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Generating Sets of the Complete Semigroup of Binary Relations Defined by Semilattices of the Class $\Sigma_8(X,5)$

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Abstract

In this article, we study generating sets of the complete semigroups of binary relations defined by X-semilattices of unions of the class $\Sigma_8(X,5)$. Found uniquely irreducible generating set for the given semigroups and when X is finite set formulas for calculating the number of elements in generating sets are derived.

Keywords

Semigroup, Semilattice, Binary Relation

1. Introduction

Let $X \neq \varnothing$, D is an X-semilattice of unions which is closed with respect to the set-theoretic union of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such α_f is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We denote by $\varnothing$ an empty subset of the set X or an empty binary relation. The condition $(x,y) \in \alpha$ will be written in the form $x\alpha y$.

Let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $\breve{D} = \bigcup_{Y \in D} Y$ and $T \in D$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^* and $V(X^*,\alpha)$ the following sets:

$$y\alpha = \{x \in X \mid y\alpha x\},\ Y\alpha = \bigcup_{y \in Y} y\alpha,\ V(D,\alpha) = \{Y\alpha \mid Y \in D\},$$
$$X^* = \{Y \mid \varnothing \ne Y \subseteq X\},\ V(X^*,\alpha) = \{Y\alpha \mid \varnothing \ne Y \subseteq X\},$$
$$D_T = \{Z \in D \mid T \subseteq Z\},\ \ Y_T^\alpha = \{y \in X \mid y\alpha = T\}.$$

Theorem 1.1. *Let* $D = \{\breve{D}, Z_1, Z_2, \cdots, Z_{m-1}\}$ *be some finite X-semilattice of unions and* $C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$ *be the family of sets of pairwise nonintersecting subsets of the set X (the set* $\varnothing$ *can be repeated several times). If* φ *is a mapping of the semilattice D on the family of sets* $C(D)$ *which satisfies the conditions*

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & \cdots & Z_{m-1} \\ P_0 & P_1 & P_2 & \cdots & P_{m-1} \end{pmatrix}$$

and $\hat{D}_Z = D \setminus D_Z$, *then the following equalities are valid*:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},$$
$$Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T).$$

In the sequel these equalities will be called formal. The parameters P_i $(0 < i \le m-1)$ there exist such parameters that cannot be empty sets for *D*. Such sets P_i are called bases sources, where sets P_j $(0 \le j \le m-1)$, which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a bases source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1] Theorem 1.1, [2] [3] chapter 11).

Definition 1.1. *The representation* $\alpha = \bigcup_{T \in D} (Y_T^\alpha \times T)$ *of binary relation* α *is called quasinormal, if* $\bigcup_{T \in D} Y_T^\alpha = X$ *and* $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$ *for any* $T, T' \in D$, $T \ne T'$ (see [1] Definition 1.2, [2], [3] chapter 1.1).

Definition 1.2. *Let* $\alpha, \beta \subseteq X \times X$. *Their product* $\delta = \alpha \circ \beta$ *is defined as follows:* $x\delta y$ $(x, y \in X)$ *if there exists an element* $z \in X$ *such that* $x\alpha z\beta y$ (see [1] Definition 1.3, [1], chapter 1.3).

Definition 1.3. *We say that an element* α *of the semigroup* $B_X(D)$ *is external if* $\alpha \ne \delta \circ \beta$ *for all* $\delta, \beta \in B_X(D) \setminus \{\alpha\}$ (see [1] Definition 1.1, [2] [3] Definition 1.15.1).

It is well known, that if *B* is all external elements of the semigroup $B_X(D)$ and B' is any generated set for the $B_X(D)$, then $B \subseteq B'$ (see [2] [3] Lemma 1.15.1).

2. Result

Let $\Sigma_8(X,5)$ be a class of all *X*-semilattices of unions, whose every element is isomorphic to an *X*-semilattice of unions $D = \{T_4, T_3, T_2, T_1, T_0\}$, which satisfies the condition:

$$T_4 \subset T_2 \subset T_0,\ T_3 \subset T_1 \subset T_0,\ T_4 \setminus T_3 \neq \varnothing,\ T_3 \setminus T_4 \neq \varnothing,$$
$$T_2 \setminus T_1 \neq \varnothing,\ T_1 \setminus T_2 \neq \varnothing,\ T_2 \setminus T_3 \neq \varnothing,\ T_3 \setminus T_2 \neq \varnothing,$$
$$T_4 \setminus T_1 \neq \varnothing,\ T_1 \setminus T_4 \neq \varnothing,\ T_4 \cup T_3 = T_4 \cup T_1 = T_3 \cup T_2 = T_1 \cup T_2 = T_0.$$

(see **Figure 1**). It is easy to see that $\tilde{D} = \{T_4, T_3, T_2, T_1\}$ is irreducible generating set of the semilattice *D*.

Let $C(D) = \{P_0, P_1, P_2, P_3, P_4\}$ is a family of sets, where $\varphi = \begin{pmatrix} T_0 & T_1 & T_2 & T_3 & T_4 \\ P_0 & P_1 & P_2 & P_3 & P_4 \end{pmatrix}$ is a mapping of the semilattice D onto the family of sets $C(D)$ and P_0, P_1, P_2, P_3, P_4 are pairwise disjoint subsets of the set *X*. Then the formal equalities of the semilattice D have a form:

$$\begin{aligned} T_0 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4, \\ T_1 &= P_0 \cup P_2 \cup P_3 \cup P_4, \\ T_2 &= P_0 \cup P_1 \cup P_3 \cup P_4, \\ T_3 &= P_0 \cup P_2 \cup P_4, \\ T_4 &= P_0 \cup P_1 \cup P_3. \end{aligned} \tag{2.1}$$

Here the element P_0 is source of completeness and the elements P_4, P_3, P_2, P_1 are basis sources of the semilattice *D*. Therefore $|X| \geq 4$ since $|P_4| \geq 1$, $|P_3| \geq 1$, $|P_2| \geq 1$, $|P_1| \geq 1$ (see Theorem 1.1).

From the formal Equalities (2.1) immediately follows

$$\begin{aligned} &P_4 = T_2 \setminus T_4,\ P_3 = (T_2 \cap T_1) \setminus T_3, \\ &P_2 = T_3 \setminus T_2 = T_0 \setminus T_2,\ P_1 = T_4 \setminus T_1,\ P_0 = T_4 \cap T_3. \end{aligned} \tag{2.2}$$

2.1. Generating Sets of the Complete Semigroup of Binary Relations Defined by Semilattices of the Class $\Sigma_8(X,5)$, When $T_4 \cap T_3 \neq \varnothing$

In the sequel, we denoted all semilattices $D = \{T_4, T_3, T_2, T_1, T_0\}$ of the class $\Sigma_8(X,5)$ by symbol $\Sigma_{8.0}(X,5)$, for which $T_4 \cap T_3 \neq \varnothing$. Of the last inequality from the formal Equalities (2.1) of a semilattise D follows that $T_4 \cap T_3 = P_0 \neq \varnothing$, *i.e.* $|X| \geq 5$.

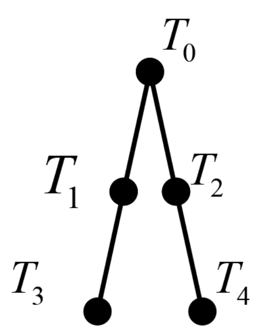

Figure 1. Diagram of the semilattice D.

We denoted by symbols $\mathfrak{A}_4,\mathfrak{A}_3,\mathfrak{A}_2,\mathfrak{A}_1$ the following sets:

$$\mathfrak{A}_4=\{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\}\},$$
$$\mathfrak{A}_3=\{\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\},\{T_4,T_2,T_0\},\{T_3,T_1,T_0\},\{T_2,T_1,T_0\}\},$$
$$\mathfrak{A}_2=\{\{T_4,T_2\},\{T_4,T_0\},\{T_3,T_1\},\{T_3,T_0\},\{T_2,T_0\}\{T_1,T_0\}\},$$
$$\mathfrak{A}_1=\{\{T_4\},\{T_3\},\{T_2\},\{T_1\},\{T_0\}\}.$$

Lemma 2.1.1. *Let* $D\in\Sigma_{8.0}(X,5)$. *Then the following statements are true*:

a) *Let* $T_3,T_4\in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$;

b) *Let* $Z\in\{T_2,T_1\}$, $Z'\in\{T_4,T_3\}$. *If* $Z'\not\subset Z$ *and* $Z,Z'\in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$;

c) *Let* $Z,Z'\in\{T_2,T_1\}$ *and* $Z\neq Z'$. *If* $V(D,\alpha)=\{T_2,T_1,T_0\}$, *then* α *is external element of the semigroup* $B_X(D)$.

Proof. Let $\alpha=\delta\circ\beta$ for some $\delta,\beta\in B_X(D)\setminus\{\alpha\}$. If quasinormal representation of binary relation δ has a form

$$\delta=(Y_4^\delta\times T_4)\cup(Y_3^\delta\times T_3)\cup(Y_2^\delta\times T_2)\cup(Y_1^\delta\times T_1)\cup(Y_0^\delta\times T_0),$$

then

$$\alpha=\delta\circ\beta=(Y_4^\delta\times T_4\beta)\cup(Y_3^\delta\times T_3\beta)\cup(Y_2^\delta\times T_2\beta)\cup(Y_1^\delta\times T_1\beta)\cup(Y_0^\delta\times T_0\beta).\quad(2.1.1)$$

From the formal Equalities (1) of the semilattice D we obtain that:

$$\begin{aligned}T_0\beta&=P_0\beta\cup P_1\beta\cup P_2\beta\cup P_3\beta\cup P_4\beta,\\T_1\beta&=P_0\beta\cup P_2\beta\cup P_3\beta\cup P_4\beta,\\T_2\beta&=P_0\beta\cup P_1\beta\cup P_3\beta\cup P_4\beta,\\T_3\beta&=P_0\beta\cup P_2\beta\cup P_4\beta,\\T_4\beta&=P_0\beta\cup P_1\beta\cup P_3\beta.\end{aligned}\quad(2.1.2)$$

where $P_k\beta\neq\varnothing$ for any $P_k\neq\varnothing$ $(k=0,1,2,3,4)$ and $\beta\in B_X(D)$. Indeed, by preposition $P_k\neq\varnothing$ for any $k=0,1,2,3,4$ and $\beta\neq\varnothing$ since $\varnothing\notin D$. Let $y\in P_k$ for some $y\in X$. Then $y\in T_0$, $\beta=\alpha_f$ for some $f:X\to D$ and $\alpha_f=\bigcup_{x\in X}(\{x\}\times f(x))\supseteq\{y\}\times f(y)$, *i.e.* there exists an element $t\in f(y)$ for which $y\alpha_f t$ and $y\beta t$. Of this and by definition of a set $P_k\beta$ we obtain that $t\in P_k\beta$ since $y\in P_k$, $y\beta t$. Thus, we have that $P_k\beta\neq\varnothing$, *i.e.* $P_k\beta\in D$ for any $k=0,1,2,3,4$.

Now, let $T_i\beta=Z$ and $T_j\beta=Z'$ for some $0\le i\neq j\le 4$ and $Z\neq Z'$, $Z,Z'\in\{T_4,T_3\}$, then from the Equalities (2.2) follows that $Z=P_0\beta=Z'$ since Z and Z' are minimal elements of the semilattice *D*. The equality $Z=Z'$ contradicts the inequality $Z\neq Z'$.

The statement a) of the Lemma 2.1.1 is proved.

