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A Growth Behavior of Szegö Type Operators

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Abstract

We define new integral operators on the Haydy space similar to Szegö projection. We show that these operators map from H^p to H^2 for some $1 \leq p < 2$, where the range of p is depending on a growth condition. To prove that, we generalize the Hausdorff-Young Theorem to multi-dimensional case.

Keywords

Szegö Projection, Hausdorff-Young Theorem, Coefficient Multiplier, Stein Interpolation Theorem

1. Introduction

Let $\mathbf{C}^n$ denote the Euclidean space of complex dimension n. The inner product on $\mathbf{C}^n$ is given by

$$\langle z, w\rangle := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$, and the associated norm is $|z| := \sqrt{\langle z, z\rangle}$. The unit ball in $\mathbf{C}^n$ is the set

$$\mathbf{B}_n := \{z \in \mathbf{C}^n : |z| < 1\}$$

and its boundary is the unit sphere

$$\mathbf{S}_n := \{z \in \mathbf{C}^n : |z| = 1\}.$$

In case $n = 1$, denote $\mathbf{D}$ in place of $\mathbf{B}_1$. Let σ_n be the normalized surface measure on $\mathbf{S}_n$.

For $0 < p < \infty$, the Hardy space $H^p(\mathbf{B}_n)$ is the space of all holomorphic function f on $\mathbf{B}_n$ for which the "norm"

$$\|f\|_{H^p} := \left\{\sup_{0<r<1} \int_{\mathbf{S}_n} |f(r\zeta)|^p \, d\sigma_n(\zeta)\right\}^{1/p}$$

is finite. As is well-known, the space $H^p(\mathbf{B}_n)$ equipped with the norm above is a Banach space for $1 \leq p < \infty$. On the other hand, it is a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f,g) \mapsto \|f-g\|_{H^p}^p$.

For a function f in $H^p(\mathbf{B}_n)$, it is known that f has a radial limit f^* almost everywhere on $\mathbf{S}_n$. Here, the radial limit f^* of f is defined by

$$f^*(\zeta) := \lim_{r\to 1^-} f(r\zeta)$$

provided that the limit exists for $\zeta \in \mathbf{S}_n$. Moreover the mapping $f \mapsto f^*$ is an isometry from $H^p(\mathbf{B}_n)$ into $L^p(\mathbf{S}_n, d\sigma_n)$. Consequently, each $H^p(\mathbf{B}_n)$ can be identified with a closed subspace of $L^p(\mathbf{S}_n, d\sigma_n)$.

Since $H^2(\mathbf{B}_n)$ can be identified with a closed subspace of $L^2(\mathbf{S}_n, d\sigma_n)$, there exists an orthogonal projection from $L^2(\mathbf{S}_n, d\sigma_n)$ onto $H^2(\mathbf{B}_n)$. By using a reproducing kernel function, which is called the Szegö kernel, we also obtain a function f from its radial function f^*. More precisely,

$$f(z) = T[f](z) := \int_{\mathbf{S}_n} \frac{f^*(\zeta)}{(1-\langle z,\zeta\rangle)^n}\, d\sigma_n(\zeta)$$

for $f \in H^2(\mathbf{B}_n)$. We usually call this integral operator as the Szegö projection. It is well known that for $1 < p < \infty$ the Szegö projection maps $L^p(\mathbf{S}_n, d\sigma_n)$ boundedly onto $H^p(\mathbf{B}_n)$. For more details, we refer the classical text books [1, 2].

In this paper we consider a class of integral operators defined by

$$T_{m,N}[f](z) := \int_{\mathbf{S}_n} \frac{\langle z,\zeta\rangle^{m+N}}{(1-\langle z,\zeta\rangle)^m}\, f^*(\zeta)\, d\sigma_n(\zeta) \tag{1.1}$$

for $m = 1, 2, \ldots, n$ and a positive integer N. Compared with the Szegö projection, the growth condition in the denominator factor is better. Thus these operators are bounded on $H^2(\mathbf{B}_n)$.

Interestingly these operators map from $H^1(\mathbf{B}_n)$ to $H^2(\mathbf{B}_n)$ for any positive integer N when $1 \le m < \frac{n}{2}$. More precisely we have the following result.

