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A Set of Almost Automorphic Functions and Applications

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Abstract

For a set $\mathbb{S}$ of real numbers, we introduce the concept of $\mathbb{S}$-almost automorphic functions valued in a Banach space. It generalizes in particular the space of $\mathbb{Z}$-almost automorphic functions. Considering the space of $\mathbb{S}$-almost automorphic functions, we give sufficient conditions of the existence and uniqueness of almost automorphic solutions of a differential equation with a piecewise constant argument of generalized type. This is done using the Banach fixed point theorem.

Keywords

Almost Automorphic Functions, $\mathbb{S}$-Almost Automorphic Functions, Differential Equation with Piecewise Constant Argument of Generalized Type

1. Introduction

The almost periodic functions were introduced by Bohr in 1925 and described phenomena that are similar to the periodic oscillations which can be observed in many fields, such as celestial mechanics, nonlinear vibration, electromagnetic theory, plasma physics, and engineering. An important generalization of the almost periodicity is the concept of the almost automorphy introduced in the literature [1] [2] [3] [4] by Bochner. In [5], the author presents the theory of almost automorphic functions and their applications to differential equations.

The study of differential equations with piecewise constant argument (EPCA) is an important subject because these equations have the structure of continuous dynamical systems in intervals of unit length. Therefore they combine the properties of both differential and difference equations. There have been many papers studying DEPCA, see for instance [6]-[11] and the references therein.

Some papers deal with the existence of asymptotically ω-periodic solutions (see for instance [12]), S-asymptotically ω-periodic solutions of DEPCA (see [13]). Other articles deal with the existence of almost automorphic solutions of EPCA (see [14] [15]). In this paper, we study the existence of almost automorphic solutions of the following differential equation with the piecewise constant argument of generalized (DEPCAG) type (see [16] [17] [18]):

$$x'(t) = A(t)x(\varphi(t)) + f(t, x(\varphi(t))), t \in \mathbb{R} \tag{1}$$

where φ is a step function, $A: \mathbb{R} \to \mathbb{R}^{q \times q}$ is continuous in $\mathbb{R} \setminus \mathbb{S}$ and $f: \mathbb{R} \times \mathbb{R}^q \to \mathbb{R}^q$ is continuous. More precisely, there exists a strictly increasing sequence of real numbers $t_i, i \in \mathbb{Z}$, such that $t_i \to_{-}^{+} \infty$ as $i \to_{-}^{+} \infty$ and on each interval $[t_i, t_{i+1}[$, $\varphi(t)$ is constant:

$$\varphi(t) = g_n, \quad t_n \le t < t_{n+1}.$$

In order to give sufficient conditions of existence and uniqueness of almost automorphic solutions of Equation (1), we introduce the concept of $\mathbb{S}$-almost automorphic functions that generalizes the one of $\mathbb{Z}$-almost automorphic ([19]) ones, where $\mathbb{S}$ is a subset of $\mathbb{R}$. In this paper, in order to study the almost automorphic solutions of (1), we will not consider almost automorphic sequence, but we will use the theory of fixed point.

The paper is organized as follows. In Section 2, we recall definitions and properties of almost automorphic functions and introduce the concept of $\mathbb{S}$-almost automorphic functions. In Section 3, we also study the existence and uniqueness of almost automorphic solutions of Equation (1) considering the concept of $\mathbb{S}$-almost automorphic functions and using the Banach fixed point Theorem.

2. Almost Automorphic Functions with Respect to a Set

Let $\mathbb{S}$ denote a subset of $\mathbb{R}$. For every non zero real number r we consider the function $\varphi_r: \mathbb{R} \to \mathbb{R}$ such that for every $(t, s) \in \mathbb{R} \times \mathbb{S}$:

$$\varphi_r(t+s) = \varphi_r(t) + rs. \tag{2}$$

In particular for all $s \in \mathbb{S}$ we have:

$$\varphi_r(s) = rs + \varphi_r(0).$$

Definition 2.1. *A subset A of* $\mathbb{R}$ *is said to be r-stable if it is invariant under the homothety of ratio r and center* 0.

We give an example of such a set $\mathbb{S}$ and an associated function φ_r.

Example 2.1. *Let* $\mathbb{S}$ *be a discrete subgroup of* $\mathbb{R}$, *then* $\mathbb{S} = \alpha\mathbb{Z}$ *for some* (*non negative*) *real* α, *and* $\mathbb{S}$ *is obviously r-stable for all non zero integer r. Set* $\varphi_r(t) = [rt/\alpha]\alpha + c$ *where* $[.]$ *is the integer part function and c is a constant; then it is easily seen that* (*2*) *is satisfied.*

Proposition 2.1. *The function* φ_r *satisfies the following properties*:

1) $\forall (t, s) \in \mathbb{R} \times \mathbb{S}$, $\varphi_r(t-s) = \varphi_r(t) - rs$.

2) $\forall(s_1,s_2,\cdots,s_p)\in\mathbb{S}^p$, $\forall(m_1,m_2,\cdots,m_p)\in\mathbb{Z}^p$:

$$\varphi_r(m_1s_1+\cdots+m_ps_p)=r(m_1s_1+\cdots+m_ps_p)+\varphi_r(0).$$

Proof. Substituting $t-s$ for t in (2), gives (1); and (2) is obtained by induction from $\varphi_r(t+m_ps_p)=\varphi_r(t)+m_prs_p$ where $t=m_1s_1+\cdots+m_{p-1}s_{p-1}$ and noticing that $\varphi_r(m_1s_1)=m_1rs_1+\varphi_r(0)$. □

In all the sequel $\mathbb{X}$ denotes a real or complex Banach space.