Let $T_i\beta=Z'$, where $Z'\in\{T_4,T_3\}$ and $T_j\beta=Z$, where $Z\in\{T_2,T_1\}$ for some $0\le i\neq j\le 4$. If $0\le i\le 4$, then from the formal equalities of a semilattice D we obtain that

$$T_0\beta = P_0\beta \cup P_1\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta = P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = Z',$$
$$T_1\beta = P_0\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta = P_0\beta = P_2\beta = P_3\beta = P_4\beta = Z',$$
$$T_2\beta = P_0\beta \cup P_1\beta \cup P_3\beta \cup P_4\beta = P_0\beta = P_1\beta = P_3\beta = P_4\beta = Z',$$
$$T_3\beta = P_0\beta \cup P_2\beta \cup P_4\beta = P_0\beta = P_2\beta = P_4\beta = Z',$$
$$T_4\beta = P_0\beta \cup P_1\beta \cup P_3\beta = P_0\beta = P_1\beta = P_3\beta = Z'.$$

since Z' is minimal element of the semilattice *D*. Now, let $i \neq j$.

1) If $T_0\beta = P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = Z'$ and $j = 1,2,3,4$, then we have

$$Z = T_1\beta = T_2\beta = T_3\beta = T_4\beta = Z',$$

which contradicts the inequality $Z \neq Z'$.

2) If $T_1\beta = P_0\beta = P_2\beta = P_3\beta = P_4\beta = Z'$ and $j = 0,2,3,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta, \text{ where } P_1\beta \in D$$

Last equalities are impossible, since $Z \neq Z' \cup T$ for any $T \in D$ and $Z \neq Z'$ by definition of a semilattice *D*.

3) If $T_2\beta = P_0\beta = P_1\beta = P_3\beta = P_4\beta = Z'$ and $j = 0,1,3,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta, \text{ where } P_1\beta \in D$$

Last equalities are impossible since $Z \neq Z' \cup T$ for any $T \in D$ and $Z \neq Z'$ by definition of a semilattice *D*.

4) If $T_3\beta = P_0\beta = P_2\beta = P_4\beta = Z'$ and $j = 0,1,2,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta \cup P_3\beta,$$
$$Z = T_1\beta = Z' \cup P_3\beta, \text{ where } P_1\beta, P_3\beta \in D$$

Last equalities are impossible since $Z \neq Z' \cup T \cup T'$ and $Z \neq Z' \cup T$ for any $T, T' \in D$, by definition of a semilattice *D*.

5) If $T_4\beta = P_0\beta = P_1\beta = P_3\beta = Z'$ and $j = 0,1,2,3$, then we have

$$Z = T_0\beta = T_1\beta = T_3\beta = Z' \cup P_2\beta \cup P_4\beta,$$
$$Z = T_2\beta = Z' \cup P_4\beta, \text{ where } P_2\beta, P_4\beta \in D$$

Last equalities are impossible since $Z \neq Z' \cup T \cup T'$ and $Z \neq Z' \cup T$ for any $T, T' \in D$, by definition of a semilattice *D*.

The statement b) of the Lemma 2.1.1 is proved.

Let $Z, Z' \in \{T_2, T_1\}$, $T_i\beta = Z$, $T_j\beta = Z'$ and $Z \neq Z'$. If $T_i\beta = Z$ where $0 \le i \neq j \le 4$, we consider the following cases:

6) $i = 0, j = 1,2,3,4$. Then from the Equality (2.1.2) follows that $Z \subset Z'$, which contradicts the definition of a semilattice *D*;

7) $i = 1, j = 0,2,3,4$.

If $i = 1, j = 0,3$. Then from the Equality (2.1.2) follows that $Z' \subset Z$, or $Z \subset Z'$ which contradicts the definition of a semilattice *D*;

If $i = 1, j = 2,4$. Then from the Equality (1.4) follows that

$$\begin{cases} T_1\beta = (P_0\beta \cup P_3\beta \cup P_4\beta) \cup P_2\beta, \\ T_2\beta = (P_0\beta \cup P_3\beta \cup P_4\beta) \cup P_1\beta, \end{cases}$$

where $P_0\beta \cup P_3\beta \cup P_4\beta,\ P_2\beta,\ P_1\beta \in D$, *i.e.* there exists such elements $T, T', T'' \in D$, for which $Z = T \cup T'$ and $Z' = T \cup T''$. But such element $T \in D$

don't exist by definition of a semilattice *D*.

8) $i=2, j=0,1,3,4$.

If $i=2, j=0,4$. Then from the Equality (2.1.2) follows that $Z'\subset Z$, or $Z\subset Z'$ which contradicts the definition of a semilattice *D*;

If $i=2, j=1,3$. In this case analogously for the case 7) we may prove that $Z=T\cup T'$ and $Z'=T\cup T''$. But such element $T\in D$ don't exist by definition of a semilattice *D*.

9) $i=3, j=0,1,2,4$.

If $i=3, j=0,1$. Then from the Equality (2.1.2) follows that $Z'\subset Z$, which contradicts the definition of a semilattice *D*;

If $i=3, j=2,4$. Then from the Equality (2.1.2) follows that

$$\begin{cases} T_2\beta = P_0\beta\cup\left(P_2\beta\cup P_3\beta\cup P_4\beta\right), \\ T_2\beta = P_0\beta\cup\left(P_1\beta\cup P_3\beta\right), \end{cases}$$

where $P_0\beta, P_2\beta\cup P_3\beta\cup P_4\beta, P_1\beta\cup P_3\beta\in D$, *i.e.* there exist such elements $T,T',T''\in D$, for which $Z=T\cup T'$ and $Z'=T\cup T''$. But such element $T\in D$ don't exist by definition of a semilattice *D*.

10) $i=4, j=0,1,2,3$.

If $i=4, j=0,2$. Then from the Equality (2.1.2) follows that $Z\subset Z'$ which contradicts the definition of a semilattice *D*;

If $i=4, j=1,3$. Then from the Equality (2.1.2) follows that

$$\begin{cases} T_1\beta = P_0\beta\cup\left(P_2\beta\cup P_3\beta\cup P_4\beta\right), \\ T_3\beta = P_0\beta\cup\left(P_2\beta\cup P_4\beta\right), \end{cases}$$

where $P_0\beta, P_2\beta\cup P_3\beta\cup P_4\beta, P_2\beta\cup P_4\beta\in D$, *i.e.* there exist such elements $T,T',T''\in D$, for which $Z=T\cup T'$ and $Z'=T\cup T''$. But such element $T\in D$ do not exist by definition of a semilattice *D*.

The statement c) of the Lemma 2.1.1 is proved.

Lemma 2.1.1 is proved.

Let $D\in\Sigma_{8.0}(X,5)$. By symbols $\mathfrak{A}_0$, $B(\mathfrak{A}_0)$ and B_0 we denoted the following sets:

$$\begin{aligned} \mathfrak{A}_0 = \{ & \{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\}, \\ & \{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\},\{T_2,T_1,T_0\}\}, \end{aligned}$$

$$B(\mathfrak{A}_0)=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)\in\mathfrak{A}_0\right\}; B_0=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)=D\right\}.$$

Remark, that the sets B_0 and $B(\mathfrak{A}_0)$ are external elements for the semigroup $B_X(D)$.

Lemma 2.1.2. *Let* $D\in\Sigma_{8.0}(X,5)$. *Then the following statements are true*:

a) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha=\left(Y_4^\alpha\times T_4\right)\cup\left(Y_2^\alpha\times T_2\right)\cup\left(Y_0^\alpha\times T_0\right),$$

where $Y_4^\alpha,Y_2^\alpha,Y_0^\alpha\notin\{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha=\left(Y_3^{\alpha}\times T_3\right)\cup\left(Y_1^{\alpha}\times T_1\right)\cup\left(Y_0^{\alpha}\times T_0\right),$$

where $Y_3^{\alpha},Y_1^{\alpha},Y_0^{\alpha}\notin\{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

Proof. 1). Let quasinormal representation of binary relations δ and β have a form

$$\delta=\left(Y_4^{\delta}\times T_4\right)\cup\left(Y_2^{\delta}\times T_2\right)\cup\left(Y_1^{\delta}\times T_1\right)\cup\left(Y_0^{\delta}\times T_0\right),$$
$$\beta=\left(T_4\times T_4\right)\cup\left(\left(T_2\setminus T_4\right)\times T_2\right)\cup\left(\left(T_0\setminus T_2\right)\times T_1\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_4^{\alpha},Y_2^{\alpha},Y_1^{\alpha}\notin\{\varnothing\}$,

$$\begin{aligned}&T_4\cup\left(T_2\setminus T_4\right)\cup\left(T_0\setminus T_2\right)\cup\left(X\setminus T_0\right)\\&=\left(P_0\cup P_1\cup P_3\right)\cup P_4\cup P_2\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X,\end{aligned}$$

(see Equalities (2.1) and (2.2)), then $\delta,\beta\in B(\mathfrak{A}_0)$ and

$$\begin{aligned}&T_4\beta=T_4,\ T_2\beta=\left(P_0\cup P_1\cup P_3\cup P_4\right)\beta=T_4\cup T_2=T_2,\\&T_1\beta=\left(P_0\cup P_2\cup P_3\cup P_4\right)\beta=T_4\cup T_1=T_0,\ T_0\beta=T_0.\\&\alpha=\delta\circ\beta=\left(Y_4^{\delta}\times T_4\beta\right)\cup\left(Y_2^{\delta}\times T_2\beta\right)\cup\left(Y_1^{\delta}\times T_1\beta\right)\cup\left(Y_0^{\delta}\times T_0\beta\right)\\&=\left(Y_4^{\delta}\times T_4\right)\cup\left(Y_2^{\delta}\times T_2\right)\cup\left(Y_1^{\delta}\times T_0\right)\cup\left(Y_0^{\delta}\times T_0\right)\\&=\left(Y_4^{\delta}\times T_4\right)\cup\left(Y_2^{\delta}\times T_2\right)\cup\left(\left(Y_1^{\delta}\cup Y_0^{\delta}\right)\times T_0\right)=\alpha,\end{aligned}$$

if $Y_4^{\delta}=Y_4^{\alpha}$, $Y_2^{\delta}=Y_2^{\alpha}$ and $Y_1^{\delta}\cup Y_0^{\delta}=Y_0^{\alpha}$. Last equalities are possible since $\left|Y_1^{\delta}\cup Y_0^{\delta}\right|\geq 1$ ($\left|Y_0^{\delta}\right|\geq 0$ by preposition).

The statement a) of the lemma 2.1.2 is proved.

2) Let quasinormal representation of binary relations δ and β have a form

$$\delta=\left(Y_3^{\delta}\times T_3\right)\cup\left(Y_2^{\delta}\times T_2\right)\cup\left(Y_1^{\delta}\times T_1\right)\cup\left(Y_0^{\delta}\times T_0\right),$$
$$\beta=\left(T_3\times T_3\right)\cup\left(\left(T_0\setminus T_1\right)\times T_2\right)\cup\left(\left(T_1\setminus T_3\right)\times T_1\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_3^{\alpha},Y_2^{\alpha},Y_1^{\alpha}\notin\{\varnothing\}$,

$$\begin{aligned}&T_3\cup\left(T_0\setminus T_1\right)\cup\left(T_1\setminus T_3\right)\cup\left(X\setminus T_0\right)\\&=\left(P_0\cup P_2\cup P_4\right)\cup P_1\cup P_3\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X,\end{aligned}$$

(see Equalities (2.1) and (2.2)), then $\delta,\beta\in B(\mathfrak{A}_0)$ and

$$\begin{aligned}&T_4\beta=T_3,\ T_2\beta=\left(P_0\cup P_1\cup P_3\cup P_4\right)\beta=T_3\cup T_2\cup T_1=T_0,\\&T_1\beta=\left(P_0\cup P_2\cup P_3\cup P_4\right)\beta=T_3\cup T_1=T_0,\ T_0\beta=T_0.\\&\alpha=\delta\circ\beta=\left(Y_3^{\delta}\times T_3\beta\right)\cup\left(Y_2^{\delta}\times T_2\beta\right)\cup\left(Y_1^{\delta}\times T_1\beta\right)\cup\left(Y_0^{\delta}\times T_0\beta\right)\\&=\left(Y_3^{\delta}\times T_3\right)\cup\left(Y_2^{\delta}\times T_0\right)\cup\left(Y_1^{\delta}\times T_1\right)\cup\left(Y_0^{\delta}\times T_0\right)\\&=\left(Y_3^{\delta}\times T_3\right)\cup\left(Y_1^{\delta}\times T_1\right)\cup\left(\left(Y_2^{\delta}\cup Y_0^{\delta}\right)\times T_0\right)=\alpha,\end{aligned}$$

if $Y_3^{\delta}=Y_3^{\alpha}$, $Y_1^{\delta}=Y_1^{\alpha}$ and $Y_2^{\delta}\cup Y_0^{\delta}=Y_0^{\alpha}$. Last equalities are possible since

$\left|Y_2^\delta \cup Y_0^\delta\right| \geq 1$ ($\left|Y_0^\delta\right| \geq 0$ by preposition).

