we recall some basic notations and properties, which we refer the reader to ([1] [2] [3] [4] [7] [8] [9] [10] [11]).

By a bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ we mean a nonempty set X on which are defined two topologies \mathcal{T}_1 and \mathcal{T}_2 , this idea was first introduced by Kelly ([12]). In the sequel, $(X, \mathcal{T}_1, \mathcal{T}_2)$ or in short hand X will mean a bitopological space unless stated. Given a bitopological space $(X, \mathcal{T}_i, \mathcal{T}_j)$, $i, j = 1, 2$; $i \neq j$, the interior and closure of $A \subseteq X$ with respect to the topology $\mathcal{T}_i = \mathcal{T}_j$ is denoted by $Int_{\mathcal{T}_i}(A)$ and $Cl_{\mathcal{T}_i}(A)$ respectively.

Definition 2.1. Let $(X, \mathcal{T}_i, \mathcal{T}_j)$, $i, j = 1, 2$; $i \neq j$ be a bitopological space, $A \subseteq X$.

(i) A is said to be $\mathcal{T}_i\mathcal{T}_j$ -open if $A \in \mathcal{T}_i \cup \mathcal{T}_j$. The complement of an $\mathcal{T}_i\mathcal{T}_j$ -open set is a $\mathcal{T}_i\mathcal{T}_j$ -closed set [12].

(ii) The $\mathcal{T}_i\mathcal{T}_j$ -interior of A denoted by $Int_{\mathcal{T}_i}(Int_{\mathcal{T}_j}(A))$ (or $\mathcal{T}_i\mathcal{T}_j$ - $Int(A)$) is the union of all $\mathcal{T}_i\mathcal{T}_j$ -open subsets of X contained in A . Clearly, A is $\mathcal{T}_i\mathcal{T}_j$ -open provided $A = Int_{\mathcal{T}_i}(Int_{\mathcal{T}_j}(A))$.

(iii) The $\mathcal{T}_i\mathcal{T}_j$ -closure of A denoted by $Cl_{\mathcal{T}_i}(Cl_{\mathcal{T}_j}(A))$ defined to be the

intersection of all $\mathcal{T}_i\mathcal{T}_j$ -closed subsets of X containing A . And

$$Cl_{\mathcal{T}_i}(Cl_{\mathcal{T}_j}(B)) \subseteq Cl_{\mathcal{T}_i}(B) \text{ and } Cl_{\mathcal{T}_j}(Cl_{\mathcal{T}_i}(B)) \subseteq Cl_{\mathcal{T}_j}(B).$$

(iv) A is said to be pairwise open if it is both $\mathcal{T}_1\mathcal{T}_2$ -open and $\mathcal{T}_2\mathcal{T}_1$ -open.

Definition 2.2. Let $(X, \mathcal{T}_i, \mathcal{T}_j)$, $i, j = 1, 2$; $i \neq j$ be a bitopological space and $A, B \subseteq X$.

(i) A is called $\mathcal{T}_i\mathcal{T}_j$ -semiopen in X provided there is a \mathcal{T}_i -open subset O of X such that $O \subseteq A \subseteq Cl_{\mathcal{T}_j}(O)$, equivalently $A \subseteq Cl_{\mathcal{T}_j}(Int_{\mathcal{T}_i}(A))$. Its complement is said to be $\mathcal{T}_i\mathcal{T}_j$ -semiclosed [2].

(ii) The $\mathcal{T}_i\mathcal{T}_j$ -semiinterior of A denoted by $\mathcal{T}_i\mathcal{T}_j$ - $sInt(A)$ is defined as the union of $\mathcal{T}_i\mathcal{T}_j$ -semiopen subsets of X contained in A . The $\mathcal{T}_i\mathcal{T}_j$ -semiclosure of A denoted by $\mathcal{T}_i\mathcal{T}_j$ - $sCl(A)$, is the intersection of all $\mathcal{T}_i\mathcal{T}_j$ -semiclosed sets of X containing A .

(iii) A is said to be pairwise semiopen if it is both $\mathcal{T}_1\mathcal{T}_2$ -semiopen and $\mathcal{T}_2\mathcal{T}_1$ -semiopen [11].

(iv) B is said to be a $\mathcal{T}_i\mathcal{T}_j$ -semi-neighbourhood of $x \in X$ provided there is a $\mathcal{T}_i\mathcal{T}_j$ -semiopen subset O of X such that $x \in O \subseteq B$.

The family of all $\mathcal{T}_i\mathcal{T}_j$ -semiopen and $\mathcal{T}_i\mathcal{T}_j$ -semiclosed subsets of X are denoted by $\mathcal{T}_i\mathcal{T}_j$ - $sO(X)$ and $\mathcal{T}_i\mathcal{T}_j$ - $sC(X)$ respectively.

Definition 2.3. [10] A point-to-set correspondence $F : X \rightarrow Y$ such that for each $x \in X$, $F(x)$ is a nonempty subset of Y is said to be a multifunction.

Following Berge [5], the upper and lower inverse of $G \subseteq Y$ with respect to a multifunction F , are denoted and defined respectively by:

$$F^+(G) = \{x \in X : F(x) \subseteq G\} \text{ and } F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}.$$

Generally, for the lower inverse F^- between Y and the power set $\mathcal{P}(X)$ of X , $F^-(y) = \{x \in X : y \in F(x)\}$ provided $y \in Y$. Clearly for $G \subseteq Y$, $F^-(G) = \bigcup \{F^-(y) : y \in G\}$ and also,

$$F^+(G) = X \setminus F^-(Y \setminus G) \text{ and } F^-(G) = X \setminus F^+(Y \setminus G)$$

$$\text{For any } A \subseteq X, F(A) = \bigcup_{x \in A} F(x).$$

Definition 2.4. [10] A multifunction $F : (X, \mathcal{T}_i, \mathcal{T}_j) \rightarrow (Y, \mathcal{C}_i, \mathcal{C}_j)$, $i, j = 1, 2$; $i \neq j$ is said to be

(i) Upper $\mathcal{T}_i\mathcal{T}_j$ -semi continuous at a point $x_0 \in X$ provided for any $\mathcal{C}_i\mathcal{C}_j$ -open subset $V \subseteq Y$ such that $F(x_0) \subseteq V$, there exists an $\mathcal{T}_i\mathcal{T}_j$ -semiopen set U in X with $x_0 \in U$ such that $F(U) \subseteq V$ (or $U \subseteq F^+(V)$).

(ii) Lower $\mathcal{T}_i\mathcal{T}_j$ -semi continuous at a point $x_0 \in X$ provided for any $\mathcal{C}_i\mathcal{C}_j$ -open set V in Y such that $F(x_0) \cap V \neq \emptyset$, there exists a $\mathcal{T}_i\mathcal{T}_j$ -semiopen set U in X with $x_0 \in U$ such that $F(z) \cap V \neq \emptyset$ for all $z \in U$ or $U \subseteq F^-(V)$.

(iii) Upper (resp Lower) $\mathcal{T}_i\mathcal{T}_j$ -semi continuous if it is Upper (resp Lower) $\mathcal{T}_i\mathcal{T}_j$ -semi continuous at all points of X .

Clearly, $F^+(V)$ and $F^-(V)$ are $\mathcal{T}_i\mathcal{T}_j$ -open in X for any $\mathcal{C}_i\mathcal{C}_j$ -open set V in Y . F is $\mathcal{T}_i\mathcal{T}_j$ -semicontinuous at $x_0 \in X$ provided it is both upper and lower— $\mathcal{T}_i\mathcal{T}_j$ -continuous (resp., $\mathcal{T}_i\mathcal{T}_j$ -semicontinuous) at x_0 . Generally, F

is upper (resp lower)— $\mathcal{T}_i, \mathcal{T}_j$ -semi continuous if the multifunctions induced $F_1: (X, \mathcal{T}_1) \rightarrow (Y, \mathcal{C}_1)$ and $F_2: (X, \mathcal{T}_2) \rightarrow (Y, \mathcal{C}_2)$ are both upper (resp lower) semi continuous on X .