Definition 2.2. *A function* $f:\mathbb{R}\to\mathbb{X}$ *is said to be* $\mathbb{S}$*-continuous if it is continuous in* $\mathbb{R}\setminus\mathbb{S}$, *which is referred as an* $\mathbb{S}$*-continuous function.*

The set of all $\mathbb{S}$-continuous functions $f:\mathbb{R}\to\mathbb{X}$ will be denoted by $\mathbb{S}C(\mathbb{R},\mathbb{X})$ and the set of those that are bounded by $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$. Clearly $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$ is a closed subspace of the Banach space $C_b(\mathbb{R},\mathbb{X})$ of bounded continuous functions and then it is also a Banach space.

Definition 2.3. *A bounded* $\mathbb{S}$*-continuous function* $f:\mathbb{R}\to\mathbb{X}$ *is said to be almost automorphic with respect to the set* $\mathbb{S}$ *if for every real sequence* s' *valued in* $\mathbb{S}$, *there are a subsequence s and a function* $g:\mathbb{R}\to\mathbb{X}$ *such that for all* $t\in\mathbb{R}$:

$$\lim_{n\to\infty}f(t+s_n)=g(t) \text{ and } \lim_{n\to\infty}g(t-s_n)=f(t). \quad (3)$$

Such a function f is called $\mathbb{S}$*-almost automorphic and if the above limits are uniform, it is called* $\mathbb{S}$*-almost periodic.*

The set of all $\mathbb{S}$-almost automorphic (resp. almost periodic) functions will be denoted by $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ (resp. $\mathbb{S}AP(\mathbb{R},\mathbb{X})$). Clearly $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ is a subspace of the Banach space $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$; we have the following:

Theorem 2.2. *The space* $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ *is a Banach space.*

Proof. We have just to show that $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ is a closed set in the Banach space $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$. For this purpose we use a diagonal process. Let (f_p) denote a sequence in $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ which converges to a function f in $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$ and let s' be any sequence of elements of $\mathbb{S}$. It follows from Definition 2.3 that there exists a subsequence s^1 of s' and a function g_1 such that (3) holds when we replace s and g with s^1 and g_1 respectively. Then, by induction, we can build a sequence (s^k) extracted from s^{k-1}, where s^{k-1} is a subsequence of s', and a sequence of functions g_k such that:

$$\lim_{n\to\infty}f_k(t+s_n^k)=g_k(t) \text{ and } \lim_{n\to\infty}g_k(t-s_n^k)=f_k(t).$$

Let $t\in\mathbb{R}$ and take $\varepsilon>0$. For $p,q,k,n\in\mathbb{N}^*$, we have:

$$\|g_p(t)-g_q(t)\|\le\|g_p(t)-f_p(t+s_n^k)\|+\|f_p(t+s_n^k)-f_q(t+s_n^k)\| + \|f_q(t+s_n^k)-g_q(t)\|.$$

Since (f_k) converges to f in $\mathbb{S}C_b(\mathbb{R},\mathbb{X})$, there is $q_0\in\mathbb{N}^*$ such that:

$$\|f_p(u)-f_q(u)\|\le\frac{\varepsilon}{3},\forall p,q\ge q_0,\forall u\in\mathbb{R}.$$

Therefore, if $p,q\ge q_0$, $k=p+q$ and $n\in\mathbb{N}^*$, it follows that:

$$\left\|g_p(t)-g_q(t)\right\| \le \left\|g_p(t)-f_p\left(t+s_n^k\right)\right\|+\left\|f_q\left(t+s_n^k\right)-g_q(t)\right\|+\frac{\varepsilon}{3}.$$

The condition $k=p+q$ implies that $k \ge \max(p,q)$; then s^k is a subsequence of both s^p and s^q. Then, $\left(f_p\left(t+s_n^k\right)\right)$ and $\left(f_q\left(t+s_n^k\right)\right)$ converge to $g_p(t)$ and $g_q(t)$ respectively as $n \to +\infty$. Consequently there exists $n_0 \in \mathbb{N}^*$ depending on *p* and *q* such that:

$$\max\left(\left\|g_p(t)-f_p\left(t+s_n^k\right)\right\|,\left\|f_q\left(t+s_n^k\right)-g_q(t)\right\|\right) \le \frac{\varepsilon}{3},$$

if $n \ge n_0$. Thus, given $t \in \mathbb{R}$ and $\varepsilon > 0$, we have found $q_0 \in \mathbb{N}^*$ such that $\left\|g_p(t)-g_q(t)\right\| \le \varepsilon$ if $\max(p,q) \ge q_0$. This means that $\left(g_k(t)\right)$ is a Cauchy sequence of real numbers. Thus $\left(g_k\right)$ converges to a bounded measurable function *g*. On the other hand, if $n \in \mathbb{N}^*$ and $t \in \mathbb{R}$, we can write:

$$\begin{aligned}\left\|f\left(t+s_n^n\right)-g(t)\right\| \le{}& \left\|f\left(t+s_n^n\right)-f_n\left(t+s_n^n\right)\right\|+\left\|g_n(t)-g(t)\right\| \\ &+\left\|f_n\left(t+s_n^n\right)-g_n(t)\right\|.\end{aligned}$$