Theorem 1.1. *Let m be a positive integer with $1 \le m < \frac{n}{2}$. Then there exists a constant $C = C(n) > 0$ such that*

$$\|T_{m,N}[f]\|_{H^2} \le C N^{m-\frac{n}{2}}\ \|f\|_{H^1}$$

for any positive integer N.

For $\frac{n}{2} \le m < n$, the operator $T_{m,N}$ maps from $H^p(\mathbf{B}_n)$ to $H^1(\mathbf{B}_n)$, but the range of p is depending on m, which determines the growth condition of the kernel function. Explicitly we have the following theorem.

Theorem 1.2. *Let m be a positive integer with $\frac{n}{2} \le m < n$ and $\frac{2n}{3n-2m} < p < 2$. Then there exists a constant $C = C(n,p) > 0$ such that*

$$\|T_{m,N}[f]\|_{H^2} \le C N^{p'} \|f\|_{H^p}$$

for any positive integer N. Here p' is a negative number defined by $p' := m + n(\frac{1}{p} - \frac{3}{2})$.

To prove the Theorem 1.2, we generalize the Hausdorff-Young Theorem to the multi-dimensional case using the Stein interpolation theorem.

2. Preliminary Results

We use the conventional multi-index notation. For a multi-index

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

with nonnegative integers α_i, the following are common notations;

$$|\alpha| := \alpha_1 + \cdots + \alpha_n,$$

$$\alpha! := \alpha_1! \cdots \alpha_n!.$$

For $z \in \mathbb{C}^n$, the monomial is defined as

$$z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

At first, we show that the Szegö type operators $T_{m,N}$ defined in (1.1) are actually coefficient multipliers.

Lemma 2.1. *Let m, N be positive integers with $1 \leq m \leq n$. For a multi-index α, there exists $\lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|)$ such that*

$$T_{m,N}[\zeta^\alpha](z) = \lambda_\alpha z^\alpha.$$

Proof. From the definition of $T_{m,N}$, we have

$$T_{m,N}[\zeta^\alpha](z) = \int_{\mathbf{S}_n} \frac{\langle z, \zeta\rangle^{m+N}\ \zeta^\alpha}{(1 - \langle z, \zeta\rangle)^m}\ d\sigma_n(\zeta)$$

for a multi-index α. Note that

$$\frac{1}{(1 - \langle z, \zeta\rangle)^m} = \sum_{k=0}^{\infty} \binom{k+m-1}{k} \langle z, \zeta\rangle^k.$$

Since the monomials are orthogonal on $L^2(\mathbf{S}_n, d\sigma_n)$; see ([1] Proposition 1.4.8), we have $T_{m,N}[\zeta^\alpha](z) = 0$ if $|\alpha| < m + N$. In case of $|\alpha| \geq m + N$, we have

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \int_{\mathbf{S}_n} \sum_{k=0}^{\infty} \binom{k+m-1}{k}\ \langle z, \zeta\rangle^{k+m+N} \zeta^\alpha\ d\sigma_n(\zeta) \\ &= \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \int_{\mathbf{S}_n} \langle z, \zeta\rangle^{|\alpha|}\ \zeta^\alpha\ d\sigma_n(\zeta). \end{aligned}$$

Expanding the term inside the above integral as

$$\langle z, \zeta\rangle^{|\alpha|} = \sum_{|\beta| = |\alpha|} \frac{|\alpha|!}{\beta!} z^\beta \overline{\zeta^\beta},$$

we obtain that

$$\begin{aligned} T_{m,N}[\zeta^\alpha](z) &= \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \frac{|\alpha|!}{\alpha!}\ z^\alpha \int_{\mathbf{S}_n} |\zeta^\alpha|^2\ d\sigma_n(\zeta) \\ &= \binom{|\alpha| - 1 - N}{|\alpha| - m - N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} z^\alpha, \end{aligned}$$

