Scientific Research Publishing

Applied Mathematics, 2023, 14, 545-579
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Some Implications of the Gessel Identity

Claire Levaillant

Department of Mathematics, University of Southern California, Los Angeles, CA, USA
Email: clairelevaillant@yahoo.fr

How to cite this paper: Levaillant, C. (2023) Some Implications of the Gessel Identity. *Applied Mathematics*, **14**, 545-579.
https://doi.org/10.4236/am.2023.149034

Received: April 26, 2023
Accepted: September 12, 2023
Published: September 15, 2023

Abstract

We generalize the congruences of Friedmann-Tamarkine (1909), Lehmer (1938), and Ernvall-Metsänkyla (1991) on the sums of powers of integers weighted by powers of the Fermat quotients to the next Fermat quotient power, namely to the third power of the Fermat quotient. Using this result and the Gessel identity (2005) combined with our past work (2021), we are able to relate residues of some truncated convolutions of Bernoulli numbers with some Ernvall-Metsänkyla residues to residues of some full convolutions of the same kind. We also establish some congruences concerning other related weighted sums of powers of integers when these sums are weighted by some analogs of the Teichmüller characters.

Keywords

Convolutions Involving Bernoulli Numbers, Truncated Convolutions Involving Bernoulli Numbers, Congruences, Binomial and Multinomial Convolutions of Divided Bernoulli Numbers, Multiple Harmonic Sums, Generalized Harmonic Numbers, Miki Identity, Gessel Identity, Sums of Powers of Integers Weighted by Powers of the Fermat Quotients, Generalization of Kummer's Congruences, Generalizations of Friedmann-Tamarkine, Lehmer, Ernvall-Metsänkyla's Congruences, p-Adic Numbers, Weighted Sums of Powers of Integers

1. Introduction

1.1. Scope and Summary

1.1.1. Scope

In the past century, convolutions involving Bernoulli numbers have drawn much attention. One motivation is that such convolutions arise in quantum field theory and string theory, see for instance [1]. However, truncated convolutions involving Bernoulli numbers are not as broadly studied, though they appear in particular in combinatorial number theory relating for instance to congruences

of unsigned Stirling numbers of the first kind on p letters with odd indices modulo p^3. For such congruences, only the even indices had been tackled by Glaisher in his pioneering work [2] around the turn of the twentieth century. In [3], we relate the residue modulo p of such truncated convolutions to the residue of some full convolutions. These congruences shortly after become in [4] an important tool for generalizing Wilson's theorem to the modulus p^4 pushing Z-H. Sun's expansion of [5] one p power further. These truncated convolutions appear independently in the late 1970s in Miki's work on his way to proving a celebrated identity involving binomial convolutions of Bernoulli numbers. Our work of [3] combined with Miki's work of [6] implies a generalization in some cases of the famous Kummer congruences to the modulus p^2 in terms of a sum of powers of integers weighted by the squared Fermat quotients that was originally due to Ernvall and Metsänkyla in [7]. Sums of powers of integers weighted by the Fermat quotients had already been studied earlier by the authors of [8] and had led to a criterion for finding irregular primes. Such a result became of importance as Kummer in 1850 had shown that Fermat's last theorem holds when the exponent is a regular prime. As part of our work, we find novel congruences involving sums of powers of integers weighted by cubes of Fermat quotients. By computing the residues of such sums in two different ways, we are able to relate residues of truncated convolutions of divided Bernoulli numbers with Ernvall-Metsänkyla residues to residues of full convolutions of the same kind. These novel congruences could open the path to important applications in number theory in the same way as similar congruences involving only divided Bernoulli numbers as already mentioned before had served as a crucial ingredient in the generalization of Wilson's theorem to the modulus p^4. We further prove similar congruences involving this time convolutions and truncated convolutions of ordinary Bernoulli numbers with Ernvall-Metsänkyla residues. This is a much harder computational problem where Gessel's identity, a generalization of Miki's identity, is used in order to reduce a differential term of cubic convolutions of divided Bernoulli numbers arising from multiple harmonic sums modulo p^4.

1.1.2. Detailed Summary

In 1991, Reijo Ernvall and Tauno Metsänkyla generalized the Kummer congruences by relating modulo p^2 the difference $\mathcal{B}_{p-1+t}-\mathcal{B}_t$ to a sum of t-th powers of the first $(p-1)$ integers weighted by the squared Fermat quotients. They did so for the range $4\le t\le p-3$, when t is even. In their honor, we will call the p-residue of $\left(\mathcal{B}_{p-1+t}-\mathcal{B}_t\right)/p$ the Ernvall-Metsänkyla residue. In this paper, we set a prime $p\ge 11$ and we restrict further the range for the even t. We namely impose $6\le t\le p-5$. We let n be the integer so that $t=p-1-2n$. Hence $4\le 2n\le p-7$. The main result of the paper offers a congruence modulo p relating a truncated convolution of order $2(p-1)-2n$ of Bernoulli numbers with Ervall-Metsänkyla residues to the full convolution of order $p-1-2n$. The latter congruence also involves a full convolution where the ordinary Bernoulli

numbers are replaced with the divided Bernoulli numbers. This result can be viewed as a generalization of Theorem 1 point (i) of [3] which offers a similar result when dealing with convolutions of divided Bernoulli numbers. One of the three proofs presented in [3] uses the famous Miki identity which relates convolutions of divided Bernoulli numbers to binomial convolutions of divided Bernoulli numbers. The current proof is based on Ira Gessel's identity, namely a generalization of Hiroo Miki's identity to cubic convolutions, and on our past work on the multiple harmonic sums $\mathcal{H}_{\{s\}^{2n}=1;p-1}$ modulo p^4, amongst other things. Another important tool consists of generalizing congruences proven throughout the twentieth century and which are due to Friedmann-Tamarkine [8], Lehmer [9], Miki [6] and Ernvall-Metsänkyla [7] in chronological order, on sums that are discussed right below and which involve the Fermat quotients in base a with $1 \le a \le p-1$. The latter generalization is interesting in its own sake. Though many mathematicians have been studying congruences concerning Bernoulli numbers and the Fermat quotients, their results tend to be sparse in time, like testified by the (non exhaustive) list below.

Since the beginning of the twentieth century, mathematicians have been working with sums of some fixed power of the first $(p-1)$ integers involving the Fermat quotients or the squared Fermat quotients as their weights. However, sums of some fixed power of integers have been studied since way earlier. In fact, the pioneer for the study of such sums is Johann Faulhaber of Ulm (1580-1635) whose computations by hand in the early 17th century [10] drew the attention and the admiration of many mathematicians in the subsequent centuries. Two centuries after Faulhaber, in 1834, Carl Jacobi was first to provide a rigorous proof for Faulhaber's guessed formula [11] which later became known as "the Faulhaber formula", see [12] for an excellent expository on the topic.