The statement b) of the lemma 2.1.2 is proved.

Lemma 2.1.2 is proved.

Lemma 2.1.3. *Let* $D \in \Sigma_{8.0}(X,5)$. *Then the following statements are true*:

a) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_2^\alpha \times T_2\right),$$

where $Y_4^\alpha, Y_2^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

c) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_1^\alpha \times T_1\right),$$

where $Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

d) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

e) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

f) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

g) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_2$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

h) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_1$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

i) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_0$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$.

Proof. 1) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_4 \times T_4\right) \cup \left(\left(T_0 \setminus T_4\right) \times T_2\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$.

$$T_4 \cup \left(T_0 \setminus T_4\right) \cup \left(X \setminus T_0\right)$$
$$= \left(P_0 \cup P_1 \cup P_3\right) \cup \left(P_2 \cup P_4\right) \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X.$$

Then from the statement a) of the Lemma 2.1.2 follows that β is generating by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$ and

$$T_4\beta = T_4,\ T_1\beta = T_4 \cup T_2 = T_2,\ T_0\beta = T_2.$$
$$\delta \circ \beta = \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_1^\delta \times T_1\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_2\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times T_2\right) = \alpha,$$

if $Y_4^\delta = Y_4^\alpha$, $Y_1^\delta \cup Y_0^\delta = Y_2^\alpha$. Last equalities are possible since $\left|Y_1^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

The statement a) of the lemma 2.1.3 is proved.

2) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_4 \times T_4\right) \cup \left(\left(T_0 \setminus T_4\right) \times T_3\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$.

$$T_4 \cup \left(T_0 \setminus T_4\right) \cup \left(X \setminus T_0\right)$$
$$= \left(P_0 \cup P_1 \cup P_3\right) \cup \left(P_2 \cup P_4\right) \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X.$$

Then from $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$T_4\beta = T_4,\ T_1\beta = T_4 \cup T_3 = T_0,\ T_0\beta = T_0.$$
$$\delta \circ \beta = \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_1^\delta \times T_1\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_0\right) \cup \left(Y_0^\delta \times T_0\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times T_0\right) = \alpha,$$

if $Y_4^\delta = Y_4^\alpha$, $Y_1^\delta \cup Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $\left|Y_1^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

The statement b) of the lemma 2.1.3 is proved.

3) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_3 \times T_3\right) \cup \left(\left(T_0 \setminus T_3\right) \times T_1\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_4^\delta, Y_2^\delta \notin \{\varnothing\}$.

$$T_3 \cup \left(T_0 \setminus T_3\right) \cup \left(X \setminus T_0\right)$$
$$= \left(P_0 \cup P_2 \cup P_4\right) \cup \left(P_1 \cup P_3\right) \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X.$$

Then from the statement b) of the Lemma 2.1.2 follows that β is generating by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$ and

$$T_3\beta = T_3,\ T_2\beta = T_3 \cup T_1 = T_1,\ T_0\beta = T_1.$$

$$\begin{aligned}\delta \circ \beta &= \left(Y_3^\delta \times T_3\beta\right) \cup \left(Y_2^\delta \times T_2\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)\\ &= \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_1\right)\\ &= \left(Y_3^\delta \times Z_3\right) \cup \left(\left(Y_2^\delta \cup Y_0^\delta\right) \times T_1\right) = \alpha,\end{aligned}$$

if $Y_3^\delta = Y_3^\alpha$, $Y_2^\delta \cup Y_0^\delta = Y_1^\alpha$. Last equalities are possible since $\left|Y_2^\delta \cup Y_0^\delta\right| \geq 1$ ($\left|Y_0^\delta\right| \geq 0$ by preposition).

The statement c) of the lemma 2.1.3 is proved.

4) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_0\right),$$

$$\beta = \left(T_3 \times T_3\right) \cup \left(\left(T_0 \setminus T_3\right) \times T_2\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_3^\delta, Y_2^\delta \notin \{\varnothing\}$. Then $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$T_3\beta = T_3,\ T_2\beta = T_3 \cup T_2 = T_0,\ T_0\beta = T_0.$$

$$\begin{aligned}\delta \circ \beta &= \left(Y_3^\delta \times T_3\beta\right) \cup \left(Y_2^\delta \times T_2\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)\\ &= \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_0\right) \cup \left(Y_0^\delta \times T_0\right)\\ &= \left(Y_3^\delta \times T_3\right) \cup \left(\left(Y_2^\delta \cup Y_0^\delta\right) \times T_0\right) = \alpha,\end{aligned}$$

if $Y_3^\delta = Y_3^\alpha$, $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $\left|Y_2^\delta \cup Y_0^\delta\right| \geq 1$ ($\left|Y_0^\delta\right| \geq 0$ by preposition).

The statement d) of the lemma 2.1.3 is proved.

5) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_0^\delta \times T_0\right),$$

$$\beta = \left(\left(\left(T_2 \cap T_1\right) \setminus T_3\right) \times T_4\right) \cup \left(\left(T_2 \setminus T_1\right) \times T_2\right) \cup \left(\left(X \setminus T_4\right) \times T_0\right),$$

where $Y_4^\delta, Y_0^\delta \notin \{\varnothing\}$,

$$\begin{aligned}&\left(\left(T_2 \cap T_1\right) \setminus T_3\right) \cup \left(T_2 \setminus T_1\right) \cup \left(X \setminus T_4\right)\\ &= \left(P_0 \cup P_3\right) \cup P_1 \cup \left(X \setminus T_4\right) = T_4 \cup \left(X \setminus T_4\right) = X.\end{aligned}$$

(See Equalities (2.1) and (2.2)). Then from the statement b) of the Lemma 2.1.3 follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement a) of the Lemma 2.1.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_4\beta = \left(P_0 \cup P_1 \cup P_3\right)\beta = T_4 \cup T_2 = T_2,\ T_0\beta = T_0.$$

$$\delta \circ \beta = \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right) = \left(Y_4^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_0\right) = \alpha,$$

if $Y_4^\delta = Y_2^\alpha$, $Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $\left|Y_4^\delta\right| \geq 1$ $\left|Y_0^\delta\right| \geq 1$.

The statement e) of the lemma 2.1.3 is proved.

6) Let quasinormal representation of a binary relations δ, β have a form

$$\delta=\left(Y_3^\delta\times T_3\right)\cup\left(Y_0^\delta\times T_0\right),$$
$$\beta=\left(\left(\left(T_2\cap T_1\right)\setminus T_4\right)\times T_3\right)\cup\left(\left(T_1\setminus T_2\right)\times T_1\right)\cup\left(\left(X\setminus T_3\right)\times T_0\right),$$

where $Y_3^\delta, Y_0^\delta\notin\{\varnothing\}$,

$$\left(\left(T_2\cap T_1\right)\setminus T_4\right)\cup\left(T_1\setminus T_2\right)\cup\left(X\setminus T_3\right)$$
$$=\left(P_0\cup P_4\right)\cup P_2\cup\left(X\setminus T_3\right)=T_3\cup\left(X\setminus T_3\right)=X.$$

(See Equalities (2.1) and (2.2)). Then from the statement d) of the Lemma 2.1.3 follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement b) of the Lemma 2.1.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_3\beta=\left(P_0\cup P_2\cup P_4\right)\beta=T_3\cup T_1=T_1,\ T_0\beta=T_0.$$
$$\delta\circ\beta=\left(Y_3^\delta\times T_3\beta\right)\cup\left(Y_0^\delta\times T_0\beta\right)=\left(Y_3^\delta\times T_1\right)\cup\left(Y_0^\delta\times T_0\right)=\alpha,$$

if $Y_3^\delta=Y_1^\alpha$, $Y_0^\delta=Y_0^\alpha$. Last equalities are possible since $\left|Y_4^\delta\right|\geq 1$ $\left|Y_0^\delta\right|\geq 1$.

The statement e) of the lemma 2.1.3 is proved.

7) Let quasinormal representation of a binary relations δ, β have a form

$$\delta=\left(Y_2^\delta\times T_2\right)\cup\left(Y_0^\delta\times T_0\right),$$
$$\beta=\left(T_1\times T_4\right)\cup\left(\left(T_2\setminus T_1\right)\times T_2\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_2^\delta, Y_0^\delta\notin\{\varnothing\}$,

$$T_1\cup\left(T_2\setminus T_1\right)\cup\left(X\setminus T_0\right)$$
$$=\left(P_0\cup P_2\cup P_3\cup P_4\right)\cup P_1\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X$$

(see Equalities (2.1) and (2.2)). Then from the statement e) of the Lemma 2.1.3 follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement a) of the Lemma 2.1.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_2\beta=T_4\cup T_2=T_2,\ T_0\beta=T_2$$
$$\delta\circ\beta=\left(Y_2^\delta\times T_2\beta\right)\cup\left(Y_0^\delta\times T_0\beta\right)=\left(Y_2^\delta\times T_2\right)\cup\left(Y_0^\delta\times T_2\right)=X\times T_2=\alpha,$$

since representation of a binary relation δ is quasinormal.

The statement g) of the lemma 2.1.3 is proved.

8) Let quasinormal representation of a binary relations δ, β have a form

$$\delta=\left(Y_1^\delta\times T_1\right)\cup\left(Y_0^\delta\times T_0\right),$$
$$\beta=\left(T_2\times T_3\right)\cup\left(\left(T_1\setminus T_2\right)\times T_1\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_1^\delta, Y_0^\delta\notin\{\varnothing\}$,

$$T_2\cup\left(T_1\setminus T_2\right)\cup\left(X\setminus T_0\right)$$
$$=\left(P_0\cup P_1\cup P_3\cup P_4\right)\cup P_2\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X$$

(see Equalities (2.1) and (2.2)). Then from the statement f) of the Lemma 2.1.3

follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement b) of the Lemma 2.1.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_1\beta = T_3 \cup T_1 = T_1,\ T_0\beta = T_1$$
$$\delta \circ \beta = \left(Y_1^\delta \times T_1\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right) = \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_1\right) = X \times T_1 = \alpha,$$

since representation of a binary relation δ is quasinormal.

The statement h) of the lemma 2.1.3 is proved.

9) Let quasinormal representation of a binary relation δ has a form

$$\delta = \left(T_4 \times T_1\right) \cup \left(\left(X \setminus T_4\right) \times T_0\right),$$

then

$$T_1\delta = \left(P_0 \cup P_2 \cup P_3 \cup P_4\right)\delta = T_4 \cup T_0 = T_0,\ T_0\delta = T_0$$
$$\delta \circ \delta = \left(T_4 \times T_1\delta\right) \cup \left(\left(X \setminus T_4\right) \times T_0\delta\right) = \left(T_4 \times T_0\right) \cup \left(\left(X \setminus T_4\right) \times T_0\right) = X \setminus T_0 = \alpha$$

since representation of a binary relation δ is quasinormal.

The statement i) of the lemma 2.1.3 is proved.

Lemma 2.1.3 is proved.

Lemma 2..4. *Let* $D \in \Sigma_{8.0}(X,5)$. *Then the following statements are true:*

a) *If* $\left|X \setminus T_0\right| \geq 1$ *and* $Z \in \{T_4, T_3\}$, *then binary relation* $\alpha = X \times Z$ *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If* $X = T_0$ *and* $Z \in \{T_4, T_3\}$, *then binary relation* $\alpha = X \times Z$ *is external element for the semigroup* $B_X(D)$.