see ([1] Proposition 1.4.9) for the last equality. Putting λ_α as

$$\lambda_\alpha = \lambda_\alpha(m, n, N, |\alpha|) = \begin{cases} 0 & \text{for} \quad |\alpha| < m + N \\ \binom{|\alpha|-1-N}{|\alpha|-m-N} \frac{(n-1)!|\alpha|!}{(n-1+|\alpha|)!} & \text{for} \quad |\alpha| \geq m + N, \end{cases} \tag{2.1}$$

we conclude the lemma. □

To prove the main theorems, we need the Hausdorff-Young Theorem for the multi-dimensional Hardy space. For a holomorphic function f in the unit disk, we have the Taylor series expansion as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For the Hardy space defined in the unit disk, a relationship between the functions in $H^p(\mathbf{D})$ and the growth condition of their coefficients are given by the Hausdorff-Young Theorem, see ([3] p.76, Theorem A).

Theorem 2.2 (Hausdorff-Young Theorem for $H^p(\mathbf{D})$)**.** *For $1 \le p \le \infty$, let q be the conjugate exponent, with $\frac{1}{p} + \frac{1}{q} = 1$.*

1) *If $1 \le p \le 2$, then $f \in H^p(\mathbf{D})$ implies $\{a_n\} \in l^q$, and $\|\{a_n\}\|_{l^q} \le \|f\|_{H^p}$.*
2) *If $2 \le p \le \infty$, then $\{a_n\} \in l^q$ implies $f \in H^p(\mathbf{D})$, and $\|f\|_{H^p} \le \|\{a_n\}\|_{l^q}$.*

Before proceeding, we introduce some notation. Let $\mathbb{N}_0^n$ be the product set of nonnegative integers.

Define a weight function w_n on $\mathbb{N}_0^n$ by

$$w_n(\alpha) := \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$$

for $\alpha \in \mathbb{N}_0^n$. Using the weight w_n, we define a norm on $\mathbb{N}_0^n$ by

$$\|c\|_{p,t}^p := \sum_{\alpha \in \mathbb{N}_0^n} |c(\alpha)|^p \ w_n^t(\alpha)$$

for $1 \le p < \infty$ and a positive real number t. For $p = \infty$, we define

$$\|c\|_{\infty,t} := \sup_{\alpha \in \mathbb{N}_0^n} |c(\alpha)| w_n^t(\alpha).$$

Let $l^{p,t}$ be the collection of all function c defined on $\mathbb{N}_0^n$ with the norm $\|c\|_{p,t} < \infty$.

For a holomorphic function f on $\mathbf{B}_n$, whose Taylor series is given by

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

we define the coefficient function c_f of f defind on $\mathbb{N}_0^n$ by

$$c_f(\alpha) := c_\alpha.$$

Proposition 2.3 (Hausdorff-Young Theorem for $H^p(\mathbf{B}_n)$)**.** *For $1 \le p < \infty$, let q be the conjugate exponent, with $\frac{1}{p} + \frac{1}{q} = 1$.*

1) *If $p = 1$, then $f \in H^1(\mathbf{B}_n)$ implies $c_f \in l^{\infty,1}$, and $\|c_f\|_{\infty,1} \le \|f\|_{H^1}$.*
2) *If $1 < p \le 2$, then $f \in H^p(\mathbf{B}_n)$ implies $c_f \in l^{q,q-1}$, and $\|c_f\|_{q,q-1} \le \|f\|_{H^p}$.*
3) *If $2 \le p < \infty$, then $c_f \in l^{q,q-1}$ implies $f \in H^p(\mathbf{B}_n)$, and $\|f\|_{H^p} \le \|c_f\|_{q,q-1}$.*

Proof. For a multi-index α, we note that

$$\int_{\mathbf{S}_n} |\zeta^\alpha|^2 \, d\sigma_n(\zeta) = w_n(\alpha). \tag{2.2}$$

From the orthogonality of monomials on $\mathbf{S}_n$, we get

$$\int_{\mathbf{S}_n} f(\zeta)\overline{\zeta^\alpha} \, d\sigma_n(\zeta) = c_f(\alpha) w_n(\alpha),$$

for $f \in H^1(\mathbf{B}_n)$. Thus we obtain

$$\|c_f\|_{\infty,1} \leq \|f\|_{H^1},$$

which prove the Proposition (1).

$$\|c_f\|_{\infty,1} \leq \|f\|_{H^1}, \qquad \|c_f\|_{2,1} \leq \|f\|_{H^2}, \tag{2.3}$$

and proved (1). Now we apply an interpolation theorem in the Equations (2.3). Since the defined norms have weight functions, we use the Stein interpolation theorem; see ([4] Theorem 3.6). Then we have

$$\|c_f\|_{q,q-1} \leq \|f\|_{H^p}$$

for $1 < p \leq 2$ and proved (2).