Weighted sums of powers of integers happen to be linked to the Bernoulli numbers. Conversely, the residues of the divided Bernoulli numbers, to the exception of some of them, can be expressed as residues of some weighted sums of powers of integers. Indeed, if i is a positive even integer that is prime to both p and $p-1$, then

$$\mathcal{B}_i = \sum_{a=1}^{p-1}\left(\sum_{b=1}^{p-1}\left\lfloor \frac{b^{-1}a}{p}\right\rfloor \frac{1}{b^{i-1}}\right)a^{i-1} \bmod p$$

This congruence namely follows from summing the Voronoi congruences from 1889 [13] and using the fact that p divides a sum of powers of the first $(p-1)$ integers when these powers are not multiples of $(p-1)$.

Three centuries after Faulhaber, the earlier work on the sums of some fixed power of the first $(p-1)$ integers weighted by the Fermat quotients goes back to Friedmann and Tamarkine [8] in the early twentieth century. Their congruence modulo p opens another path towards the study of irregular primes since it provides a criterion for irregular pairs. The search for irregular primes had been of importance on the route to the proof of Fermat's Last Theorem ever since

Ernst Kummer had shown that Fermat's Last Theorem holds when the exponent is a regular prime [14], thus leaving irregular primes to be analyzed individually. While Friedmann and Tamarkine only deal with even powers, Emma Lehmer in 1938 generalizes the Friedmann-Tamarkine congruence to odd powers and pushes the study one p power further. Her congruences namely hold modulo p^2. It is not until 1978 that sums of some fixed power of the first $(p-1)$ integers weighted by the squared Fermat quotients appear in the pioneering work of Hiroo Miki on his way to finding an identity relating a convolution of divided Bernoulli numbers to a binomial convolution of divided Bernoulli numbers, an identity which also involves harmonic numbers. Miki relates these sums modulo p for a restricted range of even powers to convolutions of divided Bernoulli numbers and to some of their truncations [6]. These exact same p-residues of truncated convolutions of divided Bernoulli numbers are studied independently in 2020 [3] using the p-adic analysis of some polynomial with p-adic integer coefficients whose study got initiated in [15]. In this study, the unsigned Stirling numbers of the first kind on p letters are involved. A congruence is obtained relating these residues of truncated convolutions to the residues of some full convolutions. By confronting Miki's work and our work of [3], we immediately retrieve Reijo Ernvall and Tauno Metsänkyla's congruence [7] (holding for the same range of powers) from 1991, which instead of the complicated convoluted form by Miki, provides a much simpler expression for the considered residual sum in terms of the second residue in the p-adic Hensel expansion [16] of a certain difference of two divided Bernoulli numbers. We show further that Ernvall and Metsänkyla's method does generalize to odd powers. This is the purpose of our Theorem 1.

At this point, the p-adic analysis of the unsigned Stirling numbers of the first kind on p letters resurfaces in order to generalize to base a with $1 \le a \le p-1$ some congruences of [17] that were achieved in base 2. We provide the residue of a powers of a weighted sum of divided Bernoulli numbers whose indices range between 1 and $p-2$, for a set integer a and where the powers of a coincide with the indices of the divided Bernoulli numbers. This residue is as simple in expression as a sum modulo p of the Wilson quotient and of the Fermat quotient in base a. By summing over the bases such sums after these have been multiplied by some fixed power of a and by the squared Fermat quotient q_a^2, we thus get amongst other terms a sum of powers of integers weighted by the third power of the Fermat quotients. It is the purpose of Theorem 2. By combining further Theorem 2 with Theorem 1 evoked before, we may conveniently adjust the range of indices that concerns the divided Bernoulli numbers. However, this is not yet enough to generalize the Friedmann-Tamarkine-Lehmer-Ernvall-Metsänkyla congruence in an appealing way, that is one which is computationally efficient. So, we continue working on the same sum and apply to it the Ernvall-Metsänkyla congruence. We must also generalize Ernvall and Metsänkyla's congruence to the unstudied cases when the power is respectively 0 or 2. This is the purpose of Theorems 3 and 4 respectively. Once this is done, we obtain Theorem 5 which

provides a satisfactory congruence modulo p for the sum of powers of integers weighted by the third power of the Fermat quotient in terms of only Bernoulli numbers. Independently however, we get a much simpler form—with a number of terms that is independent from the prime p—by using a result by Zhi-Hong Sun from [5] on sums of powers of integers. By comparing both expressions, we obtain new congruences on Bernoulli numbers generalizing those of [3]. The simpler version of Theorem 5 is the one which is retained for later use.

From there, we have all the ingredients to tackle the main congruence of the paper. This time, the conjugates to the Stirling numbers, namely the multiple harmonic sums on $p-1$ integers come into play. We will shortly discuss why. In [4], we had obtained by means of the Newton formulas and using some sophisticated p-adic expansions of the generalized harmonic numbers on $p-1$ integers modulo p^4 by Zhi-Hong Sun [5] a congruence providing these sums modulo p^4, see Theorem 3 of [4]. Sums that are related to those discussed in the previous paragraph are present in the expression. Another core term appearing in the expression is a differential term composed of two cubic convolutions of divided Bernoulli numbers. This is where Ira Gessel's identity from the title now enters the scene. The Gessel identity [18] is a generalization of the Miki identity to products of three divided Bernoulli numbers. It involves a multinomial convolution of three divided Bernoulli numbers, a binomial convolution of divided Bernoulli numbers, the harmonic numbers and the multiple harmonic sums. It plays a key role in processing the differential term. Because we may write the multiple harmonic sum $\mathcal{H}_{\{s\}^{2n}=1;p-1}$ in yet another way by relating it modulo p^4 to the Stirling numbers on p letters and with $2n+1$ cycles (those are computable in an easier manner than their homologs), we are able to derive the interesting congruence modulo p announced earlier.

Another part of the paper is devoted to studying congruences concerning sums of powers of integers weighted by the p-adic integer roots of some polynomials with p-adic integer coefficients, thus generalizing techniques arising from [19] and [7]. We provide these congruences modulo p^3. The paper is built in such a way that it alternates between the different interconnected projects in order to make the computations nicer to follow. For clarity and in order to distinguish them from the present results, the past results from the other authors have been numbered with 0.i concerning the i-th Theorem or Proposition.

1.2. Some Notations

We shall introduce some standard or some personal notations which we shall use all throughout the paper. Given a prime p, we denote by w_p the Wilson quotient and by q_a the Fermat quotient in base a with $1 \le a \le p-1$. When dealing with Bernoulli numbers, we denote by $\mathcal{B}_t$ the t-th divided Bernoulli number, by $\mathcal{CB}(2k)$ a convolution of divided Bernoulli numbers of order $2k$ and by $\mathcal{BBB}(2k)$ a sum of products of three divided Bernoulli numbers whose indices sum up to $2k$. Also, using the same notation as in [3], we denote by

$\mathcal{TCB}(p+1-2n, p-3)$ the following truncated convolutions of divided Bernoulli numbers:

$$\mathcal{TCB}(p+1-2n, p-3) := \sum_{k=p+1-2n}^{p-3} \mathcal{B}_k \mathcal{B}_{2(p-1)-2n-k}$$

Throughout Section 2, we also deal with binomial (resp multinomial) convolutions and thus introduce some notations for these as well. Namely,

$$b\mathcal{CB}(2n) := \sum_{i=2}^{2n-2} \binom{2n}{i} \mathcal{B}_i \mathcal{B}_{2n-i}$$

and

$$m\mathcal{BBB}(2n) := \sum_{i+j+k=2n} \binom{2n}{i,j,k} \mathcal{B}_i \mathcal{B}_j \mathcal{B}_k$$

By a theorem due to Clausen [20] and independently Von Staudt [21], the denominator of a Bernoulli number B_k consists of products of primes p of multiplicity one, such that $p-1 \mid k$. In particular, $pB_{p-1} = -1 \bmod p$. A conjecture of Agoh [22] and independently Giuga [23] claims that $nB_{n-1} = -1 \bmod n$ if and only if n is a prime.