Definition 2.5. [10] [13] A subfamily m_X of a power set $\mathcal{P}(X)$ of a non-empty set X is said to be a minimal structure (briefly m -structure) on X provided both \emptyset and X belongs to m_X . The pair (X, m_X) is called an m -space and member of (X, m_X) is said to be m_X -open.

Definition 2.6. [3] [13] A subset A of X in an m -space (X, m_X) is said to be

(i) m -semiopen if $A \subseteq mCl(mInt(A))$.

(ii) m -semiclosed if $mInt(mCl(A)) \subseteq A$.

Definition 2.7. [10] For a minimal space (X, m_X) , m_X is said to have property (\mathcal{S}) of Maki [3] if the union of any collection of m_X -open subsets of X is also m_X -open.

Lemma 2.8. [13] For an m -space (X, m_X) , an arbitrary union of m -semiopen subsets of X is also m -semiopen.

Lemma 2.9. (cf. [3] [10] [13]) For an m -space (X, m_X) and subsets A, B and F of X the properties below holds:

(i) $m_X sCl(X \setminus A) = X \setminus (m_X sInt(A))$ and $m_X sInt(X \setminus A) = X \setminus (m_X sCl(A))$

(ii) F is m_X -semiclosed if and only if $m_X sCl(F) = F$.

(iii) A is m_X -semiopen if and only if $m_X sInt(A) = A$.

(iv) $m_X sCl(\emptyset) = \emptyset$, $m_X sCl(X) = X$ and $m_X sInt(\emptyset) = \emptyset$,

$m_X sInt(X) = X$.

(v) If $A \subseteq B$, then $m_X sCl(A) \subseteq m_X sCl(B)$ and $m_X sInt(A) \subseteq m_X sInt(B)$

(vi) $m_X sCl(m_X sCl(A)) = m_X sCl(A)$ and $m_X sInt(m_X sInt(A)) = m_X sInt(A)$.

(vii) $A \subseteq m_X sCl(A)$ and $m_X sInt(A) \subseteq A$.

Lemma 2.10. (cf. [3] [10] [13]) Let (X, m_X) be an m -space and $A \subseteq X$. Then $x \in m_X sCl(A)$ if and only if for every m_X -semiopen set U containing x , $A \cap U \neq \emptyset$.

Lemma 2.11. [10] For an m -space (X, m_X) and subsets A and F of X , the following properties are equivalent:

(i) m_X has property \mathcal{S} ,

(ii) $A \in m_X$ provided $m_X Int(A) = A$,

(iii) $X \setminus F \in m_X$ provided $m_X Cl(F) = F$.

Lemma 2.12. ([10]) Let (X, m_X) be an m -space with m_X satisfying property \mathcal{S} and $A \subseteq X$. The properties below holds:

(i) $m_X Int(A) = A$ if and only if $A \in m_X$.

(ii) $m_X Cl(A) = A$ if and only if $X \setminus A \in m_X$

(iii) $m_X Int(A) \in m_X$ and $m_X Cl(A)$ is m_X -closed.

3. On m -Asymmetric Semi Open and Closed Sets

This section introduces the notions of m -asymmetric semiopen and semiclosed sets in the framework of bitopological spaces satisfying certain minimal condi-

tions.

Definition 3.1. Let $(X, \mathcal{T}_i, \mathcal{T}_j)$, $i, j = 1, 2$; $i \neq j$ be a topological space and m_X a minimal structure on X determined with respect to m_i and m_j . Then an ordered pair $((X, \mathcal{T}_i, \mathcal{T}_j), m_X)$ shall be called a minimal bitopological space (or m_{ij} -spaces).

Since in our sense the minimal structure m_X on X is determined by the left and right minimal structures m_i and m_j which are as a generalization of the two topologies \mathcal{T}_i and \mathcal{T}_j , $i, j = 1, 2$; $i \neq j$, we shall denote it by m_{ij} and call the pair $((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij})$ (or (X, m_{ij})) a minimal bitopological space unless explicitly stated.

Definition 3.2. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be a minimal bitopological space and A a subset of X . Then A is said to be m_{ij} -semiopen in X if there exists an m_i -open set U such that $U \subseteq A \subseteq Cl_{m_j}(U)$. It is said to be m_{ij} -semiclosed in X if there exists a m_i -open set U such that $Cl_{m_j}(A) \subseteq U$ whenever $A \subseteq U$.

We shall denote the collection of all m_{ij} -semiopen and m_{ij} -semiclosed sets in (X, m_{ij}) by $m_{ij}SO(X)$ and $m_{ij}SC(X)$ respectively.

Example 3.3. Let $X = \{-2, -1, 0, 1, 2\}$, and the minimal structures m_1 and m_2 be define by $m_1 = \{\emptyset, \{-1\}, \{0\}, \{0, 1\}, \{-2, 0, 1\}, X\}$ and $m_2 = \{\emptyset, \{0\}, \{2\}, \{0, 1\}, X\}$ respectively. The sets \emptyset and $\{0\}$ are m_{ij} -open sets but $\{-1\}$ and $\{-1, 1\}$ are not m_{ij} -open. Also, $\{1\}$ is neither m_{ij} -open nor m_{ij} -closed.

Remark 3.4. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be a minimal bitopological space.

- (i) if $m_i = \mathcal{T}_i$ and $m_j = \mathcal{T}_j$, the any m_{ij} -semiopen set is $\mathcal{T}_i\mathcal{T}_j$ -semiopen.
- (ii) every m_{ij} -open (resp. m_{ij} -closed) set is m_{ij} -semiopen (resp. m_{ij} -semiclosed), but the converse is not generally true as our next example illustrates.

Example 3.5. Let us define m_1 and m_2 on $X = \{-2, -1, 0, 1, 2, 3\}$ on X by $m_1 = \{\emptyset, \{-1\}, \{0\}, \{2\}, X\}$ and $m_2 = \{\emptyset, \{-2\}, \{-1\}, \{-1, 0\}, X\}$ respectively. Then the subset $A = \{-1, 3\}$ of X is m_{ij} -semiopen but not m_{ij} -open since $\{-1\} \subseteq A \subseteq Cl_{m_2}(\{-1\})$, whence $A \notin m_{ij}$.

Proposition 3.6. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space and $A \subseteq X$. Then:

- (i) $A \in m_{ij}SO(X)$ if and only if $A \subseteq Cl_{m_j}(Int_{m_i}(A))$.
- (ii) $A \in m_{ij}SC(X)$ if and only if $Int_{m_j}(Cl_{m_i}(A)) \subseteq A$.

Proof. (i) Suppose A is a m_{ij} -semiopen subset of X . By Definition 3.2 we can always find an m_i -open subset U of X such that $U \subseteq A \subseteq Cl_{m_j}(U)$. Since $U \subseteq Int_{m_i}(A)$ and $U = Int_{m_i}(U)$, we have

$$A \subseteq Cl_{m_j}(U) \subseteq Cl_{m_j}(Int_{m_i}(A)).$$

On the other hand, let $U = Cl_{m_j}(Int_{m_i}(A))$, then, referring to Lemma 2.9, we have

$$X \setminus U = X \setminus Cl_{m_j}(Int_{m_i}(A)) = Int_{m_j}(X \setminus Int_{m_i}(A)) = Int_{m_j}(Cl_{m_i}(X \setminus A)).$$

Thus,

$$Cl_{m_j}(X \setminus U) = Cl_{m_j}(Int_{m_j}(Cl_{m_i}(X \setminus A))) = Int_{m_j}(Cl_{m_i}(X \setminus A)) = X \setminus U.$$

Consequently, $X \setminus (X \setminus U) \in m_{ij}SO(X)$ and hence, $U \in m_{ij}SO(X)$. Therefore, $U \subseteq A \subseteq Cl_{m_j}(U)$. This implies that A is a m_{ij} -semiopen set.

(ii) Suppose A is a m_{ij} -semiclosed set. By definition 3.2, there is some m_j -open set U such that $Cl_{m_j}(A) \subseteq U$. Since $Cl_{m_j}(A) \subseteq Cl_{m_i}(U)$ and $U = Int_{m_i}(U)$, we have

$$Int_{m_j}(Cl_{m_i}(A)) \subseteq Int_{m_j}(A) \subseteq A.$$

and also from (i), $Cl_{m_j}(A) \subseteq U$. This implies that A is an m_{ij} -semiclosed set. \square

Generally, the m_{ij} -open sets and the m_{ij} -semiopen sets are not stable for the union. Nevertheless, for certain m_{ij} -structure, the class of m_{ij} -semiopen sets are stable under union of sets, like it is demonstrated in the following proposition.

