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On the Construction and Classification of the Common Invariant Solutions for Some $P(1,4)$-Invariant Partial Differential Equations

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Abstract

We consider the following (1 + 3)-dimensional $P(1,4)$-invariant partial differential equations (PDEs): the Eikonal equation, the Euler-Lagrange-Born-Infeld equation, the homogeneous Monge-Ampère equation, the inhomogeneous Monge-Ampère equation. The purpose of this paper is to construct and classify the common invariant solutions for those equations. For this aim, we have used the results concerning construction and classification of invariant solutions for the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation, since this equation is the simplest among the equations under investigation. The direct checked allowed us to conclude that the majority of invariant solutions of the (1 + 3)-dimensional Eikonal equation, obtained on the base of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, satisfy all the equations under investigation. In this paper, we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions.

Keywords

Symmetry Reduction, Classification of Invariant Solutions, Common Invariant Solutions, The Eikonal Equations, The Euler-Lagrange-Born-Infeld Equations, The Monge-Ampère Equations, Classification of Lie Algebras, Nonconjugate Subalgebras, Poincaré Group $P(1,4)$

1. Introduction

A solution of many problems of the geometric optics, theories of anisotropic media, theory of minimal surfaces, nonlinear electrodynamics, theories of gravity, geometry, unified field theory, string theories, black holes, cosmology, etc. is reduced to the investigation of the Eikonal equations [1] [2] [3] [4] [5], the Euler-Lagrange equations [6]-[12], the Born-Infeld equations [13]-[22], the Monge-Ampère equations [23]-[40] in the spaces of different dimensions and different types (see also the references therein).

Nowadays, there exist a lot of methods for the construction exact solutions of linear and nonlinear partial differential equations (PDEs). More details on this theme can be found in [41]-[46] (see also the references therein).

We consider the following (1 + 3)-dimensional $P(1,4)$-invariant PDEs:

- the Eikonal equation,
- the Euler-Lagrange-Born-Infeld equation,
- the homogeneous Monge-Ampère equation,
- the inhomogeneous Monge-Ampère equation.

From the results obtained by Fushchich W.I., Shtelen W.M. and Serov N.I. [40], it follows, in particular, that the common symmetry group of those equations is the generalized Poincaré group $P(1,4)$. Therefore, in the natural way arises the following question: what is the relationship between invariant solutions of the equations under study? In particular, whether those equations have common invariant solutions?

The purpose of this paper is to try to construct and classify the common invariant solutions for the equations under consideration. It is known that the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation is the simplest one among the equations under study. Therefore, we can use this fact for constructing the common invariant solutions. At the present time, we have constructed invariant solutions for the (1 + 3)-dimensional $P(1,4)$-invariant Eikonal equation obtained on the base of low-dimensional ($dimL \le 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, by using classical Lie-Ovsiannikov approach [41] [42] [43] [44]. This method, in particular, allows us to perform the symmetry reduction of the many-dimensional PDEs with non-trivial symmetry groups to differential equations with a fewer number of independent variables as well as to construct solutions, invariant with respect to nonconjugate subgroups of the symmetry groups, of the equations under study. According to this method, reduced equations (invariant solutions) should be classified with respect to the ranks of the corresponding nonconjugate subalgebras of the Lie algebras of the symmetry groups of the equations under study.

Our contribution in classical Lie-Ovsiannikov method consists in the suggestion to use, for the classification of symmetry reductions (invariant solutions) of PDEs with non-trivial symmetry groups, not only ranks of nonconjugate subalgebras, but also their structural property. Some details on this theme can be found in [47] [48].

In our paper, we have performed the suggestion for the classification of the common invariant solutions of some $P(1, 4)$-invariant PDEs by using the structural property of the low-dimensional ($dimL \le 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1, 4)$.

The direct checks allowed us to conclude that the majority of invariant solutions of the (1 + 3)-dimensional Eikonal equation, obtained on the base of low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$, satisfy all the equations under investigation. In this paper, we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions.

To present the results obtained, we give some information about the Lie algebra of the Poincaré group $P(1,4)$ and its nonconjugate subalgebras.

2. The Lie Algebra of the Poincaré Group $P(1,4)$ and Its Nonconjugate Subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the smallest group, which contains, as subgroups, the extended Galilei group $\tilde{G}(1,3)$ [49] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics).