We will denote the Agoh-Giuga quotient by

$$\left(pB_{p-1}\right)_1 := \frac{1+pB_{p-1}}{p}$$

when x is a p-adic integer, we denote by $(x)_i$ its $(i+1)$-th p-residue in its Hensel expansion (see e.g. [16]), that is:

$$x = \sum_{i=0}^{\infty} (x)_i \, p^i$$

It is important to underline that this notation does not apply to nor match with $\left(pB_{p-1}\right)_1$ which got actually defined independently by being the Agoh-Giuga quotient.

A quite present Hensel residue throughout the paper will be:

$$\mathcal{D}_i := \left(\mathcal{B}_{p-1+i} - \mathcal{B}_i\right)_1$$

Convolutions and truncated convolutions of these with ordinary or divided Bernoulli numbers are involved and we will thus use notations such as $\mathcal{CBD}(2n)$, $\mathcal{TCBD}(p+1-2n, p-3)$ and $\mathcal{CBD}(2n)$, $\mathcal{TCBD}(p+1-2n, p-3)$ to denote these respective entities.

For instance,

$$\mathcal{TCBD}(p+1-2n, p-3) := \sum_{i=p+1-2n}^{p-3} B_i \mathcal{D}_{2(p-1)-2n-i}$$

Finally, $\mathcal{H}_t$ denotes the harmonic number of order t. Using standard notations, generalized harmonic numbers are denoted by $H_{p-1,k}$ or simply H_k when it is clearly understood that the order of the sum is $p-1$. When the order of the sum is not $p-1$, we will rather denote these numbers by $\mathcal{H}_{n,k}$ where the first index refers to the order of the sum and the second index refers to the index of

the power. So, we have:

$$\mathcal{H}_t := \sum_{i=1}^{t} \frac{1}{i} \qquad H_k := \sum_{x=1}^{p-1} \frac{1}{x^k} \qquad \mathcal{H}_{n,k} := \sum_{x=1}^{n} \frac{1}{x^k}$$

Last, regarding multiple harmonic sums, we set

$$A^{\star}_{k,n} := \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{1}{i_1 \cdots i_k}$$

We recall that by some version of Wolstenholme's theorem, the following two congruences $H_1 = 0 \bmod p^2$ and $H_2 = 0 \bmod p$ hold, see [24].

1.3. Some Historical Background

Sums of powers of integers weighted by the Fermat quotients or by powers of the Fermat quotients have drawn the interest of many mathematicians since the beginning of the twentieth century.

Sums of powers of integers weighted by the Fermat quotients first appear in 1909 in [8]. Given an even integer t, the authors show that:

$$\sum_{a=1}^{p-1} q_a a^t = \begin{cases} -\mathcal{B}_t \bmod p & \text{if } t \neq 0 \left(\bmod p-1\right) \\ w_p \bmod p & \text{if } t = 0 \left(\bmod p-1\right) \end{cases}$$

Their proof is based on studying the number of numbers divisible by p in the sequence of numbers

$$1, 2, 3, \cdots, n$$

and noticing that this number is congruent modulo p to

$$\sum_{y=1}^{n} \left(1 - y^{p-1}\right)$$

A consequence of Friedmann and Tamarkine's result is the following.

A pair (p,t) is an irregular pair if and only if

$$\sum_{a=1}^{p-1} q_a a^t = 0 \bmod p$$

The search for irregular primes has been of importance [25] [26] [27] ever since Kummer [14] showed that Fermat's last theorem holds when the exponent is a regular prime. An important feature of irregular primes is that the numerators of the divided Bernoulli numbers consist of products of powers of irregular primes, except for $\mathcal{B}_n$ with $n \in \{2,4,6,8,10,14\}$ when it is 1. A study on the irregular prime divisors of the Bernoulli numbers gets pursued by Wells Johnson in [28].

Later in 1938, Emma Lehmer generalizes the Friedmann-Tamarkine congruence [9]. She proves that:

$$\sum_{a=1}^{p-1} a^{2t+1} q_a = -pB_{2t} \bmod p^2 \qquad \text{if } 2t \neq 0, 2 \left(\bmod p-1\right)$$

$$\sum_{a=1}^{p-1} a^{2t} q_a = B_{p-1+2t} - B_{2t} \bmod p^2 \quad \text{if } 2t \neq 0, 2 \left(\bmod p-1\right)$$

Her proof is based on congruences concerning sums of powers of integers and on the obvious equality $a^{p-1+t} - a^t = a^t p q_a$.

We note that the latter two congruences involve non divided Bernoulli numbers. In fact, Ernvall and Metsänkyla show in their 1991 paper that the differences $\mathcal{B}_{p-1+2t}-\mathcal{B}_{2t}$ where the Bernoulli numbers have now been replaced with divided Bernoulli numbers also relate modulo p^2 to these sums of powers of integers, but this time weighted by squared Fermat quotients. Explicitly, they generalize the Kummer congruences [29] by showing that:

Theorem 0.1. *Due to Ernvall and Metsänkyla* 1991 [7]. *Let t be an even integer. Then*,

$$\forall 4\le t\le p-3,\ \mathcal{B}_{p-1+t}=\mathcal{B}_t-\frac{p}{2}\sum_{a=1}^{p-1}q_a^2a^t \bmod p^2$$

Their theorem follows from another of their own congruences which we recall below. First, it will be necessary to introduce the Teichmüller characters. These arise for instance from factorizing the polynomial $X^{p-1}-1$ in $\mathbb{Z}_p[X]$. Indeed, by Hensel's lemma, each integer $1\le a\le p-1$ lifts to a unique p-adic integer root $\omega(a)$ such that $\left|w(a)-a\right|_p\le\frac{1}{p}$. The polynomial $f(X)$ factors as

$$f(X)=\prod_{a=1}^{p-1}\left(X-\omega(a)\right)$$

Writing for each integer a with $1\le a\le p-1$,

$$\omega(a)=a+pv_a\quad \text{some } a\in\mathbb{Z}_p,$$

Ernvall and Metsänkyla's congruence reads as follows.