For each $k \in \mathbb{N}^*$, $f_k\left(t+s_n^k\right)-g_k(t)$ converges to 0 as $n \to +\infty$, it follows that the diagonal sequence $f_n\left(t+s_n^n\right)-g_n(t)$ also converges to 0. Since the sequence $\left(f_k\right)$ converges uniformly to *f* and $g_k(t)$ converges to $g(t)$, it follows that $\lim_{n\to 0} f\left(t+s_n^n\right)=g(t)$. It remains to show that $\left(g\left(t-s_n^n\right)\right)$ converges to $f(t)$. It is sufficient to prove that the sequence $\left(g_k\right)$ converges uniformly since we can deal as before where we proved that $\lim_{n\to+\infty} f\left(t+s_n^n\right)=g(t)$. To do that, we keep the above notation with $p,q \in \mathbb{N}$ and $k=p+q$. Then, from (3) we have:

$$\lim_{n\to\infty}\left[g_p\left(t-s_n^k\right)-g_q\left(t-s_n^k\right)\right]=f_p(t)-f_q(t).$$

Let $\varepsilon > 0$. Using the uniform convergence of the sequence $\left(f_k\right)$, we get $q_0 \in \mathbb{N}$ such that $\left\|f_p(t)-f_q(t)\right\| \le \frac{\varepsilon}{2}$ for $\min(p,q) \ge q_0$ and all $t \in \mathbb{R}$. From the definition of the limit, there is $n_0 \in \mathbb{N}$ depending on *p* and *q* such that:

$$\left\|g_p\left(t-s_n^k\right)-g_q\left(t-s_n^k\right)\right\| \le \left\|f_p(t)-f_q(t)\right\|+\frac{\varepsilon}{2}, n \ge n_0.$$

It follows that:

$$\left\|g_p\left(t-s_n^k\right)-g_q\left(t-s_n^k\right)\right\| \le \varepsilon, \forall n \ge n_0, \forall t \in \mathbb{R}. \tag{4}$$

Now replacing *t* by $t+s_n^k$ in (4) yields $\left\|g_p(t)-g_q(t)\right\| \le \varepsilon$ for $\min(p,q) \ge q_0$ and all $t \in \mathbb{R}$. The uniform convergence of $\left(g_k\right)$ is thus established. Then *f* belongs to $\mathbb{S}AA(\mathbb{R},\mathbb{X})$ proving the theorem. □

Proposition 2.3. *Let* $\mathbb{S}$ *be r-stable and* $\varphi_r \in \mathbb{S}C(\mathbb{R},\mathbb{X})$. *If* $f \in AA(\mathbb{R},\mathbb{X})$ (*resp.* $AP(\mathbb{R},\mathbb{X})$), *then* $f \circ \varphi_r \in \mathbb{S}AA(\mathbb{R},\mathbb{X})$ (*resp.* $\mathbb{S}AP(\mathbb{R},\mathbb{X})$). *If* $f \in \mathbb{S}AA(\mathbb{R},\mathbb{X})$ (*resp.* $\mathbb{S}AP(\mathbb{R},\mathbb{X})$) *and* $\mathbb{S} \cap \varphi_r(\mathbb{R} \setminus \mathbb{S}) = \varnothing$, *the same conclusion holds.*

Proof. We keep the notation of Definition 2.3. For the given sequence *s'* we consider the sequence *rs'*. Then we get an associate subsequence *rs* together with a function g. It follows from (3) and the properties of the function φ_r that $f\circ\varphi_r(t+s_n)=f(\varphi_r(t)+rs_n)$ converges to $g\circ\varphi_r(t)$ and $g\circ\varphi_r(t-s_n)=g(\varphi_r(t)-rs_n)$ converges to $f\circ\varphi_r(t)$. These convergences are uniform if it is the case in (3). This proves the first part of the Proposition; the second part can be deduced straightforwardly. □

We associate to the subset $\mathbb{S}$ the following property:

(P1) There is a bounded set K_0 in $\mathbb{R}$ such that all real t can be written as $t=\alpha+\xi$ where $\alpha\in K_0$ and $\xi\in\mathbb{S}$.

Then we have the following:

Proposition 2.4. *Let* $\mathbb{S}$ *satisfy* **(P1)** *and let f be an* $\mathbb{S}$*-almost automorphic (resp.* $\mathbb{S}$*-almost periodic) function. If f is uniformly continuous, then f is almost automorphic (resp. almost periodic).*

Proof. As above we use the notation of Definition 2.3. Since $\overline{K_0}$ is a compact set we may assume that $s_n=\alpha_n+\xi_n$ for each $n\in\mathbb{N}^*$ with $\alpha_n\in K_0$, $\xi_n\in\mathbb{S}$ and $\lim_{n\to\infty}\alpha_n=\alpha$. Then we have:

$$\begin{aligned}&\left\|f(t+s_n)-g(t+\alpha)\right\|\\ &\le\left\|f(t+\alpha_n+\xi_n)-f(t+\alpha+\xi_n)\right\|+\left\|f(t+\alpha+\xi_n)-g(t+\alpha)\right\|.\end{aligned}$$

The uniform continuity of f shows that the first term on the right side tends to zero. Since f is $\mathbb{S}$-almost automorphic, it follows that the second term also converges to zero. On the other hand, f being uniformly continuous, the same holds for g. Then writing:

$$\left\|g(t+\alpha-s_n)-f(t)\right\|\le\left\|g(t-\xi_n+(\alpha-\alpha_n))-g(t-\xi_n)\right\|+\left\|g(t-\xi_n)-f(t)\right\|$$

shows that $(g(t+\alpha-s_n))$ converges to $f(t)$ which proves that f is almost automorphic. The almost periodic case follows straightforwardly. □

Remark 2.1. *We note that* $\mathbb{S}=\mathbb{Z}$ *satisfies the condition* **(P1)**: *it suffices to take* $K_0=[0,1[$, *since for every real number x,* $x-[x]\in[0,1[$.