The Lie algebra of the group $P(1,4)$ is generated by 15 bases elements $M_{\mu\nu} = -M_{\nu\mu} (\mu,\nu = 0,1,2,3,4)$ and $P_\mu (\mu = 0,1,2,3,4)$, which satisfy the commutation relations

$$\left[P_\mu, P_\nu\right] = 0, \quad \left[M_{\mu\nu}, P_\sigma\right] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu, \tag{1}$$

$$\left[M_{\mu\nu}, M_{\rho\sigma}\right] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho}, \tag{2}$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

In this paper, we consider the following representation [40] of the Lie algebra of the group $P(1,4)$:

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3}, \tag{3}$$

$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \ x_4 \equiv u. \tag{4}$$

In the following, we will use the next bases elements:

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12}, \tag{5}$$

$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a}, \quad (a = 1,2,3), \tag{6}$$

$$X_0 = \frac{1}{2}(P_0 - P_4), \quad X_k = P_k \ (k = 1,2,3), \quad X_4 = \frac{1}{2}(P_0 + P_4). \tag{7}$$

The Lie algebra of the extended Galilei group $\tilde{G}(1,3)$ is generated by the following bases elements:

$$L_1, \quad L_2, \quad L_3, \quad P_1, \quad P_2, \quad P_3, \quad X_0, \quad X_1, \quad X_2, \quad X_3, \quad X_4. \tag{8}$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimensions ≤ 3 was performed in [50].

3. On the Construction and Classification of the Common Invariant Solutions for Some (1 + 3)-Dimensional $P(1,4)$-Invariant PDEs

In this Section, We Consider the Following PDEs

- the Eikonal equation

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = 1;$$

- the Euler-Lagrange-Born-Infeld equation

$$\Box u\left(1 - u_\nu u^\nu\right) + u^\mu u^\nu u_{\mu\nu} = 0;$$

- the homogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right) = 0;$$

- the inhomogeneous Monge-Ampère equation

$$\det\left(u_{\mu\nu}\right) = \lambda\left(1 - u_\nu u^\nu\right)^3, \quad \lambda \neq 0,$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \frac{\partial u}{\partial x^\mu}$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}$, $u^\mu = g^{\mu\nu} u_\nu$, $g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu = 0,1,2,3$, $\Box$ is the d'Alembert operator.

Here, and in what follows, $M(1,3)$ is a four-dimensional Minkowski space, $R(u)$ is a real number axis of the depended variable *u*.

From the results obtained by Fushchich W.I., Shtelen W.M. and Serov N.I. [40] it follows, in particule, that the common symmetry group of those equations is the generalised Poincaré group $P(1,4)$.

In this section we present obtained common invariant solutions of the equations under study as well as the classification of those invariant solutions. To obtain those results, we used the nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$, structural properties of its low-dimensional ($dimL \leq 3$) nonconjugate subalgebras as well as the results of the classification of symmetry reductions of the eikonal equation. More details on this theme can be found in [47] [48].

Bellow we present the results obtained.

3.1. Classification of the Common Invariant Solutions for the Equations under Study Using One-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

1) $\langle G \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = -\left(1 - c_2^2 - c_3^2\right)^{1/2} x_1 + c_2 x_2 + c_3 x_3 + c_1,$$

where c_1, c_2 and c_3 are arbitrary real constants.

2) $\langle G + \alpha X_1, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$\alpha\ln\left(\frac{2\alpha\left(\sqrt{\left(c_1^2+c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}+\alpha\right)}{x_0-u}\right)$$
$$-\sqrt{\left(c_1^2+c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}-x_1+c_1x_2+c_2x_3+c_3,$$

where c_1, c_2 and c_3 are arbitrary real constants.

3) $\langle L_3\rangle$:

The common invariant solution for the equations under study:

$$u=\left(c_2^2+c_3^2+1\right)^{1/2}x_0+c_2x_3+c_3\left(x_1^2+x_2^2\right)^{1/2}+c_1,$$

where c_1, c_2 and c_3 are arbitrary real constants.

4) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle$:

The common invariant solution for the equations under study:

$$u=i\alpha c_2\operatorname{arctanh}\frac{c_2\alpha}{\left(\left(c_1^2-c_2^2+1\right)\left(x_1^2+x_2^2\right)+c_2^2\alpha^2\right)^{1/2}}$$
$$-i\left(\left(c_1^2-c_2^2+1\right)\left(x_1^2+x_2^2\right)+c_2^2\alpha^2\right)^{1/2}+c_2\left(x_0-\alpha\arctan\frac{x_1}{x_2}\right)+c_1x_3+c_3.$$

5) $\langle L_3+\alpha X_3,\alpha>0\rangle$:

The common invariant solution for the equations under study:

$$u=\sqrt{\left(c_1^2-c_2^2-1\right)\left(x_1^2+x_2^2\right)-\alpha^2c_2^2}+c_2\alpha\arctan\frac{x_1}{x_2}$$
$$-c_2\alpha\arctan\left(\frac{\sqrt{\left(c_1^2-c_2^2-1\right)\left(x_1^2+x_2^2\right)-\alpha^2c_2^2}}{c_2\alpha}\right)+c_2x_3+c_1x_0+c_3.$$

6) $\langle L_3+2X_4\rangle$:

The common invariant solution for the equations under study:

$$x_0-u+2\arctan\frac{x_2}{x_1}$$
$$=i\sqrt{\left(c_2^2+4c_1\right)\left(x_1^2+x_2^2\right)+4}-2i\operatorname{arctanh}\left(\frac{2}{\sqrt{\left(c_2^2+4c_1\right)\left(x_1^2+x_2^2\right)+4}}\right)$$
$$+c_1\left(x_0+u\right)+c_2x_3+c_3.$$

7) $\langle P_3-2X_0\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3+x_3\left(x_0+u\right)+x_0-u$$
$$=-\frac{1}{6}\left(\left(x_0+u\right)^2+4x_3-c_1^2-c_2^2\right)^{3/2}+c_1x_1+c_2x_2+c_3,$$

where c_1, c_2 and c_3 are arbitrary real constants.