Proposition 0.1. (*due to Ernvall and Metsänkyla*, [7] 1991) *Let t be an even integer with* $4\le t\le p-3$. *Then*,

$$\mathcal{B}_t=-\sum_{a=1}^{p-1}a^{t-1}v_a-\frac{t-1}{2}p\sum_{a=1}^{p-1}a^{t-2}v_a^2 \bmod p^2$$

Ernvall and Metsänkyla's proof is inspired from [19] and is simply based on Newton's formula and sums of powers of integers. Let $t\ge 1$. Denoting by σ_t the elementary symmetric polynomials in the roots of f, namely

$$\sigma_t=\sum_{1\le a_1,\cdots,a_t\le p-1}\omega(a_1)\cdots\omega(a_t)$$

and by s_t the sums

$$s_t=\sum_{a=1}^{p-1}\omega(a)^t,$$

we have by Newton's formula:

$$s_t=\sigma_1s_{t-1}-\sigma_2s_{t-2}+\cdots+(-1)^{t-1}t\sigma_t,\quad \forall t=1,\cdots,p-1$$

Moreover, if follows from the expansion

$$f(X)=X^{p-1}-\sigma_1X^{p-2}+\sigma_3X^{p-3}+\cdots-\sigma_{p-2}X+\prod_{a=1}^{p-1}\omega(a)$$

that

$$\sigma_t=0,\quad \forall t=1,\cdots,p-2$$

Therefore, we have $s_t = 0 \quad \forall t = 1, \cdots, p-2$. Substituting $w(a) = a + pv_a$ in the equation

$$\sum_{a=1}^{p-1} \omega(a)^t = 0,$$

expanding the t-th power and reducing modulo p^3 yields the result (see also [15] for the relevant developments on the sums of powers. This is where we need to exclude $t = 2$, namely $\sum_{a=1}^{p-1} a^t = pB_t \bmod p^2$ only when $t \neq 2$). Because we have $\omega(a)^{p-1} = 1$, we also have

$$\sum_{a=1}^{p-1} \omega(a)^{p-1+t} = 0$$

We thus get:

$$\mathcal{B}_{p-1+t} - \mathcal{B}_t = \sum_{a=1}^{p-1} a^{t-1}\left(-a^{p-1}+1\right)v_a + \frac{p}{2}\sum_{a=1}^{p-1} v_a^2 a^{t-2} \bmod p^2$$

Next, from expanding $(a + pv_a)^p = a + pv_a$ modulo p^2, we see that

$$v_a = aq_a \bmod p$$

The theorem follows. This result is fundamentally used in [4] where a congruence modulo p^4 gets provided for $(p-1)!$ in terms of Bernoulli numbers, generalizing Wilson, Glaisher and Sun's own congruences modulo p, p^2 and p^3 respectively.

In [6], on his way to proving his identity which has become commonly known as "Miki's identity", Hiroo Miki finds a congruence modulo p^2 for

$$\sum_{a=1}^{p-1} a^t q_a, \quad 4 \le t \le p-3$$

This congruence involves convolutions of divided Bernoulli numbers, binomial convolutions of divided Bernoulli numbers, truncated convolutions of divided Bernoulli numbers as well as the Agoh-Giuga quotient and harmonic numbers of order t. He first finds an expansion for $aq_a, 1 \le a \le p-1$ to the modulus p^2. We state Miki's result below using our own notations.

Theorem 0.2. *Due to Miki* (1978) [6]. *Let t be an even integer with $4 \le t \le p-3$. Let n be the integer such that $t = p-1-2n$. Then,*

$$\begin{aligned}\sum_{a=1}^{p-1} a^t q_a = -\mathcal{B}_t + \Big\{ &\left(\left(pB_{p-1}\right)_1 + \left(1-\left(pB_{p-1}\right)\right)t + \mathcal{H}_t\right)\mathcal{B}_t + \frac{t-2}{2}\mathcal{CB}(t) \\ &+ \frac{1}{2}b\mathcal{CB}(t) + \frac{t-1}{2}\mathcal{TCB}(p+1-2n, p-3)\Big\} p \pmod{p^2}\end{aligned} \tag{1}$$

By doing a similar work on the weighted sums

$$\sum_{a=1}^{p-1} a^t q_a^2, \quad 4 \le t \le p-3,$$

he obtains a congruence modulo p for these sums in terms of convolutions of divided Bernoulli numbers, of truncated convolutions of divided Bernoulli numbers and of the Agoh-Giuga quotient.

Independently in [3], we obtain a congruence relating residues of convolutions of divided Bernoulli numbers with residues of truncated convolutions of Bernoulli numbers. By using Ernvall and Metsänkyla's theorem, our Theorem 1 point (i) of [3] is Lemma 4 of [6] with $m = p-1-2n$ and $6 \le m \le p-5$.

Before closing this introduction, we recall below Miki's identity and its generalization by Gessel. The identity by Gessel relates a multinomial convolution of three divided Bernoulli numbers to a convolution of three divided Bernoulli numbers, while Miki's identity relates a binomial convolution of divided Bernoulli numbers to a convolution of divided Bernoulli number. We state both results below using our own notations.

Theorem 0.3. *Miki's identity* 1978 [6].

$$\sum_{i=2}^{m-2} \mathcal{B}_i \mathcal{B}_{m-i} = \sum_{i=2}^{m-2} \binom{m}{i} \mathcal{B}_i \mathcal{B}_{m-i} + 2\mathcal{H}_m \mathcal{B}_m$$

Theorem 0.4. *Gessel's identity* 2005 [18].

$$\sum_{\substack{i+j+k=n \\ i,j,k\ge 2}} \binom{n}{i,j,k} \mathcal{B}_i \mathcal{B}_j \mathcal{B}_k + 3\mathcal{H}_n \sum_{i=2}^{n-2} \binom{n}{i} \mathcal{B}_i \mathcal{B}_{n-i} + 6A^{\star}_{2,n} \mathcal{B}_n$$

$$= \mathcal{B}\mathcal{B}\mathcal{B}(n) + \frac{n^2-3n+5}{4} \mathcal{B}_{n-2}$$

In 1982, Shiratani and Yokoyama gave another proof for Miki's identity. Their proof of [30] is very different from Miki's original proof as it uses integration and analysis methods. It wasn't until 2005 that Gessel gave a much simpler proof of Miki's identity and one which generalizes as in Theorem 0.4. Gessel's proof is based on two different expressions for Stirling numbers of the second kind. An identity related to Miki's was found in 2000 by Faber and Pandharipande [31] and proven by Zagier. Some authors like [32] have then been talking about the Miki-Zagier-Gessel identity. By using his approach, Gessel could find a generalization of the Faber-Pandharipande identity, which is also a generalization of the Miki identity. Shortly after in the same year, Crabb [33] gave a very short and simple proof for this generalization originally due to Gessel, from a functional equation for the generating function

$$\sum_{n\ge 0} \frac{B_n(\lambda)}{n!} X^n$$

Still around the same time, Schubert and Dunne found out that Miki's identity arises naturally in a certain computation in perturbative quantum field theory. They use a different generating function which plays an important role in quantum field theory computations in order to prove Miki's identity. They also prove the Faber-Pandharipande identity by using yet another different generating function. By noticing that both generating functions are related in a certain way, they also find a novel convolution identity. By their approach, they also find a cubic generalization of the Faber-Pandharipande identity. Further, they outline the generalization of the method to the derivation of convolution identities of arbitrary order. Their work appears in [1].