Proof. 1) Let quasinormal representation of a binary relation δ has a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_3^\delta \times T_3\right) \cup \left(Y_0^\delta \times T_0\right),$$

where $Y_4^\delta, Y_3^\delta \notin \{\varnothing\}$, then $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. If quasinormal representation of a binary relation β has a form $\beta = \left(T_0 \times Z\right) \cup \bigcup_{t' \in X \setminus T_0} \left(\{t'\} \times f(t')\right)$, where f is any mapping of the set $X \setminus T_0$ in the set $\{T_4, T_3\} \setminus \{Z\}$. It is easy to see, that $\beta \neq \alpha$ and two elements of the set $\{T_4, T_3\}$ belong to the semilattice $V(D, \beta)$, *i.e.* $\delta \in B(\mathfrak{A}_0) \setminus \{\alpha\}$. In this case we have

$$T_4\beta = T_3\beta = T_0\beta = Z;$$
$$\begin{aligned}\delta \circ \beta &= \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_3^\delta \times T_3\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)\\ &= \left(Y_4^\delta \times Z\right) \cup \left(Y_3^\delta \times Z\right) \cup \left(Y_0^\delta \times Z\right)\\ &= \left(\left(Y_4^\delta \cup Y_3^\delta \cup Y_0^\delta\right) \times Z\right) = X \times Z = \alpha,\end{aligned}$$

since the representation of a binary relation δ is quasinormal. Thus, element α is generating by elements of the set $B(\mathfrak{A}_0)$.

The statement a) of the lemma 2.1.4 is proved.

2) Let $X = T_0$, $\alpha = X \times Z$, for some $Z \in \{T_4, T_3\}$ and $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_X(D) \setminus \{\alpha\}$. Then from the equality (2.1.1) and (2.1.2) we obtain that

$$T_4\beta = T_3\beta = T_2\beta = T_1\beta = T_0\beta = Z,\ P_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = Z,$$

since Z is minimal element of the semilattice D.

Now, let subquasinormal representations $\overline{\beta}$ of a binary relation β has a form

$$\overline{\beta}=\left(\left(P_0\cup P_1\cup P_2\cup P_3\cup P_4\right)\times Z\right)\cup\bigcup_{t'\in X\setminus T_0}\left(\{t'\}\times\overline{\beta}_2(t')\right),$$

where $\overline{\beta}_1=\begin{pmatrix}P_0 & P_1 & P_2 & P_3 & P_4\\ Z & Z & Z & Z & Z\end{pmatrix}$ is normal mapping. But complement mapping $\overline{\beta}_2$ is empty, since $X\setminus T_0=\varnothing$, *i.e.* in the given case, subquasinormal representation $\overline{\beta}$ of a binary relation β is defined uniquely. So, we have that $\beta=\overline{\beta}=X\times Z=\alpha$, which contradicts the condition $\beta\notin B_X(D)\setminus\{\alpha\}$.

Therefore, if $X=T_0$ and $\alpha=X\times Z$, for some $Z\in\{T_4,T_3\}$, then α is external element of the semigroup $B_X(D)$.

The statement b) of the lemma 2.1.4 is proved.

Lemma 2.1.4 is proved.

Theorem 2.1.1. *Let* $D\in\Sigma_{8.0}(X,5)$ *and*

$$\begin{aligned}\mathfrak{A}_0=&\{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\},\\&\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\},\{T_2,T_1,T_0\}\},\\ B(\mathfrak{A}_0)=&\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)\in\mathfrak{A}_0\right\};\ B_0=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)=D\right\}.\end{aligned}$$

Then the following statements are true:

a) *If* $|X\setminus T_0|\ge 1$, *then* $S_0=B_0\cup B(\mathfrak{A}_o)$ *is irreducible generating set for the semigroup* $B_X(D)$;

b) *If* $X=T_0$, *then* $S_1=B_0\cup B(\mathfrak{A}_o)\cup\{X\times T_4,X\times T_3\}$ *is irreducible generating set for the semigroup* $B_X(D)$.

Proof. Let $D\in\Sigma_{8.0}(X,5)$ and $|X\setminus T_0|\ge 1$. First, we proved that every element of the semigroup $B_X(D)$ is generating by elements of the set S_0. Indeed, let α be arbitrary element of the semigroup $B_X(D)$. Then quasinormal representation of a binary relation α has a form

$$\alpha=\left(Y_4^\alpha\times T_4\right)\cup\left(Y_3^\alpha\times T_3\right)\cup\left(Y_2^\alpha\times T_2\right)\cup\left(Y_1^\alpha\times T_1\right)\cup\left(Y_0^\alpha\times T_0\right),$$

where $Y_4^\alpha\cup Y_3^\alpha\cup Y_2^\alpha\cup Y_1^\alpha\cup Y_0^\alpha=X$ and $Y_i^\alpha\cap Y_j^\alpha=\varnothing$ $(0\le i\ne j\le 4)$. For the $\left|V\left(X^*,\alpha\right)\right|$ we consider the following cases:

1) $\left|V\left(X^*,\alpha\right)\right|=5$. Then $\alpha\in B_0$ and $B_0\subset S_0$ by definition of a set S_0.

2) $\left|V\left(X^*,\alpha\right)\right|=4$. Then

$$V\left(X^*,\alpha\right)\in\mathfrak{A}_4=\{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\}\}\subset\mathfrak{A}_0$$

i.e. $\alpha\in B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

3) $\left|V\left(X^*,\alpha\right)\right|=3$. Then we have

$$V\left(X^*,\alpha\right)\in\mathfrak{A}_3=\left\{\left\{T_4,T_3,T_0\right\},\left\{T_4,T_1,T_0\right\},\left\{T_3,T_2,T_0\right\},\left\{T_4,T_2,T_0\right\},\left\{T_3,T_1,T_0\right\},\left\{T_2,T_1,T_0\right\}\right\}.$$

By definition of a set $\mathfrak{A}_0$ we have $\left\{\left\{T_4,T_3,T_0\right\},\left\{T_4,T_1,T_0\right\},\left\{T_3,T_2,T_0\right\},\left\{T_2,T_1,T_0\right\}\right\}\subset\mathfrak{A}_0$, *i.e.* in this case $\alpha\in B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

If $V\left(X^*,\alpha\right)\in\left\{\left\{T_4,T_2,T_0\right\},\left\{T_3,T_1,T_0\right\}\right\}$, then from the statement a) and b) of the Lemma 2.1.2 element α is generating by elements $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

4) $\left|V\left(X^*,\alpha\right)\right|=2$. Then we have

$$V\left(X^*,\alpha\right)\in\mathfrak{A}_2=\left\{\left\{T_4,T_2\right\},\left\{T_4,T_0\right\},\left\{T_3,T_1\right\},\left\{T_3,T_0\right\},\left\{T_2,T_0\right\}\left\{T_1,T_0\right\}\right\}.$$

Then from the statement a)-f) of the Lemma 2.1.3 element α is generating by elements $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

5) $\left|V\left(X^*,\alpha\right)\right|=1$. Then we have $V\left(X^*,\alpha\right)\in\mathfrak{A}_1=\left\{\left\{T_4\right\},\left\{T_3\right\},\left\{T_2\right\},\left\{T_1\right\},\left\{T_0\right\}\right\}$.

If $V\left(X^*,\alpha\right)\in\left\{\left\{T_2\right\},\left\{T_1\right\},\left\{T_0\right\}\right\}$, then from the statements g), h) and i) of the Lemma 2.1.3 element α is generating by elements $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

If $V\left(X^*,\alpha\right)\in\left\{\left\{T_4\right\},\left\{T_3\right\}\right\}$, then from the statement a) of the Lemma 2.1.4 element α is generating by elements $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_0$ by definition of a set S_0.

Thus, we have that S_0 is generating set for the semigroup $B_X(D)$.

If $\left|X\setminus T_0\right|\geq 1$, then the set S_0 is irreducible generating set for the semigroup $B_X(D)$ since S_0 is a set external elements of the semigroup $B_X(D)$.

The statement a) of the Theorem 2.1.1 is proved.

Now, let $D\in\Sigma_{8.0}(X,5)$ and $X=\breve{D}$. First, we proved that every element of the semigroup $B_X(D)$ is generating by elements of the set S_1. The cases 1), 2), 3) and 4) are proved analogously of the cases 1), 2), 3) and 4) given above and consider case, when

$$V\left(X^*,\alpha\right)\in\mathfrak{A}_1=\left\{\left\{T_4\right\},\left\{T_3\right\},\left\{T_2\right\},\left\{T_1\right\},\left\{T_0\right\}\right\}.$$

If $V\left(X^*,\alpha\right)\in\left\{\left\{T_2\right\},\left\{T_1\right\},\left\{T_0\right\}\right\}$, then from the statements g), h) and i) of the Lemma 2.1.3 element α is generating by elements $B(\mathfrak{A}_0)$ and $B(\mathfrak{A}_0)\subset S_1$ by definition of a set S_1.

If $V\left(X^*,\alpha\right)\in\left\{\left\{T_4\right\},\left\{T_3\right\}\right\}$, then $\alpha\in S_1$ by definition of a set S_1.

Thus, we have that S_1 is generating set for the semigroup $B_X(D)$.

If $X=T_0$, then the set S_1 is irreducible generating set for the semigroup $B_X(D)$ since S_1 is a set external elements of the semigroup $B_X(D)$.

The statement b) of the Theorem 2.1.1 is proved.

Theorem 2.1.1 is proved.

Theorem 2.1.2. *Let* $n \geq 6$, $D = \{T_4, T_3, T_2, T_1, T_0\} \in \Sigma_{8.0}(X,5)$ *and*

$$\mathfrak{A}_0 = \{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\},$$
$$\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\},\{T_2,T_1,T_0\}\},$$
$$B(\mathfrak{A}_0) = \{\alpha \in B_X(D) \mid V(X^*,\alpha) \in \mathfrak{A}_0\};\ B_0 = \{\alpha \in B_X(D) \mid V(X^*,\alpha) = D\}.$$

Then the following statements are true:

a) *If* $|X \setminus T_0| \geq 1$, *then the number* $|S_0|$ *elements of the set* $S_0 = B_0 \cup B(\mathfrak{A}_0)$ *is equal to*

$$|S_0| = 5^n - 2\cdot 3^n + 1.$$

b) *If* $X = T_0$, *then the number* $|S_1|$ *elements of the set* $S_1 = B_0 \cup B(\mathfrak{A}_\circ) \cup \{X \times T_4, X \times T_3\}$ *is equal to*

$$|S_1| = 5^n - 2\cdot 3^n + 3.$$

Proof. Let number of a set *X* is equal to $n \geq 6$, *i.e.* $|X| = n \geq 6$. Let $S_n = \{\varphi_1, \varphi_2, \cdots, \varphi_{n!}\}$ is a group all one to one mapping of a set $M = \{1,2,\cdots,n\}$ on the set M and $\varphi_{i_1}, \varphi_{i_2}, \cdots, \varphi_{i_m}$ $(m \leq n)$ are arbitrary elements of the group S_n, $Y_{\varphi_1}, Y_{\varphi_2}, \cdots, Y_{\varphi_m}$ are arbitrary partitioning of a set *X*. By symbol k_n^m we denote the number elements of a set $\{Y_{\varphi_1}, Y_{\varphi_2}, \cdots, Y_{\varphi_m}\}$. It is well known, that

$$k_n^m = \sum_{i=1}^{m} \frac{(-1)^{m+i}}{(i-1)!\ \cdot (m-i)!} \cdot i^{n-1}.$$