8) $\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u = i\left(c_2^2 + c_3^2 + 1\right)^{1/2} x_1 + c_2 x_2 + c_3 x_3 + c_1,$$

where c_1, c_2 and c_3 are arbitrary real constants.

9) $\langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$x_3 = -i\left(c_2^2 + 1\right)^{1/2} x_1 + c_2 x_2 + c_1 + f\left(x_0 + u\right),$$

where: c_1, c_2 are arbitrary real constants, f is an arbitrary smooth function.

3.2. Classification of the Common Invariant Solutions for the Equations under Study Using Two-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

3.2.1. Lie Algebras of the Type $2A_1$

1) $\langle G \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = \left(1 - c_2^2\right)^{1/2} x_3 + c_2\left(x_1^2 + x_2^2\right)^{1/2} + c_1,$$

where c_1, c_2 are arbitrary real constants.

2) $\langle G + \alpha X_3, \alpha > 0 \rangle \oplus \langle L_3 \rangle$:

The common invariant solution for the equations under study:

$$x_3 - \alpha \ln\left(x_0 + u\right)$$
$$= -\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - u^2\right) + \alpha^2} + \alpha \ln\left(\frac{2\alpha\left(\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - u^2\right) + \alpha^2} + \alpha\right)}{x_0^2 - u^2}\right)$$
$$+ c_1\sqrt{x_1^2 + x_2^2} + c_2,$$

where c_1, c_2 are arbitrary real constants.

3) $\langle G \rangle \oplus \langle L_3 + \alpha X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_3 + \alpha \arctan\frac{x_1}{x_2}$$
$$= \alpha \arctan\frac{\alpha}{\sqrt{\left(c_2^2 - 1\right)\left(x_1^2 + x_2^2\right) - \alpha^2}} + c_2\left(x_0^2 - u^2\right)^{1/2}$$
$$+ \sqrt{\left(c_2^2 - 1\right)\left(x_1^2 + x_2^2\right) - \alpha^2} + c_1,$$

where c_1, c_2 are arbitrary real constants.

4) $\langle G \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - u^2\right)^{1/2} = \varepsilon\left(1 - c_2^2\right)^{1/2} x_2 + c_2 x_3 + c_1, \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

5) $\langle G + \alpha X_2, \alpha > 0 \rangle \oplus \langle X_1 \rangle$:

The common invariant solution for the equations under study:

$$x_3+\sqrt{\left(c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2c_2^2}$$
$$=\alpha c_2\operatorname{arctanh}\frac{\alpha c_2}{\sqrt{\left(c_2^2+1\right)\left(x_0^2-u^2\right)+\alpha^2c_2^2}}+\frac{\alpha c_2}{2}\ln\frac{x_0-u}{x_0+u}+c_2x_2+c_1,$$

where c_1,c_2 are arbitrary constants.

6) $\langle L_3\rangle\oplus\langle P_3+C_3\rangle$:

The common invariant solution for the equations under study:

$$\left(u^2+x_3^2\right)^{1/2}=\left(c_2^2+1\right)^{1/2}x_0+c_2\left(x_1^2+x_2^2\right)^{1/2}+c_1,$$

where c_1,c_2 are arbitrary constants.

7) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle P_3+C_3\rangle$:

The common invariant solution for the equations under study:

$$x_0-\alpha\arctan\frac{x_1}{x_2}$$
$$=\alpha\arctan\frac{\alpha}{\sqrt{\left(1-c_2^2\right)\left(x_1^2+x_2^2\right)-\alpha^2}}+c_2\sqrt{u^2+x_3^2}$$
$$+\sqrt{\left(1-c_2^2\right)\left(x_1^2+x_2^2\right)-\alpha^2}+c_1,$$

where c_1,c_2 are arbitrary constants.

8) $\langle L_3\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u=i\varepsilon\left(c_2^2+1\right)^{1/2}\left(x_1^2+x_2^2\right)^{1/2}+c_2x_3+c_1,\ \varepsilon=\pm1,$$

where c_1,c_2 are arbitrary constants.

9) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$u=\alpha\arctan\frac{x_1}{x_2}+i\sqrt{c_1^2\left(x_1^2+x_2^2\right)+\alpha^2}$$
$$-i\alpha\operatorname{arctanh}\frac{\alpha}{\sqrt{c_1^2\left(x_1^2+x_2^2\right)+\alpha^2}}-x_0+c_1x_3+c_2,$$

where c_1,c_2 are arbitrary constants.