Remark 1. *Multinomial convolutions of ordinary Bernoulli numbers were also studied by different authors, originating with Euler when the number N of Bernoulli numbers in the product equals* 2. *Then, in* [34], *Sitaramachandrarao and Davis generalized Euler's formula to the case when* $N=3,4$. *The cases* $N=5$ *and* $N=6,7$ *were further respectively studied by Sankaranaryanan* [35] *and Zhang* [36]. *The full generalization to any N was finally achieved by Petojević and Srivastava in* [37], *based also on Dilcher's results of* [38]. *Their closed formula involves some Stirling numbers of the first kind.*

Remark 2. *In* 2010 *in* [39], *the authors derive yet another type of quadratic convolution identities involving Bernoulli numbers, not obviously related to any included in* [34].

Remark 3. *In* 2016 *in* [40], *Agoh gave yet another proof of Miki's identity, based on Faulhaber's formula for the sums of powers of integers* (*sometimes known as the "Bernoulli formula"*).

Before stating the results of the current paper, we state below a weaker version of Miki's lemma 1 of [6] which we shall use several times in the current paper.

Proposition 0.2. *Weaker version of Lemma* 1 *of* [6] *by Miki. Let a be an integer with* $1\le a\le p-1$. *Then, we have*:

$$aq_a=-1+\left(1-\left(pB_{p-1}\right)_1\right)a-\sum_{k=1}^{p-3}\frac{1}{p}\binom{p}{k}B_k a^{p-k} \bmod p$$

We briefly recall a proof of this fact. Denote the sums of powers of integers by

$$S_m(n):=1^m+2^m+\cdots+(n-1)^m$$

Then, Bernoulli's formula reads (see e.g. [15]), (in the formula below $B_1=-\frac{1}{2}$),

$$(m+1)S_m(n)=\sum_{i=1}^{m+1}\binom{m+1}{i}B_{m+1-i}n^i$$

In particular, letting $m=p-1$ and $n=a$, we have for each integer a with $1\le a\le p-1$:

$$pS_{p-1}(a)=a^p+apB_{p-1}+\sum_{i=2}^{p-1}\binom{p}{i}B_{p-i}a^i$$

And after the change of indices $j=p-i$, we obtain:

$$pS_{p-1}(a)=a^p+apB_{p-1}+\sum_{j=1}^{p-2}\binom{p}{j}B_j a^{p-j}$$

Since $a^p=a+aq_a p$ and $pB_{p-1}=-1+p\left(pB_{p-1}\right)_1$, this equality rewrites as:

$$S_{p-1}(a)=aq_a+\left(pB_{p-1}\right)_1 a+\sum_{j=1}^{p-2}\frac{1}{p}\binom{p}{j}B_j a^{p-j}$$

On the other hand, since $k^{p-1}=1+q_k p$, we have:

$$S_{p-1}(a)=a-1+p\sum_{k=1}^{a-1}q_k$$

Comparing the latter two identities yields Proposition 0.2.

1.4. New Results

Our first theorem is based on a special case of Ira Gessel's identity (listed as Theorem 0.4 of Section 1.3) and on Hiroo Miki's identity (listed as Theorem 0.3 of Section 1.3). It also relies on a result by Jianqiang Zhao from 2007 [41], which combined with our work of [3] provides the harmonic number $\mathcal{H}_{p-1}$ modulo p^3. Recall that this number is zero modulo p^2 by Wolstenholme's theorem. The proof of the theorem below is saved for the very end of the paper.

Theorem 0.

$$\sum_{i=2}^{p-3} \mathcal{B}_i \mathcal{B}_{p-1-i} \mathcal{H}_{p-1-i} = 2\mathcal{B}_{p-3} \bmod p$$

Corollary 0.

$$\sum_{i=2}^{p-3} \mathcal{H}_i B_i \mathcal{B}_{p-1-i} = \sum_{i=2}^{p-3} \mathcal{H}_i \mathcal{B}_i B_{p-1-i} = -\mathcal{B}_{p-3} \bmod p$$

Our next results serve as preliminary results towards the core theorem of the current paper listed below as Theorem 6. However, these results are interesting in their own sake and generalize the works of the group of mathematicians evoked before. By generalizing Reijo Ernvall and Tauno Metsänkyla's congruences described in Proposition 0.1 and Theorem 0.1, of Section 1.3 to odd integers t, we obtain a "Lehmer type" congruence like stated in Theorem 1 below. This is a necessary step towards finding novel congruences concerning sums of powers of integers weighted by the third power of the Fermat quotient, thus pushing the study of such sums "one Fermat quotient power" further.

Theorem 1. *Let t be an odd integer with $5 \le t \le p-2$. Then,*

$$B_{p-1+t-1} - B_{t-1} = -\sum_{a=1}^{p-1} a^t q_a^2 \bmod p$$

By comparing this congruence modulo p to the one of Emma Lehmer modulo p^2, we derive in turn the following statement.

Corollary 1. *Let t be an odd integer with $5 \le t \le p-2$ and let $\omega(a) = a + pv_a$ be the Teichmüller character associated to each a with $1 \le a \le p-1$.*

The following congruences hold.

Version 1.

$$\sum_{a=1}^{p-1} q_a^2 a^t = -\sum_{a=1}^{p-1} q_a a^{t-1} \bmod p$$

Version 2.

$$\sum_{a=1}^{p-1} q_a a^{t-1} (1 + v_a) = 0 \bmod p$$

A straightforward consequence of Theorem 1 is also the following.

Corollary 2. *Let t be an even integer with $4 \le t \le p-3$. Then,*

$$\mathcal{B}_t = \sum_{a=1}^{p-1} a^{t+1} q_a^2 \bmod p$$

This is the version of Theorem 1 which we later use. The sum studied in the next

theorem below is present in the congruence for the multiple harmonic sums $A^{\star}_{2n}, 2n \le p-5$ modulo p^4 of Theorem 3 of [4]. This sum appears to be in connection with the sum of $2n$-th powers of integers weighted by the third power of the Fermat quotient, like follows.

Theorem 2. *Fix some integer n. Then*,

$$\sum_{a=1}^{p-1}\sum_{i=1}^{p-3}\frac{\mathcal{B}_i}{a^i}\frac{q_a^2}{a^{2n}} = w_p\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}} + \sum_{a=1}^{p-1}\frac{q_a^3}{a^{2n}} \bmod p$$

A joint application of Theorem 2, Corollary 2 applied with $t+1 = p-2-2n$ implies in turn the following result.

Corollary 3. *Fix some integer n. Then*,

$$\sum_{a=1}^{p-1}\sum_{i=2}^{p-3}\frac{\mathcal{B}_i}{a^i}\frac{q_a^2}{a^{2n}} = w_p\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}} + \sum_{a=1}^{p-1}\frac{q_a^3}{a^{2n}} + \frac{1}{2}\mathcal{B}_{p-3-2n} \bmod p$$

When $2n=0$, this sum plays a central role in [4] in relation to finding an expansion for $(p-1)!$ to the modulus p^4, a generalization of Wilson, Glaisher and Sun's results to the respective moduli p, p^2 and p^3.

Some parts of the next two statements serve as a preparation to part of Theorem 5 below.