Definition 2.4. *A continuous function* $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ *is said to be almost automorphic in* $t\in\mathbb{R}$ *for each* $x\in\mathbb{X}$, *if for every sequence of real numbers* (s'_n), *there exists a subsequence* (s_n) *such that for each* $t\in\mathbb{R}$ *and* $x\in\mathbb{X}$,

$$\lim_{n\to\infty}f(t+s_n,x)=g(t,x)\quad\text{and}\quad\lim_{n\to\infty}g(t-s_n,x)=f(t,x).$$

Then we have the following result.

Theorem 2.5. ([5], *Theorem* 2.2.5) *If f is almost automorphic in* $t\in\mathbb{R}$ *for each* $x\in\mathbb{X}$ *and if f is Lipschitzian in x uniformly in t, then g satisfies the same Lipschitz condition in x uniformly in t.*

Using the above theorem we obtain:

Theorem 2.6. *Let* $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ *be almost automorphic in* $t\in\mathbb{R}$ *for each* $x\in\mathbb{X}$. *Assume that f satisfies a Lipschitz condition in x uniformly in* $t\in\mathbb{R}$. *Let also* $\phi:\mathbb{R}\to\mathbb{X}$ *be almost automorphic. Then the function* $F:\mathbb{R}\to\mathbb{X}$ *defined by* $F(t)=f(t,\phi(\varphi_r(t)))$ *is* $\mathbb{S}$*-almost automorphic.*

Proof. Let (s'_n) be a sequence of $\mathbb{S}$. Using Proposition 2.3, we can extract a subsequence s_n such that:

1) $\lim_{n\to\infty} f(t+s_n,x)=g(t,x)$ for each $t\in\mathbb{R}$ and $x\in\mathbb{X}$,
2) $\lim_{n\to\infty} g(t-s_n,x)=f(t,x)$ for each $t\in\mathbb{R}$ and $x\in\mathbb{X}$,
3) $\lim_{n\to\infty} \phi(t+s_n)=\Phi(t)$ for each $t\in\mathbb{R}$,
4) $\lim_{n\to\infty} \Phi(t-s_n)=\phi(t)$ for each $t\in\mathbb{R}$,
5) $\lim_{n\to\infty} \phi(\varphi_r(t+s_n))=\Phi(\varphi_r(t))$ for each $t\in\mathbb{R}$,
6) $\lim_{n\to\infty} \Phi(\varphi_r(t-s_n))=\phi(\varphi_r(t))$ for each $t\in\mathbb{R}$.

Consider the function $G:\mathbb{R}\to\mathbb{X}$ defined by $G(t)=g(t,\Phi(\varphi_r(t)))$. From the Lipchitz condition on *f*, there exists a constant $L>0$ such that:

$$\begin{aligned}\|F(t+s_n)-G(t)\| &= \|f(t+s_n,\phi(\varphi_r(t+s_n)))-g(t,\Phi(\varphi_r(t)))\|\\ &\le \|f(t+s_n,\phi(\varphi_r(t+s_n)))-f(t+s_n,\Phi(\varphi_r(t)))\|\\ &\quad+\|f(t+s_n,\Phi(\varphi_r(t)))-g(t,\Phi(\varphi_r(t)))\|\\ &\le L\|\phi(\varphi_r(t+s_n))-\Phi(\varphi_r(t))\|\\ &\quad+\|f(t+s_n,\Phi(\varphi_r(t)))-g(t,\Phi(\varphi_r(t)))\|.\end{aligned}$$

We deduce from (1) and (5) that

$$\lim_{n\to\infty} F(t+s_n)=G(t).$$

Similarly, we have:

$$\begin{aligned}\|G(t-s_n)-F(t)\| &= \|g(t-s_n,\phi(\varphi_r(t-s_n)))-f(t,\Phi(\varphi_r(t)))\|\\ &\le \|g(t-s_n,\phi(\varphi_r(t-s_n)))-g(t-s_n,\Phi(\varphi_r(t)))\|\\ &\quad+\|g(t-s_n,\Phi(\varphi_r(t)))-f(t,\Phi(\varphi_r(t)))\|\\ &\le L\|\phi(\varphi_r(t-s_n))-\Phi(\varphi_r(t))\|\\ &\quad+\|g(t-s_n,\Phi(\varphi_r(t)))-f(t,\Phi(\varphi_r(t)))\|.\end{aligned}$$

Then we deduce from (2) and (6) that

$$\lim_{n\to\infty} G(t-s_n)=F(t).$$

Now, we show that the function $F(t)=f(t,\phi(\varphi_r(t)))$ is bounded. Since *f* is almost automorphic in *t*, then $\|f(\cdot,0)\|_\infty=\sup_{t\in\mathbb{R}}\|f(t,0)\|<+\infty$. Then we have

$$\|f(t,\phi(\varphi_r(t)))\|=\|f(t,\phi(\varphi_r(t)))-f(t,0)\|\le L\|\phi(\varphi_r(t))\|+\|f(t,0)\|.$$