10) $\langle L_3+\alpha X_3,\alpha>0\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$u=\frac{\alpha}{c_1}\arctan\left(\frac{x_1\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}-i\alpha x_2}{x_2\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}+i\alpha x_1}\right)$$
$$+\frac{i}{c_1}\sqrt{\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+\alpha^2}+\frac{x_3}{c_1}+c_2,\ c_1\neq0.$$

11) $\langle L_3+2X_4\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$x_0-u+2\arctan\frac{x_2}{x_1}$$

$$= 2i\operatorname{arctanh}\frac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)+1}} - 2i\sqrt{c_1\left(x_1^2+x_2^2\right)+1} + c_1\left(x_0+u\right) + c_2,$$

where c_1, c_2 are arbitrary constants.

12) $\langle L_3 - P_3 + 2\alpha X_0, \alpha \neq 0\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} &x_0 + u - 2\alpha \arctan\frac{x_1}{x_2} \\ &= 2i\alpha\varepsilon\sqrt{4c_2^2\left(x_1^2+x_2^2\right)+1} - 2i\alpha\varepsilon\operatorname{arctanh}\frac{1}{\sqrt{4c_2^2\left(x_1^2+x_2^2\right)+1}} \\ &\quad + c_2\left(\left(x_0+u\right)^2 + 4\alpha x_3\right) + c_1, \varepsilon = \pm 1, \end{aligned}$$

where c_1, c_2 are arbitrary constants.

13) $\langle L_3\rangle \oplus \langle P_3 - 2X_0\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3 + x_3\left(x_0+u\right) + x_0 - u = c_1\sqrt{x_1^2+x_2^2} - \frac{1}{6}\left(\left(x_0+u\right)^2 + 4x_3 - c_1^2\right)^{3/2} + c_2,$$

where c_1, c_2 are arbitrary constants.

14) $\langle P_1\rangle \oplus \langle P_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$$

15) $\langle P_1 - X_3\rangle \oplus \langle P_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2 - x_1^2 - x_2^2 - u^2 = 0.$$

16) $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2 - \beta X_3, \beta > 0, \gamma > 0\rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} &\frac{x_1\left(x_1 - 2c_2\right)}{x_0+u} + \frac{\left(x_2 - \beta c_2\right)^2 + c_2^2}{x_0+u+\gamma} + \frac{\gamma c_2^2}{\left(x_0+u\right)\left(x_0+u+\gamma\right)} \\ &-\left(c_2^2+1\right)\left(x_0+u\right) + 2c_2x_3 + 2u + c_1 = 0, \end{aligned}$$

where c_1, c_2 are arbitrary constants.

17) $\langle P_1 - X_3\rangle \oplus \langle P_2 - \gamma X_2, \gamma > 0\rangle$:

The common invariant solution for the equations under study:

$$\frac{\left(x_1 - c_2\right)^2}{x_0+u} + \frac{x_2^2}{x_0+u+\gamma} + 2u = \left(c_2^2+1\right)\left(x_0+u\right) - 2c_2x_3 + c_1,$$

where c_1, c_2 are arbitrary constants.

18) $\langle P_1\rangle \oplus \langle P_2 - X_2 - \beta X_3, \beta > 0\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2}{x_0+u} + 2u = \left(\frac{c_2^2}{4}+1\right)\left(x_0+u\right) - \frac{\left(\beta c_2 + 2x_2\right)^2}{4\left(x_0+u+1\right)} + c_2x_3 + c_1,$$

where c_1, c_2 are arbitrary constants.

19) $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+1}+2u=\left(\frac{c_2^2}{4}+1\right)(x_0+u)+c_2x_3+c_1,$$

where c_1, c_2 are arbitrary constants.

20) $\langle P_3 - 2X_0\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$u=\pm\sqrt{c_2x_2-4x_3-i\sqrt{c_2^2+16}\,x_1+c_1}-x_0,$$

where c_1, c_2 are arbitrary constants.

21) $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u=\varepsilon c_1x_2-\frac{\varepsilon}{6}\left((x_0+u)^2+4x_3-c_1^2\right)^{3/2}+c_2,$$

where c_1, c_2 are arbitrary constants.

22) $\langle L_3\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_1^2+x_2^2\right)^{1/2}=i\varepsilon x_3+f(x_0+u),\ \varepsilon=\pm1,$$

where f is an arbitrary smooth function.

23) $\langle L_3+\alpha X_3, \alpha>0\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_3+\alpha\arctan\frac{x_1}{x_2}$$
$$=i\varepsilon\sqrt{x_1^2+x_2^2+\alpha^2}-i\varepsilon\alpha\operatorname{arctanh}\frac{\alpha}{\sqrt{x_1^2+x_2^2+\alpha^2}}+f(x_0+u),\ \varepsilon=\pm1,$$

where f is an arbitrary smooth function.