Theorem 3.

$$\begin{aligned}\sum_{a=1}^{p-1} q_a^2 &= -w_p^2 - \mathcal{CB}(p-1) \bmod p\\ &= -\left(\left(pB_{p-1}\right)_1 - 1\right)^2 - \left(2p\mathcal{B}_{2(p-1)} - p^2\mathcal{B}_{p-1}^2\right)_2 \bmod p\\ &= \frac{pB_{2(p-1)} - 2pB_{p-1} + p - 1}{p^2} \bmod p\end{aligned}$$

Remark 4. *By an unpublished result of Sun* [42] *which got later generalized in a published version* [43], *we have for any non-negative integer k*,

$$pB_{k(p-1)} = -(k-1)(p-1) + kpB_{p-1} \bmod p^2$$

The last congruence of Theorem 3 *offers a proof for the base case* $k=2$. *Sun's congruence can then be shown by induction on* $k \ge 2$ *using the fact that* p^k *divides* $\sum_{a=1}^{p-1}\left(a^{p-1}-1\right)^k$. *This namely implies that*:

$$pB_{k(p-1)} = p\sum_{l=1}^{k-1}\binom{k}{l}B_{(p-1)(k-l)}(-1)^{l-1} + (-1)^{k-1}(p-1) \bmod p^2$$

Theorem 4.

$$\begin{aligned}\sum_{a=1}^{p-1} q_a^2 a^2 &= \frac{1}{6}w_p - \frac{1}{4} - \mathcal{TCB}(4, p-3) \bmod p\\ &= -2\left(\mathcal{B}_{p+1} - \mathcal{B}_2\right)_1 - \frac{1}{2} \bmod p\\ &= \frac{6pB_{2p} - 12pB_{p+1} + p(p-1)(2p-1)}{6p^2} \bmod p\end{aligned}$$

We now state the long announced generalization of the Friedmann-Tamarkin-Lehmer-Ernvall-Metsänkyla congruences.

Theorem 5. *Let p be a prime with $p \geq 7$. Set an integer n such that $2 \leq 2n \leq p-5$. Then, we have:*

$$\sum_{a=1}^{p-1} \frac{q_a^3}{a^{2n}} = -2\sum_{i=2}^{p-5-2n} \mathcal{B}_i \left(\mathcal{B}_{2(p-1)-2n-i} - \mathcal{B}_{p-1-2n-i}\right)_1 + 2w_p \left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_1$$
$$-\left(1+2\left(\mathcal{B}_{p+1}-\mathcal{B}_2\right)_1\right)\mathcal{B}_{p-3-2n} - \left(w_p^2 + \mathcal{CB}(p-1)\right)\mathcal{B}_{p-1-2n}$$
$$-2\sum_{i=p+1-2n}^{p-3} \mathcal{B}_i \left(\mathcal{B}_{3(p-1)-2n-i} - \mathcal{B}_{2(p-1)-2n-i}\right)_1 \bmod p$$
$$= \frac{pB_{4(p-1)-2n} - 3pB_{3(p-1)-2n} + 3pB_{2(p-1)-2n} - pB_{p-1-2n}}{p^3} - \mathcal{B}_{p-3-2n} \bmod p$$

and

$$w_p = \left(pB_{p-1}\right)_1 - 1 \bmod p$$

$$\mathcal{CB}(p-1) = \left(2pB_{2(p-1)} - p^2\mathcal{B}_{p-1}^2\right)_2 \bmod p$$

(*Case* $2n=0$)

$$\sum_{a=1}^{p-1} q_a^3 = -\left(1+2\left(\mathcal{B}_{p+1}-\mathcal{B}_2\right)_1\right)\mathcal{B}_{p-3} + w_p\left(w_p^2 + \mathcal{CB}(p-1)\right)$$
$$-2\sum_{k=2}^{p-5} \mathcal{B}_k \left(\mathcal{B}_{2(p-1)-k} - \mathcal{B}_{p-1-k}\right)_1 \bmod p$$
$$= \frac{pB_{3(p-1)} - 3pB_{2(p-1)} + 3pB_{p-1} - p + 1}{p^3} - \mathcal{B}_{p-3} \bmod p$$

In each case, the second congruence arises from a different perspective.

In the general case, we deduce a very nice congruence which is a generalization of the congruence of Theorem 1 point (i) of [3].

Corollary 4. *Let p be a prime with $p \geq 11$. Let n be an integer such that $4 \leq 2n \leq p-7$. Then,*

$$\mathcal{TCBD}(p+1-2n, p-3)$$
$$= -\mathcal{CBD}(p-1-2n) + w_p\left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_1 - \frac{1}{2}\left(w_p^2 + \mathcal{CB}(p-1)\right)\mathcal{B}_{p-1-2n}$$
$$-\frac{pB_{4(p-1)-2n} - 3pB_{3(p-1)-2n} + 3pB_{2(p-1)-2n} - pB_{p-1-2n}}{2p^3} \bmod p$$

In the case when $2n=0$, we also have a nice corollary.

Corollary 5. *The following congruence holds.*

$$-\frac{1}{2}\sum_{a=1}^{p-1}\sum_{k=2}^{p-5} \frac{\mathcal{B}_k}{a^k} q_a^2 = -\left(\mathcal{B}_{p+1}-\mathcal{B}_2\right)_1 \mathcal{B}_{p-3} + \frac{w_p}{2}\left(w_p^2 + \mathcal{CB}(p-1)\right)$$
$$-\frac{pB_{3(p-1)} - 3pB_{2(p-1)} + 3pB_{p-1} - p + 1}{2p^3} \bmod p$$

with the residues of w_p and $\mathcal{CB}(p-1)$ already provided above.

Up to a factor p^3, this sum was called $\mathcal{S}$ and had served as an intermediate in [4] for the calculation of the residue of the same sum with the divided Bernoulli numbers being replaced with the ordinary ones. The sum $\mathcal{S}$ was actually

never computed as this intermediate gently vanished during the computation.

Corollary 3 and Theorem 5 play a key role in the proof for Theorem 6 below. The result also relies heavily on our past work on the multiple harmonic sums $\mathcal{H}_{\{s\}^{2l}=1;p-1}$ modulo p^4 of [4] and on Gessel's identity of [18].

Theorem 6. *Let p be an odd prime with $p \geq 11$ and n be an integer with $4 \leq 2n \leq p-7$. Then,*

$$\begin{aligned}
\mathcal{TCBD} &= -\mathcal{CBD} + \mathcal{CBD} + \frac{1}{2}\left(\mathcal{CB}(p-1-2n) + \mathcal{TCB}(p+1-2n, p-3)\right)_1 \\
&\quad + 2n\left(\mathcal{B}_{3(p-1)-2n} - 2\mathcal{B}_{2(p-1)-2n} + \mathcal{B}_{p-1-2n}\right)_2 - \left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_2 \\
&\quad - \frac{1}{2}\left(2\left(\left(pB_{p-1}\right)_1 - 1\right)\mathcal{B}_{p-1-2n} + 2\left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_1\right)_1 \\
&\quad - w_p\left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_1 \\
&\quad + \left\{\left(2n + \frac{1}{2}\right)\mathcal{CB}(p-1) + w_p\left(2n - 1 + \frac{2n+1}{2}w_p\right)\right. \\
&\quad \left. + 2n\left(pB_{p-1} - p + \frac{1}{2}p^2\mathcal{B}_{p-1}^2 + p\mathcal{B}_{p-1} - p\mathcal{B}_{2(p-1)}\right)_2\right\}\mathcal{B}_{p-1-2n} \\
&\quad + \frac{2n+1}{2}\frac{pB_{4(p-1)-2n} - 3pB_{3(p-1)-2n} + 3pB_{2(p-1)-2n} - pB_{p-1-2n}}{p^3} \bmod p
\end{aligned}$$