We deduce that for every $t\in\mathbb{R}$

$$\|f(t,\phi(\varphi_r(t)))\|\le L\|\phi\|_\infty+\|f(\cdot,0)\|_\infty.$$

□

Remark 2.2. *Let* $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ *satisfy the conditions of the previous theorem. We have that the function* $G:\mathbb{R}\to\mathbb{X}$ *defined by* $G(t)=g(t,\phi(\varphi_r(t)))$ *is bounded.*

3. A Differential Equation with a General Piecewise Constant Argument

We consider the differential Equation (1) where φ is a step function,

$A:\mathbb{R}\to\mathbb{R}^{q\times q}$ is continuous in $\mathbb{R}\setminus\mathbb{S}$ and $f:\mathbb{R}\times\mathbb{R}^q\to\mathbb{R}^q$ is continuous. Thus, in the sequel $\mathbb{X}=\mathbb{R}^q$. Moreover, in addition to **(P1)**, we consider the two following conditions:

(P2) $\forall(t,s)\in\mathbb{R}\times\mathbb{S}$, $\varphi(t+s)=\varphi(t)+s$ and $\varphi(t)\le t$.

(P3) $f:\mathbb{R}\times\mathbb{X}\to\mathbb{X}$ is almost automorphic in $t\in\mathbb{R}$ for each $x\in\mathbb{X}$ and f satisfies a Lipschitz condition in x uniformly in $t\in\mathbb{R}$.

We give a consequence of **(P1)** that will be useful for the sequel.

Proposition 3.1. *Assume that* **(P1)** *is satisfied, then there exists a bounded set* K_1 *in* $\mathbb{R}$ *such that:* $\forall t\in\mathbb{R}$, $t-\varphi(t)\in K_1$.

Proof. Assume that **(P1)** is satisfied. For each $t\in\mathbb{R}$ there exists $(\alpha,s)\in K_0\times\mathbb{S}$, such that $t=\alpha+s$. Hence, we have $\varphi(t)=\varphi(\alpha+s)=\varphi(\alpha)+s$ and then:

$$t-\varphi(t)=\alpha-\varphi(\alpha).$$

Since φ is a step function, it is bounded on each bounded subset of $\mathbb{R}$. Therefore,

$$K_1=\{\alpha-\varphi(\alpha):\alpha\in K_0\}$$

is a bounded set such that $t-\varphi(t)\in K_1$ for all $t\in\mathbb{R}$. The proposition is thus proved. □

Definition 3.1. *A solution of* (*1*) *is a function* $x(t)$ *defined on* $\mathbb{R}$ *for which the following conditions hold*:

1) $x(t)$ *is continuous on* $\mathbb{R}$.

2) *The derivative* $x'(t)$ *exists at each point* $t\in\mathbb{R}$, *with possible exception at the points* $t_i, i\in\mathbb{Z}$, *where one-sided derivatives exist.*

3) *The Equation* (*1*) *is satisfied on each interval* $[t_i,t_{i+1}[$, $i\in\mathbb{Z}$.

Theorem 3.2. *Let f satisfy* **(P2)** *and* **(P3)**. *Then the solution of* (*1*) *satisfies*:

$$x(t)=x(\varphi(t))+\int_{\varphi(t)}^{t}A(s)x(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}f(s,x(\varphi(s)))\mathrm{d}s.$$

Proof. Considering the integral of

$$\begin{cases} x_1'(t)=a_{11}(t)x_1(\varphi(t))+\cdots+a_{1p}(t)x_p(\varphi(t))+f_1(t,x(\varphi(t)))\\ \vdots\\ x_p'(t)=a_{p1}(t)x_1(\varphi(t))+\cdots+a_{pp}(t)x_p(\varphi(t))+f_p(t,x(\varphi(t)))\end{cases},$$

on $[\varphi(t),t[$, we obtain:

$$\begin{cases} x_1(t)=x_1(\varphi(t))+\int_{\varphi(t)}^{t}a_{11}(s)x_1(\varphi(s))\mathrm{d}s+\cdots\\ \qquad+\int_{\varphi(t)}^{t}a_{1p}(t)x_p(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}f_1(s,x(\varphi(s)))\mathrm{d}s\\ \vdots\\ x_p(t)=x_1(\varphi(t))+\int_{\varphi(t)}^{t}a_{p1}(s)x_1(\varphi(s))\mathrm{d}s+\cdots\\ \qquad+\int_{\varphi(t)}^{t}a_{pp}(t)x_p(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}f_p(s,x(\varphi(s)))\mathrm{d}s\end{cases}$$

□

Lemma 3.3. *Assume that* **(P2)** *and* **(P3)** *are satisfied and that* $A(t)$ *is an* $\mathbb{S}$

-almost automorphic operator. Then

$$(\wedge\phi)(t)=\phi(\varphi(t))+\int_{\varphi(t)}^{t}A(s)\phi(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}f(s,\phi(\varphi(s)))\mathrm{d}s,$$

maps $\mathbb{S}AA(\mathbb{X})$ *into itself.*

Proof. Let (s_n') be a sequence of elements of $\mathbb{S}$. We have from **(P2)** that $\varphi(t+s)=\varphi(t)+s$ and $\varphi(t-s)=\varphi(t)-s$ for $(t,s)\in\mathbb{R}\times\mathbb{S}$. Then, there exists a subsequence (s_n) of (s_n') such that:

1) $\lim_{n\to\infty}\phi(t+s_n)=\Phi(t)$ for each $t\in\mathbb{R}$,
2) $\lim_{n\to\infty}\Phi(t-s_n)=\Phi(t)$ for each $t\in\mathbb{R}$,
3) $\lim_{n\to\infty}\phi(\varphi(t+s_n))=\Phi(\varphi(t))$ for each $t\in\mathbb{R}$,
4) $\lim_{n\to\infty}\Phi(\varphi(t-s_n))=\phi(\varphi(t))$ for each $t\in\mathbb{R}$,
5) $\lim_{n\to\infty}A(t+s_n)=B(t)$ for each $t\in\mathbb{R}$,
6) $\lim_{n\to\infty}B(t-s_n)=A(t)$ for each $t\in\mathbb{R}$,
7) $\lim_{n\to\infty}f(t+s_n,\phi(\varphi(t+s_n)))=g(t,\Phi(\varphi(t)))$ for each $t\in\mathbb{R}$,
8) $\lim_{n\to\infty}g(t-s_n,\Phi(\varphi(t-s_n)))=f(t,\phi(\varphi(t)))$ for each $t\in\mathbb{R}$.

We put

$$F(t)=\phi(\varphi(t))+\int_{\varphi(t)}^{t}A(s)\phi(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}f(s,\phi(\varphi(s)))\mathrm{d}s,$$

and

$$V(t)=\Phi(\varphi(t))+\int_{\varphi(t)}^{t}B(s)\Phi(\varphi(s))\mathrm{d}s+\int_{\varphi(t)}^{t}g(s,\Phi(\varphi(s)))\mathrm{d}s.$$

Then, we have

$$\begin{aligned}&\|F(t+s_n)-V(t)\|\\&\le\|\phi(\varphi(t)+s_n)-\Phi(\varphi(t))\|\\&\quad+\left\|\int_{\varphi(t+s_n)}^{t+s_n}A(\sigma)\phi(\varphi(\sigma))\mathrm{d}\sigma-\int_{\varphi(t)}^{t}B(\sigma)\Phi(\varphi(\sigma))\mathrm{d}\sigma\right\|\\&\quad+\left\|\int_{\varphi(t+s_n)}^{t+s_n}f(\sigma,\phi(\varphi(\sigma)))\mathrm{d}\sigma-\int_{\varphi(t)}^{t}g(\sigma,\phi(\varphi(\sigma)))\mathrm{d}\sigma\right\|\end{aligned}.$$

Using a change of variable and **(P2)**, we find

$$\begin{aligned}&\|F(t+s_n)-V(t)\|\\&\le\|\phi(\varphi(t)+s_n)-\Phi(\varphi(t))\|\\&\quad+\left\|\int_{\varphi(t)}^{t}A(\sigma+s_n)\phi(\varphi(\sigma+s_n))\mathrm{d}\sigma-\int_{\varphi(t)}^{t}B(\sigma)\Phi(\varphi(\sigma))\mathrm{d}\sigma\right\|,\\&\quad+\left\|\int_{\varphi(t)}^{t}\left[f(\sigma+s_n,\phi(\varphi(\sigma+s_n)))-g(\sigma,\phi(\varphi(\sigma)))\right]\mathrm{d}\sigma\right\|\end{aligned}$$

which can be written as

$$\begin{aligned}&\|F(t+s_n)-V(t)\|\\&\le\|\phi(\varphi(t)+s_n)-\Phi(\varphi(t))\|+\left\|\int_{\varphi(t)}^{t}\left[(A(\sigma+s_n)-B(\sigma))\phi(\varphi(\sigma+s_n))\right.\right.\\&\quad\left.\left.+B(\sigma)(\phi(\varphi(\sigma+s_n))-\Phi(\varphi(\sigma)))\right]\mathrm{d}\sigma\right\|\\&\quad+\left\|\int_{\varphi(t)}^{t}\left[f(\sigma+s_n,\phi(\varphi(\sigma+s_n)))-g(\sigma,\phi(\varphi(\sigma)))\right]\mathrm{d}\sigma\right\|.\end{aligned}$$

Now, using $\varphi(t) \le t$, we can write

$$\begin{aligned}
&\left\| F(t+s_n) - V(t) \right\| \\
&\le \left\| \phi(\varphi(t)+s_n) - \Phi(\varphi(t)) \right\| + \int_{\varphi(t)}^{t} \left\| A(\sigma+s_n) - B(\sigma) \right\| \|\phi\|_\infty \, \mathrm{d}\sigma \\
&\quad + \int_{\varphi(t)}^{t} \left\| \phi(\sigma+s_n) - \Phi(\sigma) \right\| \|B\|_\infty \, \mathrm{d}\sigma \\
&\quad + \int_{\varphi(t)}^{t} \left\| f\left(\sigma+s_n, \phi(\varphi(\sigma+s_n))\right) - g\left(\sigma, \phi(\varphi(\sigma))\right) \right\| \mathrm{d}\sigma.
\end{aligned}$$

Hence, using the Lebesgue Dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} \left\| F(t+s_n) - V(t) \right\| = 0.$$