Proposition 3.7. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space. Then, for a collection $\{A_\gamma : \gamma \in \Gamma\}$ of subsets of X , the following properties hold:

- (i) $\bigcup_{\gamma \in \Gamma} A_\gamma \in m_{ij}SO(X)$ provided for all $\gamma \in \Gamma$, $A_\gamma \in m_{ij}SO(X)$.
- (ii) $\bigcap_{\gamma \in \Gamma} A_\gamma \in m_{ij}SC(X)$ provided for all $\gamma \in \Gamma$, $A_\gamma \in m_{ij}SC(X)$.

Proof. (i) Suppose that $A_\gamma \in m_{ij}SO(X)$ for all $\gamma \in \Gamma$. Then, by definition 3.2 and Proposition 3.6, we have

$$A_\gamma \subseteq Cl_{m_j} Int_{m_i}(A_\gamma) \subseteq Cl_{m_j} Int_{m_i}\left(\bigcup_{\gamma \in \Gamma} A_\gamma\right),$$

which implies $\bigcup_{\gamma \in \Gamma} A_\gamma \subseteq Cl_{m_j} Int_{m_i}\left(\bigcup_{\gamma \in \Gamma} A_\gamma\right)$, so that, $\bigcup_{\gamma \in \Gamma} A_\gamma \in m_{ij}SO(X)$.

(ii) It follows from Proposition 3.7 (i) and De-Morgan's laws. \square

Remark 3.8. It should generally be noted that, the intersection of any two m_{ij} -semiopen sets may not be m_{ij} -semiopen in a minimal bitopological spaces (X, m_{ij}) , as illustrated in the next example.

Example 3.9. Define the minimal structures m_1 and m_2 on given set $X = \{-3, -2, -1, 0, 1, 2, 3\}$ by $m_1 = \{\emptyset, \{0\}, \{2\}, \{0, 2\}, X\}$ and $m_2 = \{\emptyset, \{-2\}, \{0\}, \{2\}, \{0, 2\}, X\}$. We observe that, $\{-1, 0\}$ and $\{-1, 2\}$ are m_{ij} -semiopen sets, $i, j = 1, 2$; $i \neq j$. But then, $\{-1, 0\} \cap \{-1, 2\} = \{-1\}$ which is not m_{ij} -semiopen, indeed, $\{-1\} \not\subseteq Cl_{m_i}(Int_{m_j}(\{-1\})) = \emptyset$.

Definition 3.10. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space. A subset O of X is an m_{ij} -semineighborhood of a point $x \in X$ if there exists an m_{ij} -semiopen subset U of X such that $x \in U \subseteq O$.

Definition 3.11. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space and $A \subseteq X$. Then, the m_{ij} -semiinterior and m_{ij} -semiclosure of A are respectively denoted and defined by:

- (i) $m_{ij}SInt(A) = \bigcup \{U : U \subseteq A \text{ and } U \in m_{ij}SO(X)\}$,
- (ii) $m_{ij}SCl(A) = \bigcap \{F : A \subseteq F \text{ and } F \in m_{ij}SC(X)\}$,

Remark 3.12. For any bitopological spaces $(X, \mathcal{T}_1, \mathcal{T}_2)$;

(i) $\mathcal{F}_i\mathcal{F}_j sO(X)$ is a minimal structures of X .

(ii) In the following, we denote by $m_{ij}(X)$ a minimal structure on X a generalization of \mathcal{F}_i and \mathcal{F}_j . For $A \subseteq X$, if $m_{ij}(X) = \mathcal{F}_i\mathcal{F}_j sO(X)$, then by Definition 3.11;

(a) $m_{ij}Int(A) = \mathcal{F}_i\mathcal{F}_j sInt(A)$,

(b) $m_{ij}Cl(A) = \mathcal{F}_i\mathcal{F}_j sCl(A)$.

Proposition 3.13. For any m_{ij} -space (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ and subsets A and B of X , the following m_{ij} -semiclosure and m_{ij} -semiinterior properties holds:

(i) $m_{ij} sInt(A) \subseteq A$ and $m_{ij} sCl(A) \supseteq A$.

(ii) $m_{ij} sInt(A) \subseteq m_{ij} sInt(B)$ and $m_{ij} sCl(A) \subseteq m_{ij} sCl(B)$ provided $A \subseteq B$.

(iii) $m_{ij} sInt(\emptyset) = \emptyset$, $m_{ij} sInt(X) = X$, $m_{ij} sCl(\emptyset) = \emptyset$ and $m_{ij} sCl(X) = X$.

(iv) $A = m_{ij} sInt(A)$ provided $A \in m_{ij} sO(X)$.

(v) $A = m_{ij} sCl(A)$ provided $X \setminus A \in m_{ij} sO(X)$.

(vi) $m_{ij} sInt(m_{ij} sInt(A)) = m_{ij} sInt(A)$ and $m_{ij} sCl(m_{ij} sCl(A)) = m_{ij} sCl(A)$

Proof. (i) and (ii) are obvious by the interior and closure property and using Proposition 3.6.

(iii) Follows from the interior and closure properties and also part (i) and (ii).

(iv) If $A \in m_{ij} sO(X)$, Definition 3.2 implies that, there exists an m_j -open subset U of X such that $U \subseteq A \subseteq Cl_{m_j}(U)$. Since, $U \subseteq Int_{m_i}(A)$ and $U = Int_{m_i}(U)$, we obtain following Proposition 3.6 that

$$A \subseteq Cl_{m_j}(U) \subseteq Cl_{m_j}(Int_{m_i}(A)).$$

And also $A \supseteq Cl_{m_j}(Int_{m_i}(A))$. Hence, $A = Cl_{m_j}(Int_{m_i}(A))$

(v) Suppose $m_{ij} sCl(A) = A$. Then by Definition 3.11 and Proposition 3.7 we have

$$X \setminus A = X \setminus m_{ij} sCl(A) = m_{ij} sInt(X \setminus A)$$

Hence by (i), $X \setminus A \in m_{ij} sO(X)$.

(vi) follows from (iii) and (iv). \square

Proposition 3.14. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_j -space and $A \subseteq X$. Then, for each $U \in m_{ij} sO(X)$ such that $x \in U$, $U \cap A \neq \emptyset$ if and only if $x \in m_{ij} sCl(A)$.

Proof. Suppose that $x \in m_{ij} sCl(A)$. We need to show that for all m_j -semiopen subset U of X containing x , $U \cap A \neq \emptyset$. On the contrary, suppose for some m_j -semiopen set U containing x , $U \cap A = \emptyset$. Then $A \subseteq X \setminus U$ and $X \setminus (X \setminus U) \in m_{ij} sO(X)$ and Proposition 3.13 implies $m_{ij} sCl(A) \subseteq m_{ij} sCl(X \setminus U) = X \setminus U$. Since $x \in m_{ij} sCl(A) \subseteq X \setminus U$, it follows that $x \notin U$ which contradicts our assertion. Consequently, $U \cap A \neq \emptyset$.

Conversely, suppose that for every m_j -semiopen subset U of X containing x , $U \cap A \neq \emptyset$, then we need to show that $x \in m_{ij} sCl(A)$. Suppose on the contrary, $x \notin m_{ij} sCl(A)$, then there exists an m_j -semiclosed subset F of X such that $A \subseteq F$. Proposition 3.13 then implies $x \notin m_{ij} sCl(A) \subseteq m_{ij} sCl(F) = F$, so that

$x \notin F$. Since $F \in m_{ij}SC(X)$, then $X \setminus F \in m_{ij}SO(X)$ and $x \in X \setminus F$ so that, $(X \setminus F) \cap A = \emptyset$. Setting $U = X \setminus F$, we obtaining $U \cap A = \emptyset$ which contradicts our assertion that $U \cap A \neq \emptyset$. Hence, $x \in m_{ij}SCl(A)$. \square

Proposition 3.15. For a minimal bitopological space (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ and any $A \subseteq X$, the properties below holds:

- (i) $m_{ij}SCl(X \setminus A) = X \setminus (m_{ij}SInt(A))$,
- (ii) $m_{ij}SInt(X \setminus A) = X \setminus (m_{ij}SCl(A))$.