24) $\langle P_3 - X_1\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_1-\frac{x_3}{x_0+u}=i\varepsilon x_2\sqrt{\frac{1}{(x_0+u)^2}+1}+f(x_0+u),\ \varepsilon=\pm1,$$

where f is an arbitrary smooth function.

25) $\langle P_3\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_1=i\varepsilon x_2+f(x_0+u),\ \varepsilon=\pm1,$$

where f is an arbitrary smooth function.

26) $\langle X_1\rangle \oplus \langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_3=i\varepsilon x_2+f(x_0+u),\ \varepsilon=\pm1,$$

where f is an arbitrary smooth function.

3.2.2. Lie Algebras of the Type A_2

1) $\langle -G, P_3 \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_3^2 - u^2\right)^{1/2} = \varepsilon\sqrt{1-c_2^2}\, x_1 + c_2 x_2 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

2) $\left\langle -G - \frac{1}{\lambda} L_3,\ X_4,\ \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) = i\lambda \operatorname{arctanh} \frac{\lambda}{\sqrt{c_1^2\left(x_1^2 + x_2^2\right) + \lambda^2}} - i\sqrt{c_1^2\left(x_1^2 + x_2^2\right) + \lambda^2}$$

$$- \lambda \arctan \frac{x_1}{x_2} + c_1 x_3 + c_2,$$

where c_1, c_2 are arbitrary constants.

3) $\langle -G - \alpha X_1,\ X_4,\ \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_1 - \alpha \ln\left(x_0 + u\right) = i\varepsilon\left(c_2^2 + 1\right)^{1/2} x_2 + c_2 x_3 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

4) $\left\langle -\frac{1}{\lambda}\left(L_3 + \lambda G + \alpha X_3\right),\ X_4,\ \alpha > 0, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) = i\varepsilon\sqrt{c_2^2\left(x_1^2 + x_2^2\right) + \left(\alpha c_2 - \lambda\right)^2}$$

$$- i\varepsilon\left(\alpha c_2 - \lambda\right) \operatorname{arctanh} \frac{\alpha c_2 - \lambda}{\sqrt{c_2^2\left(x_1^2 + x_2^2\right) + \left(\alpha c_2 - \lambda\right)^2}}$$

$$+ \left(\alpha c_2 - \lambda\right) \arctan \frac{x_1}{x_2} + c_2 x_3 + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

5) $\langle -G - \alpha X_1,\ P_3,\ \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_1 - \alpha \ln\left(x_0 + u\right) = \alpha \ln\left(2\alpha \frac{\sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - x_3^2 - u^2\right) + \alpha^2} + \alpha}{x_0^2 - x_3^2 - u^2}\right)$$

$$- \sqrt{\left(c_1^2 + 1\right)\left(x_0^2 - x_3^2 - u^2\right) + \alpha^2} + c_1 x_2 + c_2,$$

where c_1, c_2 are arbitrary constants.

3.3. Classification of the Common Invariant Solutions for the Equations under Study Using Three-Dimensional Nonconjugate Subalgebras of the Lie Algebra of the Group $P(1,4)$

3.3.1. Lie Algebras of the Type $3A_1$

1) $\langle P_1 - \gamma X_3, \gamma > 0 \rangle \oplus \langle P_2 - X_2 - \delta X_3, \delta \neq 0 \rangle \oplus \langle X_4 \rangle$:

The common invariant solution for the equations under study:

$$(x_0+u)^4+2(x_0+u)^3+(\gamma^2+\delta^2+1)(x_0+u)^2+2\gamma^2(x_0+u)+\gamma^2=0.$$

2) $\langle P_1\rangle\oplus\langle P_2-X_2-\delta X_3,\delta>0\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$(x_0+u)^2+2(x_0+u)+\delta^2+1=0.$$

3) $\langle P_1\rangle\oplus\langle P_2\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

4) $\langle P_3\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_3^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

5) $\langle P_1\rangle\oplus\langle P_2\rangle\oplus\langle P_3\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=c(x_0+u),$$

where c is an arbitrary constant.

6) $\langle P_1\rangle\oplus\langle P_2-X_2\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_0^2-x_1^2-u^2}{x_0+u}-\frac{x_2^2}{x_0+u+1}=c,$$

where c is an arbitrary constant.

7) $\langle P_1\rangle\oplus\langle P_2-\alpha X_2,\alpha>0\rangle\oplus\langle P_3-\gamma X_3,\gamma\neq 0\rangle$:

The common invariant solution for the equations under study:

$$2u+\frac{x_1^2}{x_0+u}+\frac{x_2^2}{x_0+u+\alpha}+\frac{x_3^2}{x_0+u+\gamma}=x_0+u+c,$$

where c is an arbitrary constant.