If $m = 2,3,4,5$, then we have

$$k_n^2 = 2^{n-1} - 1,\ k_n^3 = \frac{1}{2}\cdot 3^{n-1} - 2^{n-1} + \frac{1}{2},\ k_n^4 = \frac{1}{6}\cdot 4^{n-1} - \frac{1}{2}\cdot 3^{n-1} + \frac{1}{2}\cdot 2^{n-1} - \frac{1}{6},$$
$$k_n^5 = \frac{1}{24}\cdot 5^{n-1} - \frac{1}{6}\cdot 4^{n-1} + \frac{1}{4}\cdot 3^{n-1} - \frac{1}{6}\cdot 2^{n-1} + \frac{1}{24}.$$

If $Y_{\varphi_1}, Y_{\varphi_2}$ are any two elements partitioning of a set *X* and $\overline{\beta} = (Y_{\varphi_1} \times Z_1) \cup (Y_{\varphi_2} \times Z_2)$, where $Z_1, Z_2 \in D$ and $Z_1 \neq Z_2$. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$2 \cdot k_n^2 = 2^n - 2. \quad (2.1.3)$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}$ are any tree elements partitioning of a set *X* and

$$\overline{\beta} = (Y_{\varphi_1} \times Z_1) \cup (Y_{\varphi_2} \times Z_2) \cup (Y_{\varphi_3} \times Z_3),$$

where Z_1, Z_2, Z_3 are pairwise different elements of a given semilattice *D*. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$6 \cdot k_n^3 = 3^n - 3\cdot 2^n + 3. \quad (2.1.4)$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}, Y_{\varphi_4}$ are any four elements partitioning of a set *X* and

$$\overline{\beta} = (Y_{\varphi_1} \times Z_1) \cup (Y_{\varphi_2} \times Z_2) \cup (Y_{\varphi_3} \times Z_3) \cup (Y_{\varphi_4} \times Z_4),$$

where Z_1, Z_2, Z_3, Z_4 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$24 \cdot k_n^4 = 4^n - 4 \cdot 3^n + 3 \cdot 2^{n+1} - 4. \tag{2.1.5}$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}, Y_{\varphi_4}, Y_{\varphi_5}$ are any four elements partitioning of a set X and

$$\overline{\beta} = \left(Y_{\varphi_1} \times Z_1\right) \cup \left(Y_{\varphi_2} \times Z_2\right) \cup \left(Y_{\varphi_3} \times Z_3\right) \cup \left(Y_{\varphi_4} \times Z_4\right) \cup \left(Y_{\varphi_5} \times Z_5\right),$$

where Z_1, Z_2, Z_3, Z_4, Z_5 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$120 \cdot k_n^5 = 5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5. \tag{2.1.6}$$

If $\alpha \in B_0$, then quasinormal representation of a binary relation α has a form

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X.

If the system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X. Of this and from the equalities (2.1.4), (2.1.5) and (2.1.6) follows that

$$\begin{aligned}|B_0| &= \left(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5\right) + \left(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4\right) \\ &= 5^n - 4 \cdot 4^n + 6 \cdot 3^n - 4 \cdot 2^n + 1.\end{aligned}$$

If $\alpha \in B(\mathfrak{A}_0)$, then by definition of a set $B(\mathfrak{A}_0)$ the quasinormal representation of a binary relation α has a form:

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_2^\alpha, Y_1^\alpha \in \{\varnothing\}$, or $Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \in \{\varnothing\}$ are partitioning of the set X respectively.

Of this and from the equality (2.1.3), (2.1.4) and (2.1.5) follows that

$$\begin{aligned}\left|B\left(\mathfrak{A}_0\right)\right| &= 4\cdot\left(2^n-2\right)+8\cdot\left(3^n-3\cdot 2^n+3\right)+4\cdot\left(4^n-4\cdot 3^n+6\cdot 2^n-4\right)\\ &= 4\cdot 4^n-8\cdot 3^n+4\cdot 2^n.\end{aligned}$$

So, we have

$$\begin{aligned}\left|S_0\right| &= \left|B_0\cup B\left(\mathfrak{A}_0\right)\right| = \left(5^n-4\cdot 4^n+6\cdot 3^n-4\cdot 2^n+1\right)+\left(4\cdot 4^n-8\cdot 3^n+4\cdot 2^n\right)\\ &= 5^n-2\cdot 3^n+1,\\ \left|S_1\right| &= \left|B_0\cup B\left(\mathfrak{A}_\circ\right)\cup\left\{X\times T_4, X\times T_3\right\}\right| = 5^n-2\cdot 3^n+3\end{aligned}$$

Since

$$B_0\cap B\left(\mathfrak{A}_0\right) = B_0\cap\left\{X\times T_4, X\times T_3, X\times T_2\right\} = B\left(\mathfrak{A}_0\right)\cap\left\{X\times T_4, X\times T_3, X\times T_2\right\} = \varnothing.$$

Theorem 2.1.2 is proved.

2.2. Generating Sets of the Complete Semigroup of Binary Relations Defined by Semilattices of the Class $\Sigma_8(X,5)$, When $T_4 \cap T_3 = \varnothing$

In the sequel, we denoted all semilattices $D = \{T_4, T_3, T_2, T_1, T_0\}$ of the class $\Sigma_8(X,5)$ by symbol $\Sigma_{8.1}(X,5)$ for which $T_4 \cap T_3 = \varnothing$. Of the last equality from the formal equalities of a semilattise D follows that $T_4 \cap T_3 = P_0 = \varnothing$, *i.e.* $|X| \geq 4$ since $P_4 \neq \varnothing$, $P_3 \neq \varnothing$, $P_2 \neq \varnothing$, $P_1 \neq \varnothing$.

In this case, the formal equalities of the semilattice D have a form:

$$\begin{aligned}T_0 &= P_1\cup P_2\cup P_3\cup P_4,\\ T_1 &= P_2\cup P_3\cup P_4,\\ T_2 &= P_1\cup P_3\cup P_4,\\ T_3 &= P_2\cup P_4,\\ T_4 &= P_1\cup P_3.\end{aligned} \tag{2.2.1}$$

From the formal equalities of the semilattise D immediately follows, that:

$$P_4 = T_2 \setminus T_4,\ P_3 = T_1 \setminus T_3,\ P_2 = T_1 \setminus T_2,\ P_1 = T_2 \setminus T_1. \tag{2.2.2}$$

In this case we suppose that $D \in \Sigma_{8.1}(X,5)$.

By symbols $\mathfrak{A}_4, \mathfrak{A}_3, \mathfrak{A}_2$ and $\mathfrak{A}_1$ we denoted the following sets:

$$\mathfrak{A}_4 = \{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\}\},$$
$$\mathfrak{A}_3 = \{\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\},\{T_4,T_2,T_0\},\{T_3,T_1,T_0\},\{T_2,T_1,T_0\}\},$$
$$\mathfrak{A}_2 = \{\{T_4,T_2\},\{T_4,T_0\},\{T_3,T_1\},\{T_3,T_0\},\{T_2,T_0\}\{T_1,T_0\}\},$$
$$\mathfrak{A}_1 = \{\{T_4\},\{T_3\},\{T_2\},\{T_1\},\{T_0\}\}.$$

Lemma 2.2.1. *Let* $D \in \Sigma_{8.1}(X,5)$. *Then the following statements are true*:

a) *Let* $Z, Z' \in \{T_4, T_3, T_2\}$, $Z \neq Z'$. *If* $Z, Z' \in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$;

b) *Let* $Z \in \{T_2, T_1\}$, $Z' \in \{T_4, T_3\}$. *If* $Z \not\subset Z'$ *and* $Z, Z' \in V(D,\alpha)$, *then* α *is external element of the semigroup* $B_X(D)$.

Proof. Let $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B_X(D) \setminus \{\alpha\}$. If quasinormal representation of binary relation δ has a form

$$\delta = (Y_4^\delta \times T_4) \cup (Y_3^\delta \times T_3) \cup (Y_2^\delta \times T_2) \cup (Y_1^\delta \times T_1) \cup (Y_0^\delta \times T_0),$$

then

$$\alpha = \delta \circ \beta = (Y_4^\delta \times T_4\beta) \cup (Y_3^\delta \times T_3\beta) \cup (Y_2^\delta \times T_2\beta) \cup (Y_1^\delta \times T_1\beta) \cup (Y_0^\delta \times T_0\beta). \tag{2.2.3}$$

From the formal equalities (2.2.1) of the semilattice D we obtain that:

$$\begin{aligned} T_0\beta &= P_1\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta, \\ T_1\beta &= P_2\beta \cup P_3\beta \cup P_4\beta, \\ T_2\beta &= P_1\beta \cup P_3\beta \cup P_4\beta, \\ T_3\beta &= P_2\beta \cup P_4\beta, \\ T_4\beta &= P_1\beta \cup P_3\beta, \end{aligned} \tag{2.2.4}$$

where $P_i\beta \neq \varnothing$ for any $P_i \neq \varnothing$ $(i = 1,2,3,4)$ and $\beta \in B_X(D)$. Indeed, by preposition $P_i \neq \varnothing$ for any $i = 1,2,3,4$ and $\beta \neq \varnothing$ since $\varnothing \notin D$. Let $y \in P_i$ for some $y \in X$, then $y \in \breve{D}$, $\beta = \alpha_f$ for some $f : X \to D$ and $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)) \supseteq \{y\} \times f(y)$, *i.e.* there exists an element $z \in f(y)$ for which $y\alpha_f z$ and $y\beta z$. Of this and by definition of a set $P_i\beta$ we obtain that $z \in P_i\beta$ since $y \in P_i$, $y\beta z$. Thus, we have $P_i\beta \neq \varnothing$, *i.e.* $P_i\beta \in D$ for any $i = 1,2,3,4$.

Now, let $T_i\beta = Z$ and $T_j\beta = Z'$ for some $0 \le i \neq j \le 4$ and $Z \neq Z'$, $Z, Z' \in \{T_4, T_3\}$, then from the Equalities (2.2.4) follows that $Z = P_0\beta = Z'$ since Z and Z' are minimal elements of the semilattice D. The equality $Z = Z'$ contradicts the inequality $Z \neq Z'$.

The statement a) of the Lemma 2.2.1 is proved.

Let $T_i\beta = Z'$, where $Z' \in \{T_4, T_3\}$ and $T_j\beta = Z$, $Z \in \{T_2, T_1\}$ for some $0 \le i \neq j \le 4$. If $0 \le i \le 4$, then from the formal equalities of a semilattice D we obtain that

$$T_0\beta = P_1\beta \cup P_2\beta \cup P_3\beta \cup P_4\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = Z',$$
$$T_1\beta = P_2\beta \cup P_3\beta \cup P_4\beta = P_2\beta = P_3\beta = P_4\beta = Z',$$
$$T_2\beta = P_1\beta \cup P_3\beta \cup P_4\beta = P_1\beta = P_3\beta = P_4\beta = Z',$$
$$T_3\beta = P_2\beta \cup P_4\beta = P_2\beta = P_4\beta = Z',$$
$$T_4\beta = P_1\beta \cup P_3\beta = P_1\beta = P_3\beta = Z',$$

since Z' is minimal element of the semilattice *D*.

Now, let $i \neq j$.

1) If $T_0\beta = P_1\beta = P_2\beta = P_3\beta = P_4\beta = Z'$ and $j = 1,2,3,4$, then we have

$$Z = T_1\beta = T_2\beta = T_3\beta = T_4\beta = Z',$$

which contradicts the inequality $Z \neq Z'$.

2) If $T_1\beta = P_2\beta = P_3\beta = P_4\beta = Z'$ and $j = 0,2,3,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta, \text{ where } P_1\beta \in D;$$
$$Z = T_3\beta = Z'.$$

Last equalities are impossible since $Z \neq Z' \cup T$ for any $T \in D$ and $Z \neq Z'$ by definition of a semilattice *D*.

3) If $T_2\beta = P_1\beta = P_3\beta = P_4\beta = Z'$ and $j = 0,1,3,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta, \text{ where } P_1\beta \in D;$$
$$Z = T_3\beta = Z'.$$

Last equalities are impossible since for any $T \in D$ and $Z \neq Z'$ by definition of a semilattice *D*.