Similarly, taking into account **(P2)**, we get

$$\begin{aligned}
&\left\| V(t-s_n) - F(t) \right\| \\
&\le \left\| \Phi(\varphi(t-s_n)) - \phi(\varphi(t)) \right\| \\
&\quad + \left\| \int_{\varphi_1(t-s_n)}^{t-s_n} B(\sigma)\phi(\varphi(\sigma)) \mathrm{d}\sigma - \int_{\varphi(t)}^{t} A(\sigma)\Phi(\varphi(\sigma)) \mathrm{d}\sigma \right\| \\
&\quad + \left\| \int_{\varphi_1(t-s_n)}^{t-s_n} g\left(\sigma, \phi(\varphi(\sigma))\right) \mathrm{d}\sigma - \int_{\varphi_1(t)}^{t} f\left(\sigma, \phi(\varphi(\sigma))\right) \mathrm{d}\sigma \right\|
\end{aligned}$$

$$\begin{aligned}
&\left\| V(t-s_n) - F(t) \right\| \\
&\le \left\| \Phi(\varphi(t-s_n)) - \phi(\varphi(t)) \right\| \\
&\quad + \left\| \int_{\varphi(t)}^{t} B(\sigma-s_n)\Phi(\varphi(\sigma-s_n)) \mathrm{d}\sigma - \int_{\varphi(t)}^{t} A(\sigma)\phi(\varphi(\sigma)) \mathrm{d}\sigma \right\| \\
&\quad + \left\| \int_{\varphi(t)}^{t} \left[g\left(\sigma-s_n, \phi(\varphi(\sigma-s_n))\right) - f\left(\sigma, \phi(\varphi(\sigma))\right) \right] \mathrm{d}\sigma \right\|
\end{aligned}$$

$$\begin{aligned}
&\left\| V(t-s_n) - F(t) \right\| \\
&\le \left\| \Phi(\varphi(t-s_n)) - \phi(\varphi(t)) \right\| + \left\| \int_{\varphi(t)}^{t} \left[\left(B(\sigma-s_n) - A(\sigma) \right) \Phi(\varphi(\sigma-s_n)) \right. \right. \\
&\quad \left. \left. + A(\sigma)\left(\Phi(\varphi(\sigma-s_n)) - \phi(\varphi(\sigma)) \right) \right] \mathrm{d}\sigma \right\| \\
&\quad + \left\| \int_{\varphi(t)}^{t} \left[g\left(\sigma-s_n, \phi(\varphi(\sigma-s_n))\right) - f\left(\sigma, \phi(\varphi(\sigma))\right) \right] \mathrm{d}\sigma \right\|.
\end{aligned}$$

Since $\varphi(t) \le t$, it follows that

$$\begin{aligned}
&\left\| V(t-s_n) - F(t) \right\| \\
&\le \left\| \Phi(\varphi(t-s_n)) - \phi(\varphi(t)) \right\| + \int_{\varphi(t)}^{t} \left\| B(t-s_n) - A(\sigma) \right\| \|\Phi\|_\infty \, \mathrm{d}\sigma \\
&\quad + \int_{\varphi(t)}^{t} \left\| \Phi(\sigma-s_n) - \phi(\sigma) \right\| \|A\|_\infty \, \mathrm{d}\sigma \\
&\quad + \int_{\varphi(t)}^{t} \left\| g\left(\sigma-s_n, \Phi(\varphi(\sigma-s_n))\right) - f\left(\sigma, \phi(\varphi(\sigma))\right) \right\| \mathrm{d}\sigma.
\end{aligned}$$

Hence, using the Lebesgue Dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} \left\| V(t-s_n) - F(t) \right\| = 0.$$

□

We set $M_1 = \sup(K_1)$, where K_1 is the bounded subset of $\mathbb{R}$ introduced in

Proposition 3.1. Note that, if $\varphi(t) \le t$, then $M_1 \ge 0$.

Theorem 3.4. *Assume that* **(P1)**, **(P2)** *and* **(P3)** *are satisfied and that* $y \to \varphi(y)$ *is constant on the interval* $[\varphi(t), t]$. *If*

$$\left\| I + \int_{\varphi(t)}^{t} A(s)\,\mathrm{d}s \right\| + M_1 L < 1,$$

then (*1*) *has a unique* $\mathbb{S}$*-almost automorphic solution which is also the unique almost automorphic solution of* (1).

Proof. First Step

We define the nonlinear operator Γ by the expression

$$(\Gamma\phi)(t) = \phi(\varphi(t)) + \int_{\varphi(t)}^{t} A(s)\phi(\varphi(s))\,\mathrm{d}s + \int_{\varphi(t)}^{t} f(s, \phi(\varphi(s)))\,\mathrm{d}s.$$