8) $\langle P_1\rangle\oplus\langle P_2-\alpha X_2,\alpha>0\rangle\oplus\langle P_3\rangle$:

The common invariant solution for the equations under study:

$$2u+\frac{x_1^2+x_3^2}{x_0+u}+\frac{x_2^2}{x_0+u+\alpha}=x_0+u+c,$$

where c is an arbitrary constant.

9) $\langle G\rangle\oplus\langle X_2\rangle\oplus\langle X_1\rangle$:

The common invariant solution for the equations under study:

$$(x_0^2-u^2)^{1/2}=\varepsilon x_3+c,\ \varepsilon=\pm 1,$$

where c is an arbitrary constant.

10) $\langle G\rangle\oplus\langle L_3\rangle\oplus\langle X_3\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2-u^2\right)^{1/2}=\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

11) $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$\frac{1}{6}\left(x_0+u\right)^3+x_3\left(x_0+u\right)+x_0-u=\frac{\varepsilon}{6}\left(\left(x_0+u\right)^2+4x_3\right)^{3/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

12) $\langle G+\alpha X_3,\alpha>0\rangle\oplus\langle X_1\rangle\oplus\langle X_2\rangle$:

The common invariant solution for the equations under study:

$$x_3-\alpha\ln\left(x_0+u\right)$$
$$=\varepsilon\left(\alpha^2+x_0^2-u^2\right)^{1/2}-\frac{\alpha}{2}\ln\left(x_0^2-u^2\right)-\varepsilon\alpha\operatorname{arctanh}\frac{\left(\alpha^2+x_0^2-u^2\right)^{1/2}}{\alpha}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

13) $\langle L_3\rangle\oplus\langle P_3+C_3\rangle\oplus\langle X_0+X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_3^2+u^2\right)^{1/2}=i\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

14) $\langle L_3+\alpha\left(X_0+X_4\right),\alpha>0\rangle\oplus\langle X_3\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$x_0+u+\alpha\arctan\frac{x_2}{x_1}=i\frac{\varepsilon\alpha}{2}\ln\left(x_1^2+x_2^2\right)+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

15) $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^2+4x_3=4i\varepsilon x_2+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

16) $\langle L_3\rangle\oplus\langle -P_3+2X_0\rangle\oplus\langle 2X_4\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^2+4x_3=4i\varepsilon\left(x_1^2+x_2^2\right)^{1/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

3.3.2. Lie Algebras of the Type $A_2\oplus A_1$

1) $\langle -G,P_3\rangle\oplus\langle X_1\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2-x_3^2-u^2\right)^{1/2}=\varepsilon x_2+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

2) $\langle -G, P_3\rangle \oplus \langle L_3\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_3^2 - u^2\right)^{1/2} = \varepsilon\left(x_1^2 + x_2^2\right)^{1/2} + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3) $\langle -(G+\alpha X_2), P_3, \alpha > 0\rangle \oplus \langle X_1\rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_2 - \alpha \ln\left(x_0 + u\right) \\ & = \varepsilon\left(x_0^2 - x_3^2 - u^2 + \alpha^2\right)^{1/2} - \frac{\alpha}{2}\ln\left(x_0^2 - x_3^2 - u^2\right) \\ & \quad - \varepsilon\alpha \operatorname{arctanh}\frac{\alpha}{\left(x_0^2 - x_3^2 - u^2 + \alpha^2\right)^{1/2}} + c,\ \varepsilon = \pm 1, \end{aligned}$$

where c is an arbitrary constant.

4) $\left\langle -\frac{1}{\lambda}L_3 - G, 2X_4, \lambda > 0\right\rangle \oplus \langle X_3\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) + \lambda \arctan\frac{x_1}{x_2} = i\frac{\varepsilon\lambda}{2}\ln\left(x_1^2 + x_2^2\right) + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

5) $\langle -(G+\alpha X_3), X_4, \alpha > 0\rangle \oplus \langle L_3 + \beta X_3, \beta > 0\rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & x_3 - \alpha \ln\left(x_0 + u\right) + \beta \arctan\frac{x_1}{x_2} \\ & = -i\varepsilon\beta \operatorname{arctanh}\frac{\beta}{\left(x_1^2 + x_2^2 + \beta^2\right)^{1/2}} + i\varepsilon\left(x_1^2 + x_2^2 + \beta^2\right)^{1/2} + c,\ \varepsilon = \pm 1, \end{aligned}$$

where c is an arbitrary constant.