Proof.

(i) For each $A \subseteq X$ we obtain by Definition 3.11 and Proposition 3.13 that,

$$\begin{aligned} m_{ij}SCl(X \setminus A) &= \bigcap \{X \setminus V : X \setminus V \in m_{ij}SC(X), X \setminus A \subseteq X - V\} \\ &= \bigcap \{X \setminus V : V \in m_{ij}SO(X), V \subseteq A\} \\ &= X \setminus \bigcup \{V : V \subseteq A, V \in m_{ij}SO(X)\} \text{ by De-Morgans Law} \\ &= X \setminus (m_{ij}SInt(A)) \end{aligned}$$

(ii) Let $x \in m_{ij}SInt(X \setminus A)$. Then, is some neighborhood $U_x \in m_{ij}SO(X)$ of x for which $U \subseteq X \setminus A$. Because $x \notin A$, $U \cap A = \emptyset$. Proposition 3.14 then implies that, $x \notin m_{ij}SCl(A)$ and as a consequence, $x \in X \setminus m_{ij}SCl(A)$. Therefore,

$$m_{ij}SInt(X \setminus A) \subseteq X \setminus (m_{ij}SCl(A))$$

Convesely, if $X \setminus m_{ij}SCl(A) = \emptyset$, the inclusion is obvious. Suppose $x \in X \setminus m_{ij}SCl(A)$, then by the complementation property $x \notin m_{ij}SCl(A)$ implying $x \notin A$. By Proposition 3.14, there exists an m_{ij} -semiopen neighborhood of x , V such that $U \cap A = \emptyset$. Thus, $V \subseteq X \setminus A$ giving $x \in m_{ij}SCl(X \setminus A)$. Hence,

$$X \setminus (m_{ij}SInt(A)) \subseteq m_{ij}SCl(X \setminus A). \quad \square$$

Proposition 3.16. For an m_{ij} -space (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ and any subset $A \subseteq X$, the properties below are satisfied:

- (i) $m_{ij}SCl(A) = Int_{m_j}(Cl_{m_i}(A)) \cup A$.
- (ii) $m_{ij}SCl(A) = Int_{m_j}(Cl_{m_i}(A))$ provided $A \in m_{ij}O(X)$. The converse to this assertion is not necessary true.

Proof.

(i) Since by Proposition 3.6, $m_{ij}SCl(A)$ is m_{ij} -semiclosed, we have

$$m_{ij}SCl(A) \supseteq Int_{m_j}(Cl_{m_i}(m_{ij}SCl(A))).$$

Thus, $Int_{m_i}(Cl_{m_j}(A)) \subseteq m_{ij}SCl(A)$. Since $A \subseteq m_{ij}SCl(A)$, it holds that,

$$Int_{m_j}(Cl_{m_i}(A)) \cup A \subseteq m_{ij}SCl(A) \cup A = m_{ij}SCl(A).$$

Conversely, let $B = Int_{m_j}(Cl_{m_i}(A)) \cup A$. We aim to show that B is an m_{ij} -semiclosed set. By definition,

$$\begin{aligned}
 Int_{m_j}(Cl_{m_i}(B)) &= Int_{m_j}(Cl_{m_i}(Int_{m_j}(Cl_{m_i}(A)) \cup A)) \\
 &\subseteq Int_{m_j}(Cl_{m_i}(Cl_{m_i}(A) \cup A)) \\
 &= Int_{m_j}(Cl_{m_i}(Cl_{m_i}(A))) \\
 &= Int_{m_j}(Cl_{m_i}(A)) \\
 &\subseteq Int_{m_j}(Cl_{m_i}(A)) \cup A \\
 &= B
 \end{aligned}$$

Thus, $B = Int_{m_j}(Cl_{m_i}(A)) \cup A$ is a m_{ij} -semiclosed set. Hence,

$$Int_{m_j}(Cl_{m_i}(A)) \cup A \supseteq m_{ij}sCl(A).$$

Therefore, $m_{ij}sCl(A) = Int_{m_j}(Cl_{m_i}(A)) \cup A$.

(ii) Let A be m_{ij} -open in X , then by Definition 3.2 and Proposition 3.6,

$$A = Int_{m_j}(Int_{m_i}(A)) \subseteq Int_{m_j}(Cl_{m_i}(A))$$

Consequently, by (i),

$$m_{ij}sCl(A) = Int_{m_j}(Cl_{m_i}(A)). \quad \square$$

Example 3.17. In this example we aim to show that, the converse to part (ii) of Proposition 3.16 is not necessary true: Define the two minimal structures m_1 and m_2 on $X = \{-3, -2, -1, 0, 1, 2\}$ by $m_1 = \{\emptyset, \{-2, 1\}, X\}$ and $m_2 = \{\emptyset, \{-3\}, \{2\}, X\}$. One clearly see that, $\{-2, 1\}$ is only m_1 -open but not m_{12} -open even though $m_{12}sCl(\{-2, 1\}) = Int_{m_{12}}(Cl_{m_{12}}(\{-2, 1\}))$.

Remark 3.18. For a bitopological space $(X, \mathcal{F}_i, \mathcal{F}_j)$, $i, j = 1, 2; i \neq j$ the families $\mathcal{F}_i\mathcal{F}_jO(X)$ and $m_{ij}sO(X)$ are all m_{ij} -structures of X satisfying property \mathcal{B} .

Proposition 3.19. For an m_{ij} -space (X, m_{ij}) , $i, j = 1, 2; i \neq j$ with m_{ij} satisfying property (\mathcal{B}) and subsets A and F of X , the properties below holds:

- (i) $m_{ij}sInt(A) = A$ provided $A \in m_{ij}sO(X)$.
- (ii) If $X \setminus F \in m_{ij}sO(X)$, then $m_{ij}sCl(F) = F$.

Proof. Suppose that m_{ij} satisfying property (\mathcal{B}) :

(i) Let $A \in m_{ij}sO(X)$, then by Definition 3.11, $m_{ij}sInt(A) \in m_{ij}sO(X)$. Hence, $m_{ij}sInt(A) = A$.

(ii) If $m_{ij}sCl(F) = F$, then Definition 3.11 and Proposition 3.15 gives

$$X \setminus F = X \setminus m_{ij}sCl(F) = m_{ij}sInt(X \setminus F)$$

Thus, by part (i), we obtain $X \setminus F \in m_{ij}sO(X)$. \square

Proposition 3.20. For any m_{ij} -space (X, m_{ij}) , $i, j = 1, 2; i \neq j$ with m_{ij} satisfying property \mathcal{B} and any $A \subseteq X$, the properties outlined below holds:

- (i) $A = m_{ij}sInt(A)$ if and only if A is m_{ij} -semiopen in X .
- (ii) $A = m_{ij}sCl(A)$ if and only if $X \setminus A$ is m_{ij} -semiopen in X .
- (iii) $m_{ij}sInt(A)$ is m_{ij} -semiopen.
- (iv) $m_{ij}sCl(A)$ is m_{ij} -semiclosed.

Proof. (i) and (ii) follows m_{ij} -closed, m_{ij} -Interior, the property of \mathcal{B} and

also Proposition 3.13 and Proposition 3.19.

(iii) and (iv) follows from (i) and (ii) with the aid of Proposition 3.7, Proposition 3.13 and Proposition 3.19. \square

Proposition 3.21. Let (X, m_{ij}) , $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space with m_{ij} satisfying the property \mathcal{S} and let $\{A_\gamma : \gamma \in \Gamma\}$ be an arbitrary collection of subsets of X . Then, $\bigcup_{\gamma \in \Gamma} A_\gamma \in m_{ij}SO(X)$ provided $A_\gamma \in m_{ij}SO(X)$ for every $\gamma \in \Gamma$.