Remark 5. *In Theorem 6 just stated, in order to have the truncated convolutions well defined, it necessary to impose $2n \geq 4$. As for the upper bound, $2n \leq p-7$, it is necessary to have in order to be able to apply Theorem 1 point (i) of [3].*

Applying Theorem 6 in the special case when $2n = 4$ allows to relate the p-residue of a convolution of order $(p-5)$ of ordinary Bernoulli numbers with Ernvall-Metsänkyla residues to the second residue in the p-adic expansion of a convolution of order $(p-5)$ of divided Bernoulli numbers. Also, applying Theorem 6 in the other extremal case corresponding this time to $2n = p-7$ allows to relate the p-residue of a truncated convolution of ordinary Bernoulli numbers with Ernvall-Metsänkyla residues to the second residue in the p-adic expansion of the same truncated convolution of divided Bernoulli numbers. That is, under our notations, we relate $\left(\mathcal{CBD}(p-5)\right)_0$ and $\left(\mathcal{CB}(p-5)\right)_1$ on one hand and $\left(\mathcal{TCBD}(8, p-3)\right)_0$ and $\left(\mathcal{TCB}(8, p-3)\right)_1$ on the other hand. The results are gathered in the following corollary.

Corollary 6.

(i)
$$\begin{aligned}
&\sum_{i=2}^{p-7} B_i\left(\mathcal{B}_{p-1+p-5-i} - \mathcal{B}_{p-5-i}\right)_1 \\
&= \frac{1}{2}\left(\mathcal{CB}(p-5) + \mathcal{B}_{p-3}^2\right)_1 + 2\left(\mathcal{B}_{2p-4} - \mathcal{B}_{p-3}\right)_1\mathcal{B}_{p-3} + \left\{4\mathcal{CB}(p-1)\right. \\
&\quad \left. + w_p\left(3 + 2w_p\right) + 4\left(pB_{p-1} - p + \frac{1}{2}p^2\mathcal{B}_{p-1}^2 + p\mathcal{B}_{p-1} - p\mathcal{B}_{2(p-1)}\right)_2\right\}\mathcal{B}_{p-5} \\
&\quad + 4\left(\mathcal{B}_{3p-7} - 2\mathcal{B}_{2p-6} + \mathcal{B}_{p-5}\right)_2 - \left(\mathcal{B}_{2p-6} - \mathcal{B}_{p-5}\right)_2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(2\left(\left(pB_{p-1}\right)_1-1\right)\mathcal{B}_{p-5}+2\left(\mathcal{B}_{2p-6}-\mathcal{B}_{p-5}\right)_1\right)_1\\
&+2\frac{pB_{4p-8}-3pB_{3p-7}+3pB_{2p-6}-pB_{p-5}}{p^3}\bmod p\\
&\sum_{i=8}^{p-3}B_i\left(\mathcal{B}_{p-1+p+5-i}-\mathcal{B}_{p+5-i}\right)_1\\
&=\frac{1}{2}\left(\mathcal{CB}(6)+\mathcal{TCB}(8,p-3)\right)_1-\left(\mathcal{B}_{p-1+4}-\mathcal{B}_4\right)_1\mathcal{B}_2-3\left(\mathcal{B}_{p-1+2}-\mathcal{B}_2\right)_1\mathcal{B}_4\\
&+\left\{-\frac{13}{2}\mathcal{CB}(p-1)-w_p\left(8+3w_p\right)-7\left(pB_{p-1}-p+\frac{1}{2}p^2\mathcal{B}_{p-1}^2+p\mathcal{B}_{p-1}\right.\right.\\
&\left.\left.-p\mathcal{B}_{2(p-1)}\right)_2\right\}\mathcal{B}_6-7\left(\mathcal{B}_{2p+4}-2\mathcal{B}_{p+5}+\mathcal{B}_6\right)_2-\left(\mathcal{B}_{p+5}-\mathcal{B}_6\right)_2\\
&-\frac{1}{2}\left(2\left(\left(pB_{p-1}\right)_1-1\right)\mathcal{B}_6+2\left(\mathcal{B}_{p+5}-\mathcal{B}_6\right)_1\right)_1-w_p\left(\mathcal{B}_{p+5}-\mathcal{B}_6\right)_1\\
&-3\frac{pB_{3p+3}-3pB_{2p+4}+3pB_{p+5}-pB_6}{p^3}\bmod p
\end{aligned}
\tag{ii}
$$

Our last theorem deals with congruences concerning sums of powers of integers weighted by the Teichmüller characters and by some analogs of the Teichmüller characters. These characters (resp their analogs as defined below) arise from an application of Hensel's lemma to the polynomial $X^{p-1}-1\in\mathbb{Z}_p[X]$ (resp $X^{p-1}+(p-1)!\in\mathbb{Z}_p[X]$, $X^{p-1}+pB_{p-1}\in\mathbb{Z}_p[X]$). Given a polynomial $f(X)$ with p-adic integer coefficients, Hensel's lemma asserts that if there exists a p-adic integer x such that $f(x)\in p\mathbb{Z}_p$ and $f'(x)\notin p\mathbb{Z}_p$ (in other words, $f'(x)$ is a unit in the ring of p-adic integers), then there exists a unique p-adic integer root x_0 of f such that $x_0\equiv x \bmod p\mathbb{Z}_p$. We say that x lifts to a unique root x_0 of f. The $(p-1)$ elements of $\mathbb{F}_p^\times$ lift to $(p-1)$ distinct roots of the respective polynomials above. In the case of the first polynomial, the Teichmüller character of $a\in\mathbb{F}_p^\times$ is the unique $(p-1)$-th root of unity in $\mathbb{Z}_p$ which is congruent to a modulo p.

Theorem 7. *Let t be an integer with $4\le t\le p-2$.*

(i) *We denote by the ω_a's the $(p-1)$ p-adic integer roots of the polynomial $X^{p-1}-1\in\mathbb{Z}_p[X]$. The ω_a's are the Teichmüller characters.*

Then, we have:

$$
\sum_{a=1}^{p-1}a^{t-1}\omega_a=\begin{cases}p(t-1)\mathcal{B}_{p-1+t}\bmod p^3 & \text{if } t \text{ is even}\\ p^2\left(\frac{t}{2}-1\right)B_{t-1}\bmod p^3 & \text{if } t \text{ is odd}\end{cases}
$$

(ii) *We denote by the Ω_a's the $(p-1)$ p-adic integer roots of the polynomial $X^{p-1}+(p-1)!\in\mathbb{Z}_p[X]$.*

Then, we have:

$$
\sum_{a=1}^{p-1}a^{t-1}\Omega_a=\begin{cases}p(t-1)\left(\mathcal{B}_{p-1+t}+pw_p\mathcal{B}_t\right)\bmod p^3 & \text{if } t \text{ is even}\\ p^2\left(\frac{t}{2}-1\right)B_{t-1}\bmod p^3 & \text{if } t \text{ is odd}\end{cases}
$$

(iii) *We denote by the* γ_a*'s the* $(p-1)$ *p-adic integer roots of the polynomial* $X^{p-1}+pB_{p-1}\in\mathbb{Z}_p[X]$.