6) $\langle -(G+\alpha X_3), X_4, \alpha > 0\rangle \oplus \langle L_3\rangle$:

The common invariant solution for the equations under study:

$$x_3 - \alpha \ln\left(x_0 + u\right) = i\varepsilon\left(x_1^2 + x_2^2\right)^{1/2} + c,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.3. Lie Algebras of the Type $A_{3,1}$

1) $\langle 4X_4, P_1 - X_2 - \gamma X_3, P_2 + X_1 - \mu X_2 - \delta X_3, \gamma > 0, \delta \neq 0, \mu > 0\rangle$:

The common invariant solution for the equations under study:

$$\begin{aligned} & \left(x_0 + u\right)^4 + 2\mu\left(x_0 + u\right)^3 + \left(\gamma^2 + \mu^2 + \delta^2 + 2\right)\left(x_0 + u\right)^2 \\ & + 2\mu\left(\gamma^2 + 1\right)\left(x_0 + u\right) + \left(\gamma\mu - \delta\right)^2 + \gamma^2 + 1 = 0. \end{aligned}$$

2) $\langle 2\mu X_4, P_3 - 2X_0, X_1 + \mu X_3, \mu > 0\rangle$:

The common invariant solution for the equations under study:

$$u = 2\left(i\varepsilon x_2\sqrt{\mu^2 + 1} + \mu x_1 - x_3 + c\right)^{1/2} - x_0,\ \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3) $\langle 2X_4, P_3 - L_3 - 2\alpha X_0, X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$u = 2\alpha \arctan \frac{x_1}{x_2} + i\varepsilon\alpha \ln\left(x_1^2 + x_2^2\right) - x_0 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

4) $\langle -2\beta X_4, L_3 + \beta X_3, P_3 - 2X_0, \beta > 0 \rangle$:

The common invariant solution for the equations under study:

$$\beta \arctan \frac{x_1}{x_2} + \frac{1}{4}\left(x_0 + u\right)^2$$
$$= i\varepsilon\sqrt{x_1^2 + x_2^2 + \beta^2} - i\varepsilon\beta \operatorname{arctanh} \frac{\beta}{\sqrt{x_1^2 + x_2^2 + \beta^2}} - x_3 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

5) $\langle 2X_4, P_3, X_3 \rangle$:

The common invariant solution for the equations under study:

$$x_2 = i\varepsilon x_1 + f\left(x_0 + u\right), \varepsilon = \pm 1,$$

where f is an arbitrary smooth function.

3.3.4. Lie Algebras of the Type $A_{3,2}$

1) $\left\langle 2\alpha X_4, \lambda P_3, \frac{1}{\lambda} L_3 + G + \frac{\alpha}{\lambda} X_3, \alpha > 0, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0 + u\right) + \lambda \arctan \frac{x_1}{x_2} = i\varepsilon \frac{\lambda}{2} \ln\left(x_1^2 + x_2^2\right) + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.5. Lie Algebras of the Type $A_{3,3}$

1) $\langle P_1, P_2, G \rangle$:

The common invariant solution for the equations under study:

$$\left(x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2} = \varepsilon x_3 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle P_1, P_2, G + \alpha X_3, \alpha > 0 \rangle$:

The common invariant solution for the equations under study:

$$x_3 - \alpha \ln\left(x_0 + u\right)$$
$$= \varepsilon\left(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2\right)^{1/2} - i\varepsilon\alpha \arctan \frac{\left(x_0^2 - x_1^2 - x_2^2 - u^2 + \alpha^2\right)^{1/2}}{i\alpha}$$
$$- \frac{\alpha}{2} \ln\left(x_0^2 - x_1^2 - x_2^2 - u^2\right) + c, \quad \varepsilon = \pm 1.$$

3) $\left\langle P_3, X_4, \frac{1}{\lambda} L_3 + G, \lambda > 0 \right\rangle$:

The common invariant solution for the equations under study:

$$\ln\left(x_0+u\right)+\lambda\arctan\frac{x_1}{x_2}=i\varepsilon\frac{\lambda}{2}\ln\left(x_1^2+x_2^2\right)+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

3.3.6. Lie Algebras of the Type $A_{3,6}$

1) $\left\langle P_1-X_1, P_2-X_2, -P_3+L_3\right\rangle$:

The common invariant solution for the equations under study:

$$\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=x_0+u+c,$$

where c is an arbitrary constant.

2) $\left\langle P_1, -P_2, -\left(L_3+\alpha X_3\right), \alpha>0\right\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-u^2=c\left(x_0+u\right),$$

where c is an arbitrary constant.

3) $\left\langle P_1, P_2, -P_3+L_3\right\rangle$:

The common invariant solution for the equations under study:

$$x_0^2-x_1^2-x_2^2-x_3^2-u^2=c\left(x_0+u\right),$$

where c is an arbitrary constant.