Proof. Suppose that m_{ij} satisfies property \mathcal{S} and $A_\gamma \in m_{ij}SO(X)$ for all $\gamma \in \Gamma$. By Definition 3.2, we can find some m -open set U_γ such that $U_\gamma \subseteq A_\gamma \subseteq Cl_{m_j} U_\gamma$ for some $\gamma \in \Gamma$. Consequently,

$$\bigcup_{\gamma \in \Gamma} (U_\gamma) \subseteq \bigcup_{\gamma \in \Gamma} (A_\gamma) \subseteq \bigcup_{\gamma \in \Gamma} (Cl_{m_j} (U_\gamma)).$$

Since Cl_{m_j} is a monotone operator, $\bigcup_{\gamma \in \Gamma} (Cl_{m_j} (U_\gamma)) \subseteq Cl_{m_j} \left(\bigcup_{\gamma \in \Gamma} (U_\gamma) \right)$ and

$\bigcup_{\gamma \in \Gamma} (U_\gamma)$ is m -open as m_{ij} satisfies property \mathcal{S} . As a consequence,

$\bigcup_{\gamma \in \Gamma} (U_\gamma) \subseteq \bigcup_{\gamma \in \Gamma} (A_\gamma) \subseteq Cl_{m_j} \left(\bigcup_{\gamma \in \Gamma} (U_\gamma) \right)$. Therefore, $\bigcup_{\gamma \in \Gamma} A_\gamma \in m_{ij}SO(X)$. \square

Proposition 3.22. Let $((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij})$, $i, j = 1, 2$; $i \neq j$ be an m_{ij} -space and $A \subseteq X$. Provided m_{ij} satisfy property \mathcal{S} , then:

(i) $m_{ij}Cl(A) = A \cup Int_{m_i} (Cl_{m_i} (A))$, and

(ii) $m_{ij}Int(A) = A \cap Cl_{m_i} (Int_{m_i} (A))$ holds.

Proof. (i) Since m_{ij} satisfy property \mathcal{S} , then $m_{ij}Cl(A)$ is an m_{ij} -semiclosed set. By Definition 3.2 and Proposition 3.6, we obtain that

$$Int_{m_j} (Cl_{m_i} (m_{ij}Cl(A))) \subseteq m_{ij}Cl(A).$$

Therefore, $Int_{m_j} (Cl_{m_i} (A)) \subseteq m_{ij}Cl(A)$ and it follows that,

$$A \cup Int_{m_j} (Cl_{m_i} (A)) \subseteq m_{ij}Cl(A).$$

On the other hand, we observe that

$$\begin{aligned} & Int_{m_j} (Cl_{m_i} (A \cup Int_{m_j} (Cl_{m_i} (A)))) \\ &= Int_{m_i} (Cl_{m_i} (A)) \cup Int_{m_j} (Cl_{m_i} (Int_{m_j} (Cl_{m_i} (A)))) \\ &\subseteq (Cl_{m_i} (A)) \cup Int_{m_j} (Cl_{m_i} (Int_{m_j} (Cl_{m_i} (A)))) \\ &= Cl_{m_i} (A) \cup Int_{m_j} (Cl_{m_i} (A)) \\ &= Cl_{m_i} (A). \end{aligned}$$

Hence, Definition 3.2 and Proposition 3.6 implies,

$$Int_{m_j} (Cl_{m_i} (A \cup Int_{m_j} (Cl_{m_i} (A)))) \subseteq Int_{m_j} (Cl_{m_i} (A)) \subseteq A \cup Int_{m_j} (Cl_{m_i} (A)).$$

and so,

$$Int_{m_j} \left(Cl_{m_i} \left(A \cup Int_{m_j} \left(Cl_{m_j} (A) \right) \right) \right) \subseteq A \cup Int_{m_j} \left(Cl_{m_j} (A) \right).$$

Consequently, $A \cup Int_{m_j} \left(Cl_{m_j} (A) \right)$ is an m_j -semiclosed set and so $m_j sCl(A) \subseteq A \cup Int_{m_j} \left(Cl_{m_j} (A) \right)$. This completes the proof.

(ii) The proof follows from Theorems 3.13 and 3.20. \square

If the property \mathcal{S} of Make is removed in the previous theorem, the equality does not necessarily hold as show in the example below:

Example 3.23. Defined two minimal structures m_1 and m_2 on \mathbb{N} as:

$$m_1 = \{ \emptyset, P(\{2n-1 : n \in \mathbb{N}\}), \{2\}, \mathbb{N} \} \text{ and } m_2 = \{ \emptyset, P(\{2n-1 : n \in \mathbb{N}\}), \{2\}, \mathbb{N} \}.$$

Then the collection of all m_j -closed and m_j -open subsets of \mathbb{N} are:

$$m_j C(\mathbb{N}) = \{ \emptyset, P(\{2n-1 : n \in \mathbb{N}\})^c, \mathbb{N} - \{2\}, \mathbb{N} \}$$

$$\text{and } m_j O(\mathbb{N}) = \{ \emptyset, P(\{2n-1 : n \in \mathbb{N}\}), \{2\}, V, \mathbb{N} \},$$

where $V \cap \{2n-1 : n \in \mathbb{N}\} \neq \emptyset$. Take $A = \{4\}$. Then $m_j sCl(A) = \{4\}$, $m_j Cl(A) = \{2n : n \in \mathbb{N}\}$ and $m_j Int(A) = \{2n : n \in \mathbb{N}\} = \{2\}$. Further,

$$A \cup int_{m_2} \left(Cl_{m_1} (A) \right) = \{2, 4\}.$$

Consequently, $m_j sO(A) \subseteq A \cup int_{m_2} \left(Cl_{m_1} (A) \right)$.

4. On M -Asymmetric Semi Continuous Multifunctions

In this part of our paper, we introduce and investigate some properties of the notion of upper and lower M asymmetric semi-continuous multifunctions.

Definition 4.1. Let $\left((X, \mathcal{F}_i, \mathcal{F}_j), m_{ij}(X) \right)$ and $\left((Y, \mathcal{G}_1, \mathcal{G}_2), m_{ij}(Y) \right)$, $i, j = 1, 2; i \neq j$ be minimal bitopological spaces. We shall call a multifunction $F : \left((X, \mathcal{F}_i, \mathcal{F}_j), m_{ij}(X) \right) \rightarrow \left((Y, \mathcal{G}_j, \mathcal{G}_i), m_{ij}(Y) \right)$ to be:

(i) Upper $m_{ij}(X)$ -semi-continuous at some point $x_o \in X$ provided for any $m_{ij}(Y)$ -open subset $V \subseteq Y$ satisfying $F(x_o) \subseteq V$, there is an $m_{ij}(X)$ -semiopen set U in X with $x_o \in U$ for which $F(U) \subseteq V$ (or $U \subseteq F^+(V)$).

(ii) Lower $m_{ij}(X)$ -semi-continuous at some point $x_o \in X$ provided for each $m_{ij}(Y)$ -open subset $V \subseteq Y$ satisfying $F(x_o) \cap V \neq \emptyset$, we can find an $m_{ij}(Y)$ -semiopen set U in X with $x_o \in U$ such that for all $z \in U$, $F(z) \cap V \neq \emptyset$.

(iii) Upper (resp Lower) $m_{ij}(X)$ -semi continuous if it is Upper (resp Lower) $m_{ij}(X)$ -semi continuous at each and every point of X .

Example 4.2. Let us define the two minimal structures $m_1(X)$ and $m_2(X)$ on $X = \{-2, -1, 0, 1, 2, 3\}$ by $m_1(X) = \{ \emptyset, \{-1\}, \{2\}, \{-1, 2\}, X \}$ and $m_2(X) = \{ \emptyset, \{-1\}, \{2\}, \{-1, 2\}, X \}$. Also, let the minimal structure $m_1(Y)$ and $m_2(Y)$ on $Y = \{-3, -2, -1, 0, 1, 2, 3\}$ be defined by $m_1(Y) = \{ \emptyset, \{-3\}, \{-2, 0, 1, 3\}, X \}$ and $m_2(Y) = \{ \emptyset, \{-2\}, \{0\}, \{-2, 0\}, \{-2, 0, 1, 3\}, X \}$. Defined a multifunction

$F : ((X, \mathcal{T}_1, \mathcal{T}_2), m_{ij}) \rightarrow ((Y, \mathcal{C}_1, \mathcal{C}_2), m_{ij})$ by:

$$f(x) = \begin{cases} \{-2, 0\} & x = -1 \\ \{-3, 1, 3\} & x = 1 \\ \{-3, -2\} & x = 2 \end{cases} \quad (1)$$

Clearly, F is upper and lower M -asymmetric semi-continuous.