According to Theorem 2.6, the function $t \mapsto f(t, \phi(\varphi(t)))$ belongs to $\mathbb{S}AA(\mathbb{R}, \mathbb{X})$. According to Lemma 3.3 the operator Γ maps $\mathbb{S}AA(\mathbb{R}, \mathbb{X})$ into itself. Since $t - \varphi(t) \le M_1$ for all $t \in \mathbb{R}$, we have:

$$\begin{aligned}
\|(\Gamma\phi)(t) - (\Gamma\psi)(t)\| &= \Big\| \Big(I + \int_{\varphi(t)}^{t} A(s)\,\mathrm{d}s\Big)\big(\phi(\varphi(t)) - \psi(\varphi(t))\big) \\
&\qquad + \int_{\varphi(t)}^{t} f(s, \phi(\varphi(s))) - f(s, \psi(\varphi(s)))\,\mathrm{d}s \Big\| \\
&\le \Big\| I + \int_{\varphi(t)}^{t} A(s)\,\mathrm{d}s \Big\| \big\| \phi(\varphi(t)) - \psi(\varphi(t)) \big\| \\
&\qquad + \Big\| \int_{\varphi(t)}^{t} f(s, \phi(\varphi(s))) - f(s, \psi(\varphi(s)))\,\mathrm{d}s \Big\| \\
&\le \Big\| I + \int_{\varphi(t)}^{t} A(s)\,\mathrm{d}s \Big\| \|\phi - \psi\|_\infty \\
&\qquad + \int_{\varphi(t)}^{t} L \big\| \phi(\varphi(s)) - \psi(\varphi(s)) \big\|\,\mathrm{d}s
\end{aligned}$$

$$\|(\Gamma\phi)(t) - (\Gamma\psi)(t)\| \le \left(\left\| I + \int_{\varphi(t)}^{t} A(s)\,\mathrm{d}s \right\| + M_1 L \right) \|\phi - \psi\|_\infty .$$

This proves that Γ is a contraction. We conclude that Γ has a unique fixed point in $\mathbb{S}AA(\mathbb{R}, \mathbb{X})$. We denote by z the unique $\mathbb{S}$-almost automorphic solution of (1).

Second Step

We show that z is an almost automorphic solution of (1). Since z is $\mathbb{S}$-almost automorphic, using Proposition 2.3, it suffices to prove that z is uniformly continuous. Consider the set $D := \{t_i : i \in \mathbb{Z}\}$ of possible points of discontinuity of z'. We have

$$z'(t) = A(t) z(\varphi(t)) + f(t, z(\varphi(t))),$$

and then

$$\|z'(t)\| \le \|A(t)\| \|z(\varphi(t))\| + \|f(t, z(\varphi(t))) - f(t, 0)\| + \|f(t, 0)\|$$

for all $t \in \mathbb{R} \setminus D$. If we set

$$M = \|A\|_\infty \|z\|_\infty + L \|z\|_\infty + \|f(\cdot, 0)\|_\infty ,$$

it follows that $\|z'(t)\| \le M$ for all $t \in \mathbb{R} \setminus D$. Therefore, since z is continuous and D is countable, the mean value Theorem (see [20], Theorem 8.5.2) asserts that

$$\|z(t_2)-z(t_1)\|\le M(t_2-t_1),$$

for all $t_1,t_2\in\mathbb{R}$ with $t_1<t_2$. This means that z is lipschitzian and then uniformly continuous. Thus, z is an almost automorphic function.

The function z is necessarily the unique almost automorphic solution of (1). In fact, an almost automorphic function is also $\mathbb{S}$-almost automorphic and (1) has a unique such solution. The theorem is thus proved. □

Corollary 3.5. *Let* $A(t)$ *be a* $\mathbb{Z}$*-almost automorphic operator and assume that* **(P3)** *is satisfied. If*

$$\left\|I+\int_{[t]}^{t}A(s)\mathrm{d}s\right\|+L<1,$$

then the following equation:

$$x'(t)=A(t)x([t])+f(t,x([t]))\mathrm{d}t,t\in\mathbb{R}$$

has a unique $\mathbb{Z}$*-almost automorphic solution which is also his unique almost automorphic solution*.

Proof. We have $\varphi(t)=t\le t$, $K_0=K_1=[0,1[$ and $M_1=1$. □

Corollary 3.6. *Suppose that* $A(t)$ *is a* $\alpha h\mathbb{Z}$*-almost automorphic operator and that* **(P3)** *is satisfied. If*

$$\left\|I+\int_{\left[\frac{t}{\alpha h}\right]\alpha h}^{t}A(s)\mathrm{d}s\right\|+\alpha hL<1,$$

then the following equation:

$$x'(t)=A(t)x\left(\left[\frac{t}{\alpha h}\right]\alpha h\right)+f\left(t,x\left(\left[\frac{t}{\alpha h}\right]\alpha h\right)\right)\mathrm{d}t,t\in\mathbb{R},$$

has a unique $\alpha h\mathbb{Z}$*-almost automorphic solution which is also its unique almost automorphic solution.*

Proof. We have that $\varphi(t)=\left[\frac{t}{\alpha h}\right]\alpha h$. Then φ is constant on each interval $[n\alpha h,(n+1)\alpha h[$ where $n\in\mathbb{Z}$. We observe also that

$$\begin{aligned}\varphi(t+\alpha hn)&=\left[\frac{t+\alpha hn}{\alpha h}\right]\alpha h=\left[\frac{t}{\alpha h}+n\right]\alpha h\\&=\left[\frac{t}{\alpha h}\right]\alpha h+\alpha hn=\varphi(t)+\alpha hn.\end{aligned}$$

If $t\in[n\alpha h,(n+1)\alpha h[$ where $n\in\mathbb{Z}$, then $\varphi(t)=\alpha hn$, $\varphi(t)\le t$ $t-\varphi(t)\in[0,\alpha h]$ and $M_0=\alpha h$. All real t can be written as $t=\beta+\zeta$ where $\beta\in[0,\alpha h]$ and $\zeta\in\alpha h\mathbb{Z}$. □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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