4) If $T_3\beta = P_2\beta = P_4\beta = Z'$ and $j = 0,1,2,4$, then we have

$$Z = T_0\beta = T_2\beta = T_4\beta = Z' \cup P_1\beta \cup P_3\beta,$$
$$Z = T_1\beta = Z' \cup P_3\beta, \text{ where } P_1\beta, P_3\beta \in D.$$

Last equalities are impossible since $Z \neq Z' \cup T \cup T'$ and $Z \neq Z' \cup T$ for any $T, T' \in D$, by definition of a semilattice *D*.

5) If $T_4\beta = P_1\beta = P_3\beta = Z'$ and $j = 0,1,2,3$, then we have

$$Z = T_0\beta = T_1\beta = T_3\beta = Z' \cup P_2\beta \cup P_4\beta,$$
$$Z = T_2\beta = Z' \cup P_4\beta, \text{ where } P_2\beta, P_4\beta \in D.$$

Last equalities are impossible since $Z \neq Z' \cup T \cup T'$ and $Z \neq Z' \cup T$ for any $T, T' \in D$, by definition of a semilattice *D*.

The statement b) of the Lemma 2.2.1 is proved.

Lemma 2.2.1 is proved.

Let $D \in \Sigma_{8.1}(X,5)$. We denoted the following sets by symbols $\mathfrak{A}_0$, $B(\mathfrak{A}_0)$ and B_0:

$$\mathfrak{A}_0 = \{\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\},$$
$$\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\}\},$$
$$B(\mathfrak{A}_0) = \{\alpha \in B_X(D) \mid V(X^*,\alpha) \in \mathfrak{A}_0\};\ B_0 = \{\alpha \in B_X(D) \mid V(X^*,\alpha) = D\}.$$

Remark, that the sets B_0 and $B(\mathfrak{A}_0)$ are external elements for the semigroup $B_X(D)$.

Lemma 2.2.2. *Let* $D \in \Sigma_{8.1}(X,5)$ *. Then the following statements are true:*

a) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

c) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B_0 \cup B(\mathfrak{A}_0)$.

Proof. 1). Let quasinormal representation of binary relations δ and β have a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_4 \times T_4\right) \cup \left(\left(T_2 \setminus T_4\right) \times T_2\right) \cup \left(\left(T_0 \setminus T_2\right) \times T_1\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$,

$$\begin{aligned} & T_4 \cup \left(T_2 \setminus T_4\right) \cup \left(T_0 \setminus T_2\right) \cup \left(X \setminus T_0\right) \\ & = \left(P_1 \cup P_3\right) \cup P_4 \cup P_2 \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X, \end{aligned}$$

(see Equalities (2.2.1) and (2.2.2)), then $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$\begin{aligned} & T_4\beta = T_4,\ T_2\beta = \left(P_1 \cup P_3 \cup P_4\right)\beta = T_4 \cup T_2 = T_2, \\ & T_1\beta = \left(P_2 \cup P_3 \cup P_4\right)\beta = T_4 \cup T_1 = T_0,\ T_0\beta = T_0. \\ & \alpha = \delta \circ \beta = \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_2^\delta \times T_2\beta\right) \cup \left(Y_1^\delta \times T_1\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right) \\ & \quad = \left(Y_4^\delta \times T_4\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_1^\delta \times T_0\right) \cup \left(Y_0^\delta \times T_0\right) \\ & \quad = \left(Y_4^\delta \times T_4\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times T_0\right) = \alpha, \end{aligned}$$

if $Y_4^\delta = Y_4^\alpha$, $Y_2^\delta = Y_2^\alpha$ and $Y_1^\delta \cup Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $\left|Y_1^\delta \cup Y_0^\delta\right| \geq 1$ ($\left|Y_0^\delta\right| \geq 0$ by preposition).

The statement a) of the lemma 2.2.2 is proved.

2) Let quasinormal representation of binary relations δ and β have a form

$$\delta = \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_3 \times T_3\right) \cup \left(\left(T_0 \setminus T_1\right) \times T_2\right) \cup \left(\left(T_1 \setminus T_3\right) \times T_1\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$,

$$T_3 \cup (T_0 \setminus T_1) \cup (T_1 \setminus T_3) \cup (X \setminus T_0)$$
$$= (P_2 \cup P_4) \cup P_1 \cup P_3 \cup (X \setminus T_0) = T_0 \cup (X \setminus T_0) = X,$$

(see Equalities (2.2.1) and (2.2.2)), then $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$T_4\beta = T_3,\ T_2\beta = (P_1 \cup P_3 \cup P_4)\beta = T_3 \cup T_2 \cup T_1 = T_0,$$
$$T_1\beta = (P_2 \cup P_3 \cup P_4)\beta = T_3 \cup T_1 = T_0,\ T_0\beta = T_0.$$
$$\alpha = \delta \circ \beta = (Y_3^\delta \times T_3\beta) \cup (Y_2^\delta \times T_2\beta) \cup (Y_1^\delta \times T_1\beta) \cup (Y_0^\delta \times T_0\beta)$$
$$= (Y_3^\delta \times T_3) \cup (Y_2^\delta \times T_0) \cup (Y_1^\delta \times T_1) \cup (Y_0^\delta \times T_0)$$
$$= (Y_3^\delta \times T_3) \cup (Y_1^\delta \times T_1) \cup ((Y_2^\delta \cup Y_0^\delta) \times T_0) = \alpha,$$

if $Y_3^\delta = Y_3^\alpha$, $Y_1^\delta = Y_1^\alpha$ and $Y_2^\delta \cup Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $|Y_2^\delta \cup Y_0^\delta| \geq 1$ ($|Y_0^\delta| \geq 0$ by preposition).

The statement b) of the lemma 2.2.2 is proved.

3) Let quasinormal representation of binary relations δ and β have a form

$$\delta = (Y_4^\delta \times T_4) \cup (Y_3^\delta \times T_3) \cup (Y_0^\delta \times T_0),$$
$$\beta = ((T_2 \setminus T_1) \times T_4) \cup ((T_1 \setminus T_2) \times T_3) \cup ((T_1 \setminus T_3) \times T_2)$$
$$\cup ((T_2 \setminus T_4) \times T_1) \cup ((X \setminus T_0) \times T_0),$$

where $Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$,

$$(T_2 \setminus T_1) \cup (T_1 \setminus T_2) \cup (T_1 \setminus T_3) \cup (T_2 \setminus T_4) \cup (X \setminus T_0)$$
$$= P_1 \cup P_2 \cup P_3 \cup P_4 \cup (X \setminus T_0) = T_0 \cup (X \setminus T_0) = X,$$

(see Equalities (2.2.1) and (2.2.2)), then $\delta \in B(\mathfrak{A}_0)$, $\beta \in B_0$ and

$$T_4\beta = (P_1 \cup P_3)\beta = T_4 \cup T_2 = T_2,$$
$$T_3\beta = (P_2 \cup P_4)\beta = T_3 \cup T_1 = T_1,\ T_0\beta = T_2 \cup T_1 = T_0,$$
$$\alpha = \delta \circ \beta = (Y_4^\delta \times T_4\beta) \cup (Y_3^\delta \times T_3\beta) \cup (Y_0^\delta \times T_0\beta)$$
$$= (Y_4^\delta \times T_2) \cup (Y_3^\delta \times T_1) \cup (Y_0^\delta \times T_0) = \alpha,$$

if $Y_4^\delta = Y_2^\alpha$, $Y_3^\delta = Y_1^\alpha$ and $Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $|Y_4^\delta| \geq 1$, $|Y_3^\delta| \geq 1$ and $|Y_0^\delta| \geq 0$.

The statement c) of the lemma 2.2.2 is proved.

Lemma 2.2.2 is proved.

Lemma 2.2.3. *Let* $D \in \Sigma_{8.1}(X,5)$. *Then the following statements are true:*

a) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_4^\alpha \times T_4) \cup (Y_2^\alpha \times T_2),$$

where $Y_4^\alpha, Y_2^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

b) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_4^\alpha \times T_4) \cup (Y_0^\alpha \times T_0),$$

where $Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

c) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_3^\alpha \times T_3) \cup (Y_1^\alpha \times T_1),$$

where $Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

d) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_3^\alpha \times T_3) \cup (Y_0^\alpha \times T_0),$$

where $Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

e) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_2^\alpha \times T_2) \cup (Y_0^\alpha \times T_0),$$

where $Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

f) *If quasinormal representation of a binary relation* α *has a form*

$$\alpha = (Y_1^\alpha \times T_1) \cup (Y_0^\alpha \times T_0),$$

where $Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

g) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_2$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

h) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_1$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$;

i) *If quasinormal representation of a binary relation* α *has a form* $\alpha = X \times T_0$, *then* α *is generating by elements of the elements of set* $B(\mathfrak{A}_0)$.

Proof. 1) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = (Y_4^\delta \times T_4) \cup (Y_1^\delta \times T_1) \cup (Y_0^\delta \times T_0),$$
$$\beta = (T_4 \times T_4) \cup ((T_0 \setminus T_4) \times T_2) \cup ((X \setminus T_0) \times T_0),$$

where $Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$.

$$T_4 \cup (T_0 \setminus T_4) \cup (X \setminus T_0)$$
$$= (P_1 \cup P_3) \cup (P_2 \cup P_4) \cup (X \setminus T_0) = T_0 \cup (X \setminus T_0) = X.$$

Then from the statement a) of the Lemma 2.2.2 follows that β is generating by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$ and

$$T_4\beta = T_4,\ T_1\beta = T_4 \cup T_2 = T_2,\ T_0\beta = T_2.$$
$$\delta \circ \beta = (Y_4^\delta \times T_4\beta) \cup (Y_1^\delta \times T_1\beta) \cup (Y_0^\delta \times T_0\beta)$$
$$= (Y_4^\delta \times T_4) \cup (Y_1^\delta \times T_2) \cup (Y_0^\delta \times T_2)$$
$$= (Y_4^\delta \times T_4) \cup ((Y_1^\delta \cup Y_0^\delta) \times T_2) = \alpha,$$

If $Y_4^\delta = Y_4^\alpha$, $Y_1^\delta \cup Y_0^\delta = Y_2^\alpha$. Last equalities are possible since $\left|Y_1^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

The statement a) of the lemma 2.2.3 is proved.

2) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_4 \times T_4\right) \cup \left(\left(T_0 \setminus T_4\right) \times T_3\right) \cup \left(\left(X \setminus T\right) \times T_0\right),$$

where $Y_4^\delta, Y_1^\delta \notin \{\varnothing\}$.

$$T_4 \cup \left(T_0 \setminus T_4\right) \cup \left(X \setminus T_0\right)$$
$$= \left(P_1 \cup P_3\right) \cup \left(P_2 \cup P_4\right) \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X.$$

Then from $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$T_4\beta = T_4,\ T_1\beta = T_4 \cup T_3 = T_0,\ T_0\beta = T_0.$$
$$\delta \circ \beta = \left(Y_4^\delta \times T_4\beta\right) \cup \left(Y_1^\delta \times T_1\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(Y_1^\delta \times T_0\right) \cup \left(Y_0^\delta \times T_0\right)$$
$$= \left(Y_4^\delta \times T_4\right) \cup \left(\left(Y_1^\delta \cup Y_0^\delta\right) \times T_0\right) = \alpha,$$

if $Y_4^\delta = Y_4^\alpha$, $Y_1^\delta \cup Y_0^\delta = Y_0^\alpha$. Last equalities are possible since $\left|Y_1^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

The statement b) of the lemma 2.2.3 is proved.

3) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_3 \times T_3\right) \cup \left(\left(T_0 \setminus T_3\right) \times T_1\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_4^\delta, Y_2^\delta \notin \{\varnothing\}$.

$$T_3 \cup \left(T_0 \setminus T_3\right) \cup \left(X \setminus T_0\right)$$
$$= \left(P_2 \cup P_4\right) \cup \left(P_1 \cup P_3\right) \cup \left(X \setminus T_0\right) = T_0 \cup \left(X \setminus T_0\right) = X.$$

Then from the statement b) of the Lemma 2.2.2 follows that β is generating by elements of the set $B(\mathfrak{A}_0)$, $\delta \in B(\mathfrak{A}_0)$ and

$$T_3\beta = T_3,\ T_2\beta = T_3 \cup T_1 = T_1,\ T_0\beta = T_1.$$
$$\delta \circ \beta = \left(Y_2^\delta \times T_3\beta\right) \cup \left(Y_2^\delta \times T_2\beta\right) \cup \left(Y_0^\delta \times T_0\beta\right)$$
$$= \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_1\right) \cup \left(Y_0^\delta \times T_1\right)$$
$$= \left(Y_3^\delta \times T_3\right) \cup \left(\left(Y_2^\delta \cup Y_0^\delta\right) \times T_1\right) = \alpha,$$

if $Y_3^\delta = Y_3^\alpha$, $Y_2^\delta \cup Y_0^\delta = Y_1^\alpha$. Last equalities are possible since $\left|Y_2^\delta \cup Y_0^\delta\right| \ge 1$ ($\left|Y_0^\delta\right| \ge 0$ by preposition).

The statement c) of the lemma 2.2.3 is proved.

4) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_3^\delta \times T_3\right) \cup \left(Y_2^\delta \times T_2\right) \cup \left(Y_0^\delta \times T_0\right),$$
$$\beta = \left(T_3 \times T_3\right) \cup \left(\left(T_0 \setminus T_3\right) \times T_2\right) \cup \left(\left(X \setminus T_0\right) \times T_0\right),$$

where $Y_3^{\delta}, Y_2^{\delta} \notin \{\varnothing\}$. Then $\delta, \beta \in B(\mathfrak{A}_0)$ and

$$T_3\beta = T_3,\ T_2\beta = T_3 \cup T_2 = T_0,\ T_0\beta = T_0.$$
$$\begin{aligned}\delta \circ \beta &= \left(Y_3^{\delta} \times T_3\beta\right) \cup \left(Y_2^{\delta} \times T_2\beta\right) \cup \left(Y_0^{\delta} \times T_0\beta\right)\\ &= \left(Y_3^{\delta} \times T_3\right) \cup \left(Y_2^{\delta} \times T_0\right) \cup \left(Y_0^{\delta} \times T_0\right)\\ &= \left(Y_3^{\delta} \times T_3\right) \cup \left(\left(Y_2^{\delta} \cup Y_0^{\delta}\right) \times T_0\right) = \alpha,\end{aligned}$$

if $Y_3^{\delta} = Y_3^{\alpha}$, $Y_2^{\delta} \cup Y_0^{\delta} = Y_0^{\alpha}$. Last equalities are possible since $\left|Y_2^{\delta} \cup Y_0^{\delta}\right| \geq 1$ ($\left|Y_0^{\delta}\right| \geq 0$ by preposition).

The statement d) of the lemma 2.2.3 is proved.

5) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_4^{\delta} \times T_4\right) \cup \left(Y_0^{\delta} \times T_0\right),$$
$$\beta = \left(\left(\left(T_2 \cap T_1\right) \setminus T_3\right) \times T_4\right) \cup \left(\left(T_2 \setminus T_1\right) \times T_2\right) \cup \left(\left(X \setminus T_4\right) \times T_0\right),$$

where $Y_4^{\delta}, Y_0^{\delta} \notin \{\varnothing\}$,

$$\left(\left(T_2 \cap T_1\right) \setminus T_3\right) \cup \left(T_2 \setminus T_1\right) \cup \left(X \setminus T_4\right) = P_3 \cup P_1 \cup \left(X \setminus T_4\right) = T_4 \cup \left(X \setminus T_4\right) = X.$$

(See Equalities (2.2.1) and (2.2.2)). Then from the statement b) of the Lemma 2.2.3 follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement a) of the Lemma 2.2.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_4\beta = \left(P_1 \cup P_3\right)\beta = T_4 \cup T_2 = T_2,\ T_0\beta = T_0.$$
$$\delta \circ \beta = \left(Y_4^{\delta} \times T_4\beta\right) \cup \left(Y_0^{\delta} \times T_0\beta\right) = \left(Y_4^{\delta} \times T_2\right) \cup \left(Y_0^{\delta} \times T_0\right) = \alpha,$$

if $Y_4^{\delta} = Y_2^{\alpha}$, $Y_0^{\delta} = Y_0^{\alpha}$. Last equalities are possible since $\left|Y_4^{\delta}\right| \geq 1$ $\left|Y_0^{\delta}\right| \geq 1$.

The statement e) of the lemma 2.2.3 is proved.

6) Let quasinormal representation of a binary relations δ, β have a form

$$\delta = \left(Y_3^{\delta} \times T_3\right) \cup \left(Y_0^{\delta} \times T_0\right),$$
$$\beta = \left(\left(\left(T_2 \cap T_1\right) \setminus T_4\right) \times T_3\right) \cup \left(\left(T_1 \setminus T_2\right) \times T_1\right) \cup \left(\left(X \setminus T_3\right) \times T_0\right),$$

where $Y_3^{\delta}, Y_0^{\delta} \notin \{\varnothing\}$,

$$\left(\left(T_2 \cap T_1\right) \setminus T_4\right) \cup \left(T_1 \setminus T_2\right) \cup \left(X \setminus T_3\right) = P_4 \cup P_2 \cup \left(X \setminus T_3\right) = T_3 \cup \left(X \setminus T_3\right) = X.$$

(see Equalities (2.2.1) and (2.2.2)). Then from the statement d) of the Lemma 2.2.3 follows that δ is generating by elements of the set $B(\mathfrak{A}_0)$ and from the statement b) of the Lemma 2.2.2 element β is generating by elements of the set $B(\mathfrak{A}_0)$ and

$$T_3\beta = \left(P_2 \cup P_4\right)\beta = T_3 \cup T_1 = T_1,\ T_0\beta = T_0.$$
$$\delta \circ \beta = \left(Y_3^{\delta} \times T_3\beta\right) \cup \left(Y_0^{\delta} \times T_0\beta\right) = \left(Y_3^{\delta} \times T_1\right) \cup \left(Y_0^{\delta} \times T_0\right) = \alpha,$$

if $Y_3^{\delta} = Y_1^{\alpha}$, $Y_0^{\delta} = Y_0^{\alpha}$. Last equalities are possible since $\left|Y_4^{\delta}\right| \geq 1$ $\left|Y_0^{\delta}\right| \geq 1$.

The statement e) of the lemma 2.2.3 is proved.

7) Let quasinormal representation of a binary relations δ, β have a form

$$\delta=\left(Y_2^{\delta}\times T_2\right)\cup\left(Y_0^{\delta}\times T_0\right),$$
$$\beta=\left(T_1\times T_4\right)\cup\left(\left(T_2\setminus T_1\right)\times T_2\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_2^{\delta},Y_0^{\delta}\notin\{\varnothing\}$,

$$T_1\cup\left(T_2\setminus T_1\right)\cup\left(X\setminus T_0\right)=\left(P_2\cup P_3\cup P_4\right)\cup P_1\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X$$

(see Equalities (2.2.1) and (2.2.2)). Then from the statement e) of the Lemma 2.2.3 follows that δ is generating by elements of the set $B\left(\mathfrak{A}_0\right)$ and from the statement a) of the Lemma 2.2.2 element β is generating by elements of the set $B\left(\mathfrak{A}_0\right)$ and

$$T_2\beta=T_4\cup T_2=T_2,T_0\beta=T_2$$
$$\delta\circ\beta=\left(Y_2^{\delta}\times T_2\beta\right)\cup\left(Y_0^{\delta}\times T_0\beta\right)=\left(Y_2^{\delta}\times T_2\right)\cup\left(Y_0^{\delta}\times T_2\right)=X\times T_2=\alpha,$$

since representation of a binary relation δ is quasinormal.

The statement g) of the lemma 2.2.3 is proved.

8) Let quasinormal representation of a binary relations δ, β have a form

$$\delta=\left(Y_1^{\delta}\times T_1\right)\cup\left(Y_0^{\delta}\times T_0\right),$$
$$\beta=\left(T_2\times T_3\right)\cup\left(\left(T_1\setminus T_2\right)\times T_1\right)\cup\left(\left(X\setminus T_0\right)\times T_0\right),$$

where $Y_1^{\delta},Y_0^{\delta}\notin\{\varnothing\}$,

$$T_2\cup\left(T_1\setminus T_2\right)\cup\left(X\setminus T_0\right)=\left(P_1\cup P_3\cup P_4\right)\cup P_2\cup\left(X\setminus T_0\right)=T_0\cup\left(X\setminus T_0\right)=X$$

(see Equalities (2.2.1) and (2.2.2)). Then from the statement f) of the Lemma 2.2.3 follows that δ is generating by elements of the set $B\left(\mathfrak{A}_0\right)$ and from the statement b) of the Lemma 2.2.2 element β is generating by elements of the set $B\left(\mathfrak{A}_0\right)$ and

$$T_1\beta=T_3\cup T_1=T_1,T_0\beta=T_1$$
$$\delta\circ\beta=\left(Y_1^{\delta}\times T_1\beta\right)\cup\left(Y_0^{\delta}\times T_0\beta\right)=\left(Y_1^{\delta}\times T_1\right)\cup\left(Y_0^{\delta}\times T_1\right)=X\times T_1=\alpha,$$

since representation of a binary relation δ is quasinormal.

The statement h) of the lemma 2.2.3 is proved.

9) Let quasinormal representation of a binary relation δ has a form

$$\delta=\left(T_4\times T_1\right)\cup\left(\left(X\setminus T_4\right)\times T_0\right),$$

then

$$T_1\delta=\left(P_2\cup P_3\cup P_4\right)\delta=T_4\cup T_0=T_0,\ T_0\delta=T_0$$
$$\delta\circ\delta=\left(T_4\times T_1\delta\right)\cup\left(\left(X\setminus T_4\right)\times T_0\delta\right)=\left(T_4\times T_0\right)\cup\left(\left(X\setminus T_4\right)\times T_0\right)=X\setminus T_0=\alpha$$

since representation of a binary relation δ is quasinormal.

The statement i) of the lemma 2.2.3 is proved.

Lemma 2.2.3 is proved.

Lemma 2.2.4. *Let* $D\in\Sigma_{8.1}(X,5)$. *Then the following statements are true.*

a) *If* $\left|X\setminus T_0\right|\geq 1$ *and* $Z\in\left\{T_4,T_3\right\}$, *then binary relation* $\alpha=X\times Z$ *is gene-*

rating by elements of the elements of set $B(\mathfrak{A}_0)$;

b) *If* $X=T_0$ *and* $Z\in\{T_4,T_3\}$, *then binary relation* $\alpha=X\times Z$ *is external element for the semigroup* $B_X(D)$.

Proof. 1) Let quasinormal representation of a binary relation δ has a form

$$\delta=\left(Y_4^\delta\times T_4\right)\cup\left(Y_3^\delta\times T_3\right)\cup\left(Y_0^\delta\times T_0\right),$$

where $Y_4^\delta, Y_3^\delta\notin\{\varnothing\}$, then $\delta\in B(\mathfrak{A}_0)\setminus\{\alpha\}$. If quasinormal representation of a binary relation β has a form $\beta=(T_0\times T)\cup\bigcup_{t'\in X\setminus T_0}\left(\{t'\}\times f(t')\right)$, where f is any mapping of the set $X\setminus T_0$ in the set $\{T_4,T_3\}\setminus\{Z\}$. It is easy to see, that $\beta\neq\alpha$ and two elements of the set $\{T_4,T_3\}$ belong to the semilattice $V(D,\beta)$, *i.e.* $\delta\in B(\mathfrak{A}_0)\setminus\{\alpha\}$. In this case we have

$$\begin{aligned}
&T_4\beta=T_3\beta=T_0\beta=Z;\\
&\delta\circ\beta=\left(Y_4^\delta\times T_4\beta\right)\cup\left(Y_3^\delta\times T_3\beta\right)\cup\left(Y_0^\delta\times T_0\beta\right)\\
&\quad=\left(Y_4^\delta\times Z\right)\cup\left(Y_3^\delta\times Z\right)\cup\left(Y_0^\delta\times Z\right)\\
&\quad=\left(\left(Y_4^\delta\cup Y_3^\delta\cup Y_0^\delta\right)\times Z\right)=X\times Z=\alpha,
\end{aligned}$$

since the representation of a binary relation δ is quasinormal. Thus, element α is generating by elements of the set $B(\mathfrak{A}_0)$.