In this first part of these sections we discuss some characterizations of upper m_{ij} -semicontinuous multifunctions.

Theorem 4.3. Let $F : ((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij}(X)) \rightarrow ((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$, $i, j = 1, 2$; $i \neq j$ be a multifunction where, $((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ satisfies property \mathcal{S} . F is upper m_{ij} -semicontinuous at some point x of X if and only if for every $m_{ij}(Y)$ -open set V in Y with $F(x) \subseteq V$, $x \in m_{ij} sInt(F^+(V))$.

Proof. Let F be upper m_{ij} -semicontinuous at a point x of X and V be an $m_{ij}(Y)$ -open set such that $F(x) \subseteq V$. By Definition 4.1, there is some $m_{ij}(X)$ -semiopen set U containing x for which $F(U) \subseteq V$. Therefore, $x \in U \subseteq F^+(V)$. The $m_{ij}(X)$ -semiopenness of U implies, $U = m_{ij}(X) sInt(U)$. Because $((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ satisfies property \mathcal{S} , we have from Definition 2.3 that, $x \in m_{ij}(X) sInt(F^+(V))$.

Conversely, if V is an $m_{ij}(Y)$ -open set in Y with $F(x) \subseteq V$ for which $x \in m_{ij}(X) sInt(F^+(V))$, then by definition, there is some $m_{ij}(X)$ -semiopen set U in X with $x \in U$ satisfying $U \subseteq F^+(V)$. This implies, $F(U) \subseteq V$ and so F is upper m_{ij} -semicontinuous at a point x of X . \square

Theorem 4.4. Provided $((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$, $i, j = 1, 2$; $i \neq j$ satisfies property \mathcal{S} , the following properties are equivalent for a multifunction

$F : ((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij}(X)) \rightarrow ((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$:

- (i) F is upper m_{ij} -semicontinuous;
- (ii) For every $m_{ij}(Y)$ -open set V , the set $F^+(V)$ is $m_{ij}(X)$ -semiopen;
- (iii) For every $m_{ij}(Y)$ -closed set K , the set $F^-(K)$ is $m_{ij}(X)$ -semiclosed;
- (iv) For every $m_{ij}(X)$ -open set, the set inclusion $F(m_{ij}(X) sCl(A)) \subseteq m_{ij}(Y) Cl(F(A))$ is true;
- (v) For any subset B of Y , the set inclusion $m_{ij}(X) sCl(F^-(B)) \subseteq F^-(m_{ij}(Y) Cl(B))$ holds true;
- (vi) For any subset B of Y , the set inclusion $F^+(m_{ij}(Y) Int(B)) \subseteq m_{ij}(X) sInt(F^+(B))$ holds.

Proof. (i) \Rightarrow (ii): let V be a $m_{ij}(Y)$ -open set of Y and $x \in F^+(V)$ for all $x \in X$ then, $F(x) \subseteq V$. From the hypothesis, we can find an $m_{ij}(X)$ -semi-open set U with $x \in U$ for which $F(U) \subseteq V$. And so, one gets $x \in U \subseteq F^+(V)$ for every x belonging to $F^+(V)$. Consequently by Proposition 3.6,

$$x \in U \subseteq Cl_{m_j} (Int_{m_i} (U)) \subseteq Cl_{m_j} (Int_{m_i} (F^+ (V))).$$

This implies that $F^+ (V) \subseteq m_{ij} (X) sInt (F^+ (V))$. By Proposition 3.13, we have that, $F^+ (V) \supseteq m_{ij} sInt (F^+ (V))$. This gives, $F^+ (V) = m_{ij} sInt (F^+ (V))$ and so, $F^+ (V)$ is $m_{ij} (X)$ semi-open in X .

(ii) \Rightarrow (iii): Let $K \subset m_{ij} sC (Y)$. Since $Y \setminus K \in m_{ij} (Y) O (Y)$ then, we have from $F^+ (Y \setminus K) = X \setminus F^- (K)$, Proposition 3.13 and Proposition 3.19 that,

$$\begin{aligned} X \setminus F^- (K) &= F^+ (Y \setminus K) = m_{ij} (X) sInt (F^+ (Y \setminus K)) \\ &= m_{ij} (X) sInt (X \setminus F^- (K)) \\ &= X \setminus m_{ij} (X) sCl (F^- (K)) \\ \\ X \setminus m_{ij} (X) sCl (F^- (K)) &= m_{ij} (X) sInt (X \setminus F^- (K)) \\ &= m_{ij} (X) sInt (F^+ (Y \setminus K)) \\ &= F^+ (Y \setminus K) \\ &= X \setminus F^- (K) \end{aligned}$$

Hence,

$$F^- (K) = m_{ij} (X) sCl (F^- (K)),$$

implying $F^- (K)$ is $m_{ij} (X)$ -semiclosed.

(iii) \Rightarrow (iv): If A is some subset of X , Definition 3.11 implies,

$$\begin{aligned} &F^- (m_{ij} (Y) Cl (F (A))) \\ &= F^- (\bigcap \{K \subseteq Y : F (A) \subseteq K \text{ and } K \in m_{ij} C (Y)\}) \\ &\supseteq \bigcap \{F^- (K) \subseteq X : A \subseteq F^- (K) \text{ and } F^- (K) \in m_{ij} sC (X)\} \\ &= \bigcap \{L \subseteq X : A \subseteq L \text{ and } L \in m_{ij} sC (X)\} \\ &= m_{ij} (X) sCl (A). \end{aligned}$$

And so, $m_{ij} (Y) Cl (F (A)) \supseteq F (m_{ij} (X) sCl (A))$.

(iv) \Rightarrow (v): Proposition 3.13 and Proposition 3.20 implies $m_{ij} (Y) Cl (B)$ is a $m_{ij} (Y)$ -closed set for if B is any subset of Y . Proposition 3.13 and (iv) of this theorem gives:

$$\begin{aligned} &F^- (m_{ij} (Y) Cl (B)) \\ &= F^- (\bigcap \{Q \subseteq Y : B \subseteq Q \text{ and } Q \in m_{ij} C (Y)\}) \\ &= \bigcap \{F^- (Q) \subseteq X : F^- (B) \subseteq F^- (Q) \text{ and } F^- (Q) \in m_{ij} sC (X)\} \\ &\supseteq \bigcap \{R \subseteq X : F^- (B) \subseteq R \text{ and } R \in m_{ij} sC (X)\} \\ &= m_{ij} (X) sCl (F^- (B)). \end{aligned}$$

and the implication follows.

(v) \Rightarrow (vi): Using Definition 2.3, $m_{ij} (Y) Int (B) = Y \setminus m_{ij} (Y) Cl (Y \setminus B)$, Proposition 3.13 and (v), it follows for a subset B of Y that,

$$\begin{aligned}
X \setminus m_{ij}(X) sInt(F^+(B)) &= m_{ij}(X) sCl(X \setminus F^+(B)) \\
&= m_{ij}(X) sCl(F^-(Y \setminus B)) \\
&\subseteq F^-(m_{ij}(Y) Cl(Y \setminus B)) \\
&= F^-(Y \setminus m_{ij}(Y) Int(B)) \\
&= X \setminus F^+(m_{ij}(Y) Int(B)).
\end{aligned}$$

Consequently, $F^+(m_{ij}(Y) Int(B)) \subseteq m_{ij}(X) sInt(F^+(B))$.