Now we prove (3). For $1 < q \leq 2$ we let c_f be given with $\|c_f\|_{q,q-1} < \infty$ and define

$$f_k(z) := \sum_{|\alpha| \leq k} c_\alpha z^\alpha$$

for a positive integer k. Since each $f_k \in H^p(\mathbf{B}_n)$, for any $g \in H^q(\mathbf{B}_n)$ with coefficient function c_g we have

$$\begin{aligned}
&\left| \int_{\mathbb{S}_n} f_k \bar{g} \, d\sigma_n \right| = \left| \sum_{|\alpha| \leq k} c_f(\alpha) c_g(\alpha) w_n(\alpha) \right| \\
&\leq \left(\sum_\alpha |c_f(\alpha)|^q w_n(\alpha)^{q-1} \right)^{1/q} \left(\sum_\alpha |c_g(\alpha)|^p w_n(\alpha)^{p-1} \right)^{1/p} \\
&\leq \|c_f\|_{q,q-1} \|g\|_{H^q}
\end{aligned}$$

by above proved Proposition (2). Since for $1 < p < \infty$ the dual space of $H^p(\mathbf{B}_n)$ is $H^q(\mathbf{B}_n)$, we have

$$\|f_k\|_{H^p} \leq \|c_f\|_{q,q-1}$$

for any positive integer k. Moreover $w_n \leq 1$ and $q \leq 2$ implies that

$$\|c_f\|_{q,q/2} \leq \|c_f\|_{q,q-1} < \infty,$$

so we have

$$\|f\|_{H^2} = \|c_f\|_{2,1} < \infty.$$

Consequently $\|f - f_k\|_{H^2}$ goes to zero as k increase. Hence f_k converges to f pointwise and by applying Fatou's lemma we finish the proof. □

3. Proofs

For a holomorphic function f on $\mathbf{B}_n$ with Taylor series

$$f(z) = \sum_{\alpha} c_\alpha z^\alpha,$$

we define j^{th} partial sum of f by

$$S_j f(z) := \sum_{|\alpha|\le j} c_\alpha z^\alpha \tag{3.1}$$

for a positive integer j.

Now we are ready to prove the Theorem 1.1. Here we restate the Theorem 1.1 for convenience.

Theorem 3.1. *For $1 \le m < \frac{n}{2}$, there exists a constant $C = C(n)$ depending only on n such that*

$$\|T_{m,N}[f]\|_{H^2} \le C N^{m-\frac{n}{2}} \ \|f\|_{H^1}$$

for any positive integer N.

Proof. For a given $z \in \mathbf{B}_n$, the kernel function $\frac{\langle z,\zeta\rangle^{m+N}}{(1-\langle z,\zeta\rangle)^m}$ is bounded in $\zeta \in \mathbf{S}_n$. So for $p > 1$, we have

$$\begin{aligned}
&|T_{m,N}[f](z) - T_{m,N}[S_j f](z)|^p \\
&\le \int_{\mathbf{S}_n} \left| \frac{\langle z,\zeta\rangle^{m+N}}{(1-\langle z,\zeta\rangle)^m} \right|^p |f(\zeta) - S_j f(\zeta)|^p \ d\sigma_n(\zeta) \\
&\le C_z \int_{\mathbf{S}_n} |f(\zeta) - S_j f(\zeta)|^p \ d\sigma_n(\zeta) \\
&= C_z \ \|f - S_j f\|_{H^p}^p,
\end{aligned}$$