4) $\left\langle X_1, -X_2, P_3-L_3-2\alpha X_0, \alpha>0\right\rangle$:

The common invariant solution for the equations under study:

$$\left(x_0+u\right)^3+6\alpha x_3\left(x_0+u\right)+6\alpha^2\left(x_0-u\right)=\varepsilon\left(\left(x_0+u\right)^2+4\alpha x_3\right)^{3/2}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

5) $\left\langle X_1, -X_2, -L_3-\frac{1}{2}\left(P_3+C_3\right)-\alpha\left(X_0+X_4\right), \alpha>0\right\rangle$:

The common invariant solution for the equations under study:

$$\alpha\arctan\frac{x_3}{u}-x_0=\varepsilon\sqrt{x_3^2+u^2-\alpha^2}+\varepsilon\alpha\arctan\frac{\alpha}{\sqrt{x_3^2+u^2-\alpha^2}}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

6) $\left\langle X_1, X_2, L_3+\frac{\lambda}{2}\left(P_3+C_3\right)+\alpha\left(X_0+X_4\right), \alpha>0, 0<\lambda<1\right\rangle$:

The common invariant solution for the equations under study:

$$\alpha\arctan\frac{x_3}{u}-\lambda x_0$$
$$=\varepsilon\sqrt{\lambda^2\left(x_3^2+u^2\right)-\alpha^2}+\varepsilon\alpha\arctan\frac{\alpha}{\sqrt{\lambda^2\left(x_3^2+u^2\right)-\alpha^2}}+c,\ \varepsilon=\pm1,$$

where c is an arbitrary constant.

7) $\left\langle X_1, X_2, L_3+\lambda G+\alpha X_3, \alpha>0, \lambda>0\right\rangle$:

The common invariant solution for the equations under study:

$$\lambda x_3 - \alpha \ln(x_0 + u)$$
$$= \varepsilon\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2} - \varepsilon\alpha \operatorname{arctanh}\frac{\alpha}{\sqrt{\lambda^2(x_0^2 - u^2) + \alpha^2}}$$
$$-\frac{\alpha}{2}\ln(x_0^2 - u^2) + c, \varepsilon = \pm 1.$$

where c is an arbitrary constant.

8) $\langle P_1, P_2, L_3 \rangle$:

The common invariant solution for the equations under study:

$$x_3 = c_1 \ln(x_0 + u) - \varepsilon(x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2)^{1/2}$$
$$+ \varepsilon c_1 \operatorname{arctanh}\frac{\sqrt{x_0^2 - x_1^2 - x_2^2 - u^2 + c_1^2}}{c_1}$$
$$-\frac{c_1}{2}\ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c_2, \varepsilon = \pm 1, c_1 \neq 0.$$

3.3.7. Lie Algebras of the Type $A_{3,7}^{a}$

1) $\langle P_1, P_2, L_3 + \lambda G, \lambda > 0 \rangle$:

The common invariant solution for the equations under study:

$$(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varepsilon x_3 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle P_1, P_2, L_3 + \lambda G + \alpha X_3, \alpha > 0, \lambda > 0 \rangle$:

The common invariant solution for the equations under study:

$$\lambda x_3 - \alpha \ln(x_0 + u)$$
$$= \varepsilon(\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2)^{1/2} - \varepsilon\alpha \operatorname{arctanh}\frac{\sqrt{\lambda^2(x_0^2 - x_1^2 - x_2^2 - u^2) + \alpha^2}}{\alpha}$$
$$-\frac{\alpha}{2}\ln(x_0^2 - x_1^2 - x_2^2 - u^2) + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

3.3.8. Lie Algebras of the Type $A_{3,8}$

$\langle P_3, G, -C_3 \rangle$:

The common invariant solution for the equations under study:

$$(x_0^2 - x_3^2 - u^2)^{1/2} = \varepsilon(1 - c_2^2)^{1/2} x_1 + c_2 x_2 + c_1, \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

3.3.9. Lie Algebras of the Type $A_{3,9}$

1) $\left\langle -\frac{1}{2}\left(L_3 + \frac{1}{2}(P_3 + C_3)\right), \frac{1}{2}\left(L_2 + \frac{1}{2}(P_2 + C_2)\right), \frac{1}{2}\left(L_1 + \frac{1}{2}(P_1 + C_1)\right) \right\rangle$:

The common invariant solution for the equations under study:

$$(x_1^2 + x_2^2 + x_3^2 + u^2)^{1/2} = \varepsilon x_0 + c, \varepsilon = \pm 1,$$

where c is an arbitrary constant.

2) $\langle -L_3, -L_2, -L_1 \rangle$:

The common invariant solution for the equations under study:

$$u = \varepsilon\left(c_2^2 + 1\right)^{1/2} x_0 + c_2\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2} + c_1,\ \varepsilon = \pm 1,$$

where c_1, c_2 are arbitrary constants.

4. Conclusions

In this paper, we have presented obtained common invariant solutions of the following (1 + 3)-dimensional equations: the Eikonal equations, the Euler-Lagrange-Born-Infeld equation, the homogeneous Monge-Ampère equation and the inhomogeneous Monge-Ampère equation. We have used the structural properties of the low-dimensional ($dimL \leq 3$) nonconjugate subalgebras of the same ranks of the Lie algebra of the Poincaré group $P(1,4)$ for classification of the obtained common invariant solutions.

Since the group $P(1,4)$ contains, as subgroups, the extended Galilei group $\tilde{G}(1,3)$ [49] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics), the results obtained can be useful in construction and investigation of corresponding physical models.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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