(vi) \Rightarrow (ii): Since $Y \setminus V$ is a $m_{ij}(Y)$ -closed set for any $m_{ij}(Y)$ -open set V in Y , (vi) and Proposition 3.13 implies;

$$\begin{aligned}
X \setminus F^+(V) &\supseteq X \setminus F^+(m_{ij}(Y) Int(V)) \\
&= F^-(Y \setminus m_{ij}(Y) Int(V)) \\
&= F^-(m_{ij}(Y) Cl(Y \setminus V)) \\
&\supseteq m_{ij}(X) sCl(F^-(Y \setminus V)) \quad \text{by (iv)} \\
&= m_{ij}(X) sCl(X \setminus F^+(V)) \\
&= X \setminus m_{ij}(X) sInt(F^+(V)).
\end{aligned}$$

Consequently, we obtain

$F^+(V) \subseteq F^+(m_{ij}(Y) Int(V)) \subseteq m_{ij}(X) sInt(F^+(V))$, implying $F^+(V)$ is an $m_{ij}(X)$ -semiopen set in X .

(ii) \Rightarrow (i): For any point $x \in X$ and $m_{ij}(Y)$ -open set V in Y such that $F^+(x) \subseteq V$, part (ii) of this theorem implies, $F^+(V)$ is a $m_{ij}(X)$ -semiopen set in X with $x \in F^+(V)$. Setting $x \in F^+(V)$, we have by the $F(U) \subseteq V$ and consequently, by Definition 4.1, F is upper m_{ij} -semicontinuous. \square

This second part of the sections gives some characterizations of lower m_{ij} -semicontinuous multifunctions.

Theorem 4.5. Provided $((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ $i, j = 1, 2$; $i \neq j$, satisfies property \mathcal{B} , a multifunction $F : ((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij}(X)) \rightarrow ((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ is lower m_{ij} -semicontinuous at $x \in X$ if and only if $x \in m_{ij} sInt(F^-(V))$ for every $m_{ij}(Y)$ -open set V of Y such that $F(x) \cap V \neq \emptyset$.

Proof. Suppose F is lower m_{ij} -semicontinuous at $x_o \in X$. Let V be an $m_{ij}(Y)$ -open set satisfying $F(x) \cap V \neq \emptyset$ for $x_o \in X$. By Definition 4.1, there is an $m_{ij}(X)$ -semiopen set U with $x_o \in U$ such that $F(x) \cap V \neq \emptyset$ for all $x \in U$. Thus, from Definition 2.3, $x \in U \subseteq F^-(V)$. Because U is $m_{ij}(X)$ -semiopen, $U = m_{ij}(X) sInt(U)$. Since $((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ satisfies property \mathcal{B} , we get

$$x \in m_{ij}(X) sInt(U) \subseteq m_{ij}(X) sInt(F^-(V)).$$

Conversely, suppose V is an $m_{ij}(Y)$ -open set with $F(x) \cap V \neq \emptyset$. From the assumption, $x \in m_{ij}(X) sInt(F^-(V))$. By Definition 2.3 and 4.1, there exists a $m_{ij}(X)$ -semiopen set O in X containing x such that for all $z \in O$, $F(z) \cap V \neq \emptyset$. Clearly, $O \subseteq F^-(V)$ so that, $F(O) \subseteq V$ by Definition 2.3. Consequently, F is

lower m_{ij} -semicontinuous at a point x of X . \square

Theorem 4.6. If $\left(\left(Y, \mathcal{C}_j, \mathcal{C}_i\right), m_{ij}(Y)\right)$, $i, j = 1, 2 ; i \neq j$, satisfies property \mathcal{B} , then the following properties are equivalent for a multifunction

$$F : \left(\left(X, \mathcal{T}_i, \mathcal{T}_j\right), m_{ij}(X)\right) \rightarrow \left(\left(Y, \mathcal{C}_j, \mathcal{C}_i\right), m_{ij}(Y)\right):$$

(i) F is lower m_{ij} -semicontinuous;
 (ii) For any $m_{ij}(Y)$ -open set V in Y , the set $F^-(V)$ is $m_{ij}(X)$ -semiopen in X

(iii) For any $m_{ij}(Y)$ -closed set K in Y , the set $F^+(K)$ is $m_{ij}(X)$ -semiclosed in X ;

$$(iv) \text{ Given any subset } B \text{ of } Y, m_{ij}(X) sCl(F^+(B)) \subseteq F^+(m_{ij}(Y) Cl(B));$$

$$(v) \text{ Given any every } A \in m_{ij}(X), F(m_{ij} sCl(A)) \subseteq Cl(F(A));$$

(vi) Given any subset $B \in Y$, the inclusion

$$F^-(m_{ij}(Y) Int(B)) \subseteq m_{ij}(X) sInt(F^-(B)) \text{ holds true.}$$

Proof. (i) \Rightarrow (ii): let V be a $m_{ij}(Y)$ -open set of Y such that $F^-(x) \cap V \neq \emptyset$ for a point $x \in X$. By the hypothesis, we can find an $m_{ij}(X)$ -semiopen set O with $x \in O$ such that for any $z \in O, F(z) \cap V \neq \emptyset$ holds. Since O is $m_{ij}(X)$ -semiopen, $O = m_{ij}(X) sInt(U)$. Because $\left(\left(Y, \mathcal{C}_j, \mathcal{C}_i\right), m_{ij}(Y)\right)$ satisfies property \mathcal{B} , we then obtain;

$$x \in m_{ij}(X) sInt(U) \subseteq m_{ij}(X) sInt(F^-(V)),$$

implies $F^-(V) \subseteq m_{ij}(X) sInt(F^-(V))$. By Proposition 3.13,

$$F^-(V) \supseteq m_{ij} sInt(F^-(V)) \text{ and so, } F^-(V) \text{ is } m_{ij}(X) \text{ semiopen.}$$

(ii) \Rightarrow (iii): If K is any $m_{ij}(Y)$ -closed set in Y then, $Y - K \in m_{ij}(Y) O(Y)$. Thus,

$$\begin{aligned} X \setminus F^+(K) &= F^-(Y \setminus K) = m_{ij}(X) sInt(F^-(Y \setminus K)) \\ &= m_{ij}(X) sInt(X \setminus F^+(K)) \\ &= X \setminus m_{ij}(X) sCl(F^+(K)). \end{aligned}$$

Which shows that, $F^+(K) = m_{ij}(X) sCl(F^+(K))$ implying $F^+(K)$ is $m_{ij}(X)$ -semiclosed.

(iii) \Rightarrow (iv): By Proposition 3.13 and Proposition 3.20, $m_{ij}(Y) Cl(B)$ is $m_{ij}(Y)$ -closed in Y for any subset B of Y . From (iii), we obtain that

$$\begin{aligned} &m_{ij}(X) sCl(F^+(B)) \\ &= \bigcap \{F^+(K) \subseteq X : F^+(B) \subseteq F^+(K) \text{ and } F^+(K) \in m_{ij} sC(X)\} \\ &\subseteq F^+\left(\bigcap \{K \subseteq Y : B \subseteq K \text{ and } K \in m_{ij} C(Y)\}\right) \\ &= F^+(m_{ij}(Y) Cl(B)) \end{aligned}$$

$$\text{Thus, } m_{ij}(X) sCl(F^+(B)) \subseteq F^+(m_{ij}(Y) Cl(B)).$$

(iv) \Rightarrow (v): For $A \subseteq X$ we have for (iv) that,

$$\begin{aligned} & F^+(m_{ij}(Y)Cl(F(A))) \\ &= F^+\left(\bigcap\{K \subseteq Y : F(A) \subseteq K \text{ and } K \in m_{ij}C(Y)\}\right) \\ &= \bigcap\{F^+(K) \subseteq X : A \subseteq F^+(K) \text{ and } F^+(K) \in m_{ij}sC(X)\} \\ &\supseteq \bigcap\{D \subseteq X : A \subseteq D \text{ and } D \in m_{ij}sC(X)\} \\ &= m_{ij}(X)sCl(A). \end{aligned}$$

Consequently, $F(m_{ij}(X)sCl(A)) \subseteq m_{ij}(Y)Cl(F(A))$.

(v) \Rightarrow (vi): Because $m_{ij}(Y)Int(B) = Y \setminus m_{ij}(Y)Cl(Y \setminus B)$ for a set $B \subseteq Y$, Proposition 3.13 and part (v), then implies:

$$\begin{aligned} F^-(m_{ij}(Y)Int(B)) &= F^-(Y \setminus m_{ij}(Y)Cl(Y \setminus B)) \\ &= X \setminus F^+(m_{ij}(Y)Cl(Y \setminus B)) \\ &\subseteq X \setminus m_{ij}(X)sCl(F^+(Y \setminus B)) \\ &= X \setminus m_{ij}(X)sCl(X \setminus F^-(B)) \\ &= m_{ij}(X)sInt(F^-(B)). \end{aligned}$$

Consequently, $F^+(m_{ij}(Y)Int(B)) \subseteq m_{ij}(X)sInt(F^-(B))$.