The statement a) of the lemma 2.2.4 is proved.

2) Let $X=T_0$, $\alpha=X\times Z$, for some $Z\in\{T_4,T_3\}$ and $\alpha=\delta\circ\beta$ for some $\delta,\beta\in B_X(D)\setminus\{\alpha\}$. Then from the Equalities (2.2.3) and (2.2.4) we obtain that

$$T_4\beta=T_3\beta=T_2\beta=T_1\beta=T_0\beta=Z,\ P_1\beta=P_2\beta=P_3\beta=P_4\beta=Z,$$

since Z is minimal element of the semilattice D.

Now, let subquasinormal representations $\overline{\beta}$ of a binary relation β has a form

$$\overline{\beta}=\left((P_1\cup P_2\cup P_3\cup P_4)\times Z\right)\cup\bigcup_{t'\in X\setminus T_0}\left(\{t'\}\times\overline{\beta}_2(t')\right),$$

where $\overline{\beta}_1=\begin{pmatrix}P_0 & P_1 & P_2 & P_3 & P_4\\ \varnothing & Z & Z & Z & Z\end{pmatrix}$ is normal mapping. But complement mapping $\overline{\beta}_2$ is empty, since $X\setminus T_0=\varnothing$, *i.e.* in the given case, subquasinormal representation $\overline{\beta}$ of a binary relation β is defined uniquely. So, we have that $\beta=\overline{\beta}=X\times Z=\alpha$, which contradicts the condition $\beta\notin B_X(D)\setminus\{\alpha\}$.

Therefore, if $X=T_0$ and $\alpha=X\times Z$, for some $Z\in\{T_4,T_3\}$, then α is external element of the semigroup $B_X(D)$.

The statement b) of the lemma 2.2.4 is proved.

lemma 2.2.4 is proved.

Theorem 2.2.1. *Let* $D\in\Sigma_{8.1}(X,5)$ *and*

$$\begin{aligned}
\mathfrak{A}_0=\{&\{T_4,T_3,T_2,T_0\},\{T_4,T_3,T_1,T_0\},\{T_4,T_2,T_1,T_0\},\{T_3,T_2,T_1,T_0\},\\
&\{T_4,T_3,T_0\},\{T_4,T_1,T_0\},\{T_3,T_2,T_0\}\},
\end{aligned}$$

$$B(\mathfrak{A}_0)=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)\in\mathfrak{A}_0\right\};\ B_0=\left\{\alpha\in B_X(D)\,|\,V\left(X^*,\alpha\right)=D\right\}.$$

Then the following statements are true:

a) *If* $\left|X \setminus T_0\right| \geq 1$, *then* $S_0 = B_0 \cup B(\mathfrak{A}_\circ)$ *is irreducible generating set for the semigroup*.

b) *If* $X = T_0$, *then* $S_1 = B_0 \cup B(\mathfrak{A}_\circ) \cup \{X \times T_4, X \times T_3\}$ *is irreducible generating set for the semigroup* $B_X(D)$.

Proof. The theorem 2.2.1 we may prove analogously of the theorems 2.1.1.

Theorem 2.2.2. *Let* $n \geq 6$, $D = \{T_4, T_3, T_2, T_1, T_0\} \in \Sigma_{8.1}(X, 5)$ *and*

$$\mathfrak{A}_0 = \{\{T_4, T_3, T_2, T_0\}, \{T_4, T_3, T_1, T_0\}, \{T_4, T_2, T_1, T_0\}, \{T_3, T_2, T_1, T_0\}, \{T_4, T_3, T_0\}, \{T_4, T_1, T_0\}, \{T_3, T_2, T_0\}\},$$

$$B(\mathfrak{A}_0) = \left\{\alpha \in B_X(D) \mid V\left(X^*, \alpha\right) \in \mathfrak{A}_0\right\};\ B_0 = \left\{\alpha \in B_X(D) \mid V\left(X^*, \alpha\right) = D\right\}.$$

Then the following statements are true:

a) *If* $\left|X \setminus T_0\right| \geq 1$, *then the number* $|S_0|$ *elements of the set* $S_0 = B_0 \cup B(\mathfrak{A}_0)$ *is equal to*

$$|S_0| = 5^n - 3 \cdot 3^n + 2 \cdot 2^n + 2.$$

b) *If* $X = T_0$, *then the number* $|S_1|$ *elements of the set* $S_1 = B_0 \cup B(\mathfrak{A}_\circ) \cup \{X \times T_4, X \times T_3\}$ *is equal to*

$$|S_1| = 5^n - 3 \cdot 3^n + 2 \cdot 2^n + 4.$$

Proof. Let number of a set *X* is equal to $n \geq 6$, *i.e.* $|X| = n \geq 6$. Let $S_n = \{\varphi_1, \varphi_2, \cdots, \varphi_{n!}\}$ is a group all one to one mapping of a set $M = \{1, 2, \cdots, n\}$ on the set *M* and $\varphi_{i_1}, \varphi_{i_2}, \cdots, \varphi_{i_m}$ $(m \leq n)$ are arbitrary elements of the group S_n, $Y_{\varphi_1}, Y_{\varphi_2}, \cdots, Y_{\varphi_m}$ are arbitrary partitioning of a set *X*. By symbol k_n^m we denote the number elements of a set $\left\{Y_{\varphi_1}, Y_{\varphi_2}, \cdots, Y_{\varphi_m}\right\}$. It is well known, that

$$k_n^m = \sum_{i=1}^{m} \frac{(-1)^{m+i}}{(i-1)! \cdot (m-i)!} \cdot i^{n-1}.$$

If $m = 2, 3, 4, 5$, then we have

$$k_n^2 = 2^{n-1} - 1,\ k_n^3 = \frac{1}{2} \cdot 3^{n-1} - 2^{n-1} + \frac{1}{2},\ k_n^4 = \frac{1}{6} \cdot 4^{n-1} - \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} \cdot 2^{n-1} - \frac{1}{6},$$

$$k_n^5 = \frac{1}{24} \cdot 5^{n-1} - \frac{1}{6} \cdot 4^{n-1} + \frac{1}{4} \cdot 3^{n-1} - \frac{1}{6} \cdot 2^{n-1} + \frac{1}{24}.$$

If $Y_{\varphi_1}, Y_{\varphi_2}$ are any two elements partitioning of a set *X* and $\overline{\beta} = \left(Y_{\varphi_1} \times Z_1\right) \cup \left(Y_{\varphi_2} \times Z_2\right)$, where $Z_1, Z_2 \in D$ and $Z_1 \neq Z_2$. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$2 \cdot k_n^2 = 2^n - 2. \tag{2.2.5}$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}$ are any tree elements partitioning of a set *X* and

$$\overline{\beta} = \left(Y_{\varphi_1} \times Z_1\right) \cup \left(Y_{\varphi_2} \times Z_2\right) \cup \left(Y_{\varphi_3} \times Z_3\right),$$

where Z_1, Z_2, Z_3 are pairwise different elements of a given semilattice *D*. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$6 \cdot k_n^3 = 3^n - 3 \cdot 2^n + 3. \tag{2.2.6}$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}, Y_{\varphi_4}$ are any four elements partitioning of a set X and

$$\overline{\beta} = \left(Y_{\varphi_1} \times Z_1\right) \cup \left(Y_{\varphi_2} \times Z_2\right) \cup \left(Y_{\varphi_3} \times Z_3\right) \cup \left(Y_{\varphi_4} \times Z_4\right),$$

where Z_1, Z_2, Z_3, Z_4 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$24 \cdot k_n^4 = 4^n - 4 \cdot 3^n + 3 \cdot 2^{n+1} - 4. \tag{2.2.7}$$

If $Y_{\varphi_1}, Y_{\varphi_2}, Y_{\varphi_3}, Y_{\varphi_4}, Y_{\varphi_5}$ are any four elements partitioning of a set X and

$$\overline{\beta} = \left(Y_{\varphi_1} \times Z_1\right) \cup \left(Y_{\varphi_2} \times Z_2\right) \cup \left(Y_{\varphi_3} \times Z_3\right) \cup \left(Y_{\varphi_4} \times Z_4\right) \cup \left(Y_{\varphi_5} \times Z_5\right),$$

where Z_1, Z_2, Z_3, Z_4, Z_5 are pairwise different elements of a given semilattice D. Then number of different binary relations $\overline{\beta}$ of a semigroup $B_X(D)$ is equal to

$$120 \cdot k_n^5 = 5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5. \tag{2.2.8}$$

If $\alpha \in B_0$, then quasinormal representation of a binary relation α has a form

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X.

If the system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha$, or a system $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha$ are partitioning of the set X. Of this from the Equalities (2.2.7) and (2.2.8) follows that

$$\begin{aligned} |B_0| &= \left(5^n - 5 \cdot 4^n + 10 \cdot 3^n - 10 \cdot 2^n + 5\right) + \left(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4\right) \\ &= 5^n - 4 \cdot 4^n + 6 \cdot 3^n - 4 \cdot 2^n + 1. \end{aligned}$$

If $\alpha \in B(\mathfrak{A}_0)$, then by definition of a set $B(\mathfrak{A}_0)$ the quasinormal representation of a binary relation α has a form:

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_3^\alpha \times T_3\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_3^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_4^\alpha \times T_4\right) \cup \left(Y_1^\alpha \times T_1\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_4^\alpha, Y_1^\alpha \notin \{\varnothing\}$, or $Y_4^\alpha, Y_1^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively;

$$\alpha = \left(Y_3^\alpha \times T_3\right) \cup \left(Y_2^\alpha \times T_2\right) \cup \left(Y_0^\alpha \times T_0\right),$$

where $Y_3^\alpha, Y_2^\alpha \notin \{\varnothing\}$, or $Y_3^\alpha, Y_2^\alpha, Y_0^\alpha \notin \{\varnothing\}$ are partitioning of the set X respectively.

Of this and from the Equality (2.2.5), (2.2.6) and (2.2.7) follows that

$$\begin{aligned}\left|B(\mathfrak{A}_0)\right| &= 3\cdot\left(2^n - 2\right) + 7\cdot\left(3^n - 3\cdot 2^n + 3\right) + 4\cdot\left(4^n - 4\cdot 3^n + 6\cdot 2^n - 4\right)\\ &= 4\cdot 4^n - 9\cdot 3^n + 6\cdot 2^n + 1.\end{aligned}$$

So, we have that:

$$\begin{aligned}|S_0| &= \left|B_0 \cup B(\mathfrak{A}_0)\right|\\ &= \left(5^n - 4\cdot 4^n + 6\cdot 3^n - 4\cdot 2^n + 1\right) + \left(4\cdot 4^n - 9\cdot 3^n + 6\cdot 2^n + 1\right)\\ &= 5^n - 3\cdot 3^n + 2\cdot 2^n + 2,\\ |S_1| &= \left|B_0 \cup B(\mathfrak{A}_0) \cup \{X\times T_4, X\times T_3\}\right| = 5^n - 3\cdot 3^n + 2\cdot 2^n + 4.\end{aligned}$$

Since

$$B_0 \cap B(\mathfrak{A}_0) = B_0 \cap \{X\times T_4, X\times T_3, X\times T_2\} = B(\mathfrak{A}_0) \cap \{X\times T_4, X\times T_3, X\times T_2\} = \varnothing.$$

Theorem 2.2.2 is proved.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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