where C_z is a constant depending on $z \in \mathbf{B}_n$. However the Talyor series $S_j f$ converges to f in the norm H^p when $p > 1$; we refer ([5] Theorem 1.1). Thus for $f \in H^p(\mathbf{B}_n)$ with $p > 1$ we have

$$\begin{aligned}
T_{m,N}[f](z) &= \lim_{j\to\infty} T_{m,N}\left[\sum_{|\alpha|\le j} c_f(\alpha)\zeta^\alpha\right](z) \\
&= \lim_{j\to\infty} \sum_{|\alpha|\le j} c_f(\alpha) \ T_{m,N}[\zeta^\alpha](z) \\
&= \sum_{\alpha} c_f(\alpha) \ \lambda(m,n,N,|\alpha|) \ z^\alpha,
\end{aligned}$$

where we used the Lemma 2.1. So we have

$$\begin{aligned}
\|T_{m,N}[f]\|_{H^2}^2 &= \sum_{|\alpha|\ge m+N} \lambda(m,n,N,|\alpha|)^2 |c_f(\alpha)|^2 w_n(\alpha) \qquad (3.2)\\
&\le \|c_f\|_{\infty,1}^2 \cdot \sum_{|\alpha|\ge m+N} \lambda(m,n,N,|\alpha|)^2 w_n(\alpha)^{-1}
\end{aligned}$$

By Proposition 2.3 and the Equation (2.1), we have

$$\|T_{m,N}[f]\|_{H^2}^2 \le C(n) \ \|f\|_{H^1}^2 \cdot \sum_{|\alpha|\ge m+N} \frac{1}{|\alpha|^{2(n-m)}} \frac{(n-1+|\alpha|)!}{\alpha!}$$

where $C(n)$ is a constant depending on n. By the multinomial theorem we know that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{2^n}{k!},$$

so we get

$$\begin{aligned}
&\sum_{|\alpha|\geq m+N} \frac{1}{|\alpha|^{2(n-m)}} \frac{(n-1+|\alpha|)!}{\alpha!} \\
&\leq C(n) \sum_{k=m+N}^{\infty} \frac{1}{k^{2(n-m)}} \frac{(n-1+k)!}{k!} \\
&\leq C(n) \sum_{k=N}^{\infty} \frac{1}{k^{n-2m+1}} \\
&\leq C(n) N^{2m-n},
\end{aligned}$$

where the series is bounded by the assumption $m < \frac{n}{2}$. And the constant $C(n)$ is also depending only on n. Thus we have

$$\|T_{m,N}[f]\|_{H^2} \leq C(n) N^{m-\frac{n}{2}} \|f\|_{H^1},$$

for $f \in H^p(\mathbf{B}_n)$ with $p > 1$.

Now we fix $p_0 > 1$. Since H^{p_0} is dense in H^1 and $T_{m,N}$ is uniformly continuous from H^{p_0} to H^2, there exists the unique continuous extension $\widetilde{T}_{m,N}$ from H^1 to H^2 defined as

$$\widetilde{T}_{m,N}[f] := \lim_{k\to\infty} T_{m,N}[f_k],$$

for any $f_k \to f$ with each f_k is in H^{p_0}. In particular, $\widetilde{T}_{m,N}[f] = \lim_{r\to 1^-} T_{m,N}[f^r]$ where $f^r(z) := f(rz)$ for $0 < r < 1$. By Fatou's lemma we have $\widetilde{T}_{m,N}[f](z) = \lim_{r\to 1^-} T_{m,N}[f^r](z)$ for any $z \in \mathbf{B}_n$.

Moreover using the similar argument in the beginning of the proof, for any $z \in \mathbf{B}_n$ we have

$$|T_{m,N}[f](z) - T_{m,N}[f^r](z)| \leq C_z \|f - f^r\|_{H^1},$$

where C_z is a constant depending on z. Since $\|f - f^r\|_{H^1} \to 0$ as $r \to 1^-$, we also have $T_{m,N}[f](z) = \lim_{r\to 1^-} T_{m,N}[f^r](z)$. Consequently we have $\widetilde{T}_{m,N}[f](z) = T_{m,N}[f](z)$ for any $f \in H^1$ and

$$\|T_{m,N}[f]\|_{H^2} \leq C(n) N^{m-\frac{n}{2}} \|f\|_{H^1},$$

for any $f \in H^1(\mathbf{B}_n)$. □

Since $T_{m,N}$ is an integral operator with conjugate symmetric kernel, we have the following corollary by using its adjoint operator.