(vi) \Rightarrow (i): From (vi), we obtain for $x \in X$ and $m_{ij}(Y)$ -open set V in Y satisfying $F^+(x) \cap V \neq \emptyset$ that,

$$x \in F^-(V) = F^-(m_{ij}(Y)Int(V)) \subseteq m_{ij}(X)sInt(F^-(V)).$$

Thus, there is an $m_{ij}(X)$ -semiopen set O with $x \in O$ for which $F^+(z) \cap V \neq \emptyset$ for all $z \in O$. This implies $F^-(V)$ is $m_{ij}(X)$ -semiopen and so, F is a lower m_{ij} -semicontinuous multifunction. \square

Lemma 4.7. For a minimal bitopological space $((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij}(X))$, $i, j = 1, 2$; $i \neq j$, and for a subset A of X , the properties below holds:

- (i) $Cl_{m_j}(Int_{m_i}(A)) \subseteq Cl_{m_j}(Int_{m_i}(m_{ij}sCl(A))) \subseteq m_{ij}sCl(A)$.
- (ii) $m_{ij}sInt(A) \subseteq Int_{m_j}(Cl_{m_i}(m_{ij}sInt(A))) \subseteq Int_{m_j}(Cl_{m_i}(A))$.

Proof. (i) For $A \subseteq X$, Proposition 3.13, $m_{ij}sCl(A)$ is an m_{ij} -semiclosed set. Hence from Proposition 3.6, we have

$$Cl_{m_j}(Int_{m_i}(A)) \subseteq Cl_{m_j}(Int_{m_i}(m_{ij}sCl(A))) \subseteq m_{ij}sCl(A).$$

(ii) Similarly, by Proposition 3.13, $m_{ij}sInt(A)$ is an m_{ij} -semiopen set and by Proposition 3.6, we have

$$m_{ij}sInt(A) \subseteq Int_{m_j}(Cl_{m_i}(m_{ij}sInt(A))) \subseteq Int_{m_j}(Cl_{m_i}(A)). \quad \square$$

The concepts of Theorem 4.4 and 4.6 and, Lemma 4.7, leads us the theorem that follows:

Theorem 4.8. For a multifunction

$$F : ((X, \mathcal{T}_i, \mathcal{T}_j), m_{ij}(X)) \rightarrow ((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y)), \quad i, j = 1, 2; \quad i \neq j \quad \text{with}$$

$((Y, \mathcal{C}_j, \mathcal{C}_i), m_{ij}(Y))$ satisfying property \mathcal{S} , the statements below are equivalent

lent:

- (i) F is lower m_{ij} semicontinuous;
- (ii) $F^-(V) \subseteq Cl_{m_j} (Int_{m_i} (F^-(V)))$ for any $m_{ij}(Y)$ -open set V in Y ,
- (iii) $Int_{m_j} (Cl_{m_i} (F^-(K))) \subseteq F^-(K)$ for any $m_{ij}(Y)$ -closed set K in Y ,
- (iv) For $A \in X$, $F(Int_{m_j} (Cl_{m_i} (A))) \subseteq m_{ij}(Y)Cl(F(A))$;
- (v) $Int_{m_j} (Cl_{m_i} (F^-(B))) \subseteq F^-(m_{ij}(Y)Cl(B))$ for any subset B of Y ;
- (vi) $F^-(m_{ij}(Y)Int(B)) \subseteq Cl_{m_j} (Int_{m_i} (F^-(B)))$ for any subset B of Y .

Proof. (i) \Rightarrow (ii): From Theorem 4.6 and Proposition 3.6, we obtain

$$x \in m_{ij}(X) sInt(U) \subseteq m_{ij}(X) sInt(F^-(V)).$$

Consequently, $F^-(V) \subseteq Cl_{m_j} (Int_{m_i} (F^-(V)))$.

(ii) \Rightarrow (iii): If K is a $m_{ij}(Y)$ -closed subset in Y , then by Theorem 4.4 and Proposition 3.6,

$$\begin{aligned} X \setminus F^-(K) &= F^+(Y \setminus K) \subseteq m_{ij}(X) sInt(F^+(Y \setminus K)) \\ &= m_{ij}(X) sInt(X \setminus F^-(K)) \\ &= X \setminus m_{ij}(X) sCl(F^-(K)). \end{aligned}$$

Consequently, $F^-(K) \supseteq Cl_{m_j} (Int_{m_i} (F^-(K)))$.

(iii) \Rightarrow (iv): By Theorem 4.4 and Lemma 4.7, we obtain for a subset A of X that,

$$Int_{m_j} (Cl_{m_i} (A)) \subseteq m_{ij}(X) sCl(A) \subseteq F^-(Cl_{m_i} (F(A))).$$

This shows that, $F(Int_{m_j} (Cl_{m_i} (A))) \subseteq Cl_{m_i} (F(A))$.

(iv) \Rightarrow (v): From (iv) and Proposition 3.13, we obtain for any $B \subseteq Y$ that,

$$\begin{aligned} &F^-(m_{ij}(Y)Cl(B)) \\ &= F^-(\bigcap \{K \subseteq Y : B \subseteq K \text{ and } K \in m_{ij}C(Y)\}) \\ &= \bigcap \{F^-(K) \subseteq X : F^-(B) \subseteq F^-(K) \text{ and } F^-(K) \in m_{ij} sC(X)\} \\ &\supseteq \bigcap \{Q \subseteq X : F^-(B) \subseteq Q \text{ and } Q \in m_{ij} sC(X)\} \\ &= m_{ij}(X) sCl(F^-(B)). \end{aligned}$$

By Proposition 3.6, $Int_{m_j} (Cl_{m_j} (F^-(B))) \subseteq F^-(m_{ij}(Y)Cl(B))$.

(v) \Rightarrow (vi): From Proposition 3.13 and (v), one obtain

$$\begin{aligned} F^-(m_{ij}(Y)Int(B)) &= F^-(Y \setminus m_{ij}(Y)Cl(Y \setminus B)) \\ &= X \setminus F^+(m_{ij}(Y)Cl(Y \setminus B)) \\ &\subseteq X \setminus m_{ij}(X) sCl(F^+(Y \setminus B)) \\ &= X \setminus m_{ij}(X) sCl(X \setminus F^-(B)) \\ &= m_{ij}(X) sInt(F^-(B)). \end{aligned}$$

Thus, $F^-(m_{ij}(Y)Int(B)) \subseteq Cl_{m_j} (Int_{m_i} (F^-(B)))$.

(vi) \Rightarrow (i): For an $m_{ij}(Y)$ -open set V in Y we obtain from (vi) and Proposition 3.13 that,

$$F^-(V) = F^-(m_{ij}(Y) \text{Int}(V)) \subseteq Cl_{m_j} \left(\text{Int}_{m_i} \left(F^-(V) \right) \right).$$

As such, $F^-(V)$ is $m_{ij}(X)$ -semiopen in X and by (ii), F is lower m_{ij} -semi-continuous. \square

5. Conclusion

In the first part of this paper, we have investigated and then generalized Maki [3], Prasad [2] and Bose [4] extensions of the weak forms of sets: the semiopen sets, to asymmetric topologies with sets satisfying certain minimal conditions. In the second part of our work, we have to use Noiri and Popa's [10] ideas to successfully introduce and investigate several fundamental properties of upper and lower M -asymmetric semicontinuous multifunctions defined between sets satisfying some minimal conditions in the realm of bitopological spaces. We have observed that the M -asymmetric semicontinuous multifunctions have properties similar to those of upper and lower continuous functions and M -continuous multifunctions between topological spaces, with the only difference that, the semiopen sets use in our results belonging to two topologies.

Acknowledgements

The authors wish to acknowledge the support of Mulungushi University and the refereed authors for their helpful work towards this paper. They also thank the anonymous referees for their helpful comments towards the improvement of the paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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