Corollary 3.2. *For $1 \leq m < \frac{n}{2}$, there exists a constant $C = C(n)$ depending only on n such that*

$$\|T_{m,N}[f]\|_{BMO} \leq C N^{m-\frac{n}{2}} \|f\|_{H^2}$$

for any positive integer N. Here $\|\cdot\|_{BMO}$ means the BMOA norm.

Remark. For $0 < p_1 < p_2 < \infty$, we have

$$\|f\|_{H^{p_1}} \leq \|f\|_{H^{p_2}} \leq \|f\|_{BMO}.$$

By Theorem 3.1 and Corollary 3.2 we obtain that if $1 \leq m < \frac{n}{2}$,

$$\|T_{m,N}[f]\|_{H^p(\text{or } BMO)} \lesssim \frac{\|f\|_{H^p(\text{or } BMO)}}{N^{\frac{n}{2}-m}},$$

for $1 \leq p < \infty$. That is, the Szegö type operators makes the norm decrease quickly as N goes large when $m < \frac{n}{2}$.

We prove the Theorem 1.2. Here we restate the Theorem 1.2 for convenience.

Theorem 3.3. *Let $\frac{n}{2} \leq m < n$ and $\frac{2n}{3n-2m} < p < 2$. Then there exists a constant $C = C(n,p)$ such that*

$$\|T_{m,N}[f]\|_{H^2} \leq CN^{p'}\|f\|_{H^p}$$

for any positive integer N. Here p' is a negative number defined by $p' := m + n(\frac{1}{p} - \frac{3}{2})$.

Proof. We begin with the Equation (3.2). Then we have

$$\begin{aligned}\|T_{m,N}[f]\|^2_{H^2} &= \sum_{|\alpha|\geq m+N} \lambda(m,n,N,|\alpha|)^2 |c_f(\alpha)|^2 w_n(\alpha) \\ &\leq \left(\sum_{|\alpha|\geq m+N} |c_f(\alpha)|^q w_n(\alpha)^{q-1}\right)^{2/q} \\ &\quad \left(\sum_{|\alpha|\geq m+N} \lambda(m,n,N,|\alpha|)^{2r} w_n(\alpha)^{-1}\right)^{1/r},\end{aligned}$$

where q is the conjugate index of p and r is of $q/2$. By Proposition 2.3 and the Equation (2.1), we have

$$\|T_{m,N}[f]\|^2_{H^2} \leq C(n)\ \|f\|^2_{H^p} \cdot \left(\sum_{|\alpha|\geq m+N} \frac{1}{|\alpha|^{2r(n-m)}} \frac{(n-1+|\alpha|)!}{\alpha!}\right)^{1/r},$$

where $C(n)$ is a constant depending on n. By the multinomial theorem, we know that

$$\sum_{|\alpha|=k} \frac{1}{\alpha!} = \frac{2^n}{k!}.$$

So we get

$$\begin{aligned}&\sum_{|\alpha|\geq m+N} \frac{1}{|\alpha|^{2r(n-m)}} \frac{(n-1+|\alpha|)!}{\alpha!} \\ &\leq C(n) \sum_{k=m+N}^{\infty} \frac{1}{k^{2r(n-m)}} \frac{(n-1+k)!}{k!} \\ &\leq C(n) \sum_{k=N}^{\infty} \frac{1}{k^{2r(n-m)-n+1}},\end{aligned}$$

where a constant $C(n)$ is depending on n. Since $p > \frac{2n}{3n-2m}$, we have $r > \frac{n}{2(n-m)}$. So the above series converges and bounded by

$$\frac{N^{-2r(n-m)+n}}{2r(n-m)-n}.$$

Thus we prove

$$\|T_{m,N}[f]\|_{H^2} \leq C(n,p)\ N^{m+n(\frac{1}{p}-\frac{3}{2})}\|f\|_{H^p}.$$

□

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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