Then, we have:

$$\sum_{a=1}^{p-1}a^{t-1}\gamma_a=\begin{cases}p(t-1)\left(\mathcal{B}_{p-1+t}+p\left(pB_{p-1}\right)_1\mathcal{B}_t\right)\bmod p^3 & \text{if } t \text{ is even}\\ p^2\left(\frac{t}{2}-1\right)B_{t-1}\bmod p^3 & \text{if } t \text{ is odd}\end{cases}$$

2. Proofs of the Theorems

2.1. Where We Apply Gessel's Identity

This is the most technical part of the discussion. We build upon the work of [4] which provides expressions for the Stirling numbers $\begin{bmatrix}p\\2n+1\end{bmatrix}$ (also denoted by A_{p-1-2n} for the sum of products of $p-1-2n$ distinct integers chosen amongst the first $p-1$ integers) and the multiple harmonic sums $\mathcal{H}_{\{s\}^{2n}=1;p-1}$ (also denoted by $A_{2n}^{\star}$ for the sum of products of $2n$ distinct reciprocals of integers chosen amongst the first $p-1$ integers) modulo p^4. We will go straight into the technical details. On one hand, the conjunction of Theorem 1 and Theorem 2 point (i) of [4] allow to write:

$$\begin{aligned}A_{2n}^{\star}=&\frac{p^3}{6}\mathcal{BBB}(p-1-2n)+p\left(1+pw_p\left(1+pw_p\right)\right)\mathcal{B}_{p-1-2n}\\&+\frac{4(2n+1)^2+6(2n+1)+5}{24}p^3\mathcal{B}_{p-3-2n}\\&-\frac{p^2}{2}\left(1+pw_p\right)\mathcal{CB}(p-1-2n)\bmod p^4\end{aligned}\tag{2}$$

On the other hand, we may list the main contributors from Theorem 3 of [4] as follows.

$$\begin{aligned}A_{2n}^{\star}=&\frac{p^3}{6}\mathcal{BBB}(p-1-2n)-\frac{2n-1}{12n}p^3\left(\mathcal{BBB}\left(2(p-1)-2n\right)-\mathcal{BBB}(p-1-2n)\right)\\&+\frac{p^3}{4n}\sum_{a=1}^{p-1}\left(\sum_{i=p+1-2n}^{p-5}\frac{(2n+1)\mathcal{B}_i+B_i}{a^i}+\sum_{i=2}^{p-7-2n}\frac{\mathcal{B}_i+B_i}{a^i}\right)\frac{q_a^2}{a^{2n}}+OT\bmod p^4\end{aligned}\tag{3}$$

where the other terms OT must be copied from the theorem itself.

We will study modulo p^4 the differential term:

$$p^3\Delta:=p^3(\mathcal{BBB}\left(2(p-1)-2n\right)-\mathcal{BBB}(p-1-2n)$$

By using the Gessel identity, we may partly reduce this study to that of another simpler differential term, this time composed of multinomial cubic convolutions, namely:

$$p^3m\Delta:=p^3\left(m\mathcal{BBB}\left(2(p-1)-2n\right)-m\mathcal{BBB}(p-1-2n)\right)$$

Before we start the computation, it is worth noting that for integers n,i,j,k such that $i+j+k=n$, we have:

$$\binom{n}{i,j,k}=\frac{n!}{i!\,j!(n-i-j)!}=\frac{n!(n-i)(n-i-1)\cdots(n-i-j+1)}{i!(n-i)!\,j!}=\binom{n}{i}\binom{n-i}{j}$$

We will split $m\mathcal{BBB}(2(p-1)-2n)$ into three sums, say S_1, S_2 and S_3, namely:

$$S_1=\binom{2(p-1)-2n}{p-1}\mathcal{B}_{p-1}\sum_{j=2}^{p-3-2n}\binom{p-1-2n}{j}\mathcal{B}_j\mathcal{B}_{p-1-2n-j}$$

$$S_2=\sum_{i=2}^{p-3}\binom{2(p-1)-2n}{i}\mathcal{B}_i\sum_{j=2}^{2(p-1)-2n-i-2}\binom{2(p-1)-2n-i}{j}\mathcal{B}_j\mathcal{B}_{2(p-1)-2n-i-j}$$

$$S_3=\sum_{i=p+1}^{2(p-1)-2n-4}\binom{2(p-1)-2n}{i}\mathcal{B}_i\sum_{j=2}^{2(p-1)-2n-i-2}\binom{2(p-1)-2n-i}{j}\mathcal{B}_j\mathcal{B}_{2(p-1)-2n-i-j}$$

$$=\sum_{s=4}^{p-3-2n}\binom{2(p-1)-2n}{s}\mathcal{B}_{2(p-1)-2n-s}\sum_{j=2}^{s-2}\binom{s}{j}\mathcal{B}_j\mathcal{B}_{s-j}$$

By Von Staudt-Clausen's theorem, the sum S_3 may be treated modulo p. When $s\le p-3-2n$, Kummer's congruence applies and yields $\mathcal{B}_{2(p-1)-2n-s}=\mathcal{B}_{p-1-2n-s} \bmod p$. Moreover, by a straightforward application of the Chu-Vandermonde identity and from other well known binomial congruences and identities (see e.g. [3]), we easily derive:

$$\binom{2(p-1)-2n}{s}=\binom{p-1-2n}{s}+\frac{s}{2n+1}\binom{p-1-2n}{s}\bmod p$$

And since we also have,

$$m\mathcal{BBB}(p-1-2n)$$

$$=\sum_{i=2}^{p-5-2n}\binom{p-1-2n}{i}\mathcal{B}_i\sum_{j=2}^{p-1-2n-i-2}\binom{p-1-2n-i}{j}\mathcal{B}_j\mathcal{B}_{p-1-2n-i-j}$$

$$=\sum_{s=4}^{p-3-2n}\binom{p-1-2n}{s}\mathcal{B}_{p-1-2n-s}\sum_{j=2}^{s-2}\binom{s}{j}\mathcal{B}_j\mathcal{B}_{s-j},$$

we therefore obtain:

$$S_3-m\mathcal{BBB}(p-1-2n)$$

$$=\frac{1}{2n+1}\sum_{s=4}^{p-3-2n}s\binom{p-1-2n}{s}\mathcal{B}_{p-1-2n-s}\sum_{j=2}^{s-2}\binom{s}{j}\mathcal{B}_j\mathcal{B}_{s-j}$$

After a change of indices, the right hand side above rewrites as:

$$\frac{1}{2n+1}\sum_{l=2}^{p-5-2n}(p-1-2n-l)\binom{p-1-2n}{l}\mathcal{B}_l\sum_{j=2}^{p-1-2n-l-2}\binom{p-1-2n-l}{j}\mathcal{B}_j\mathcal{B}_{p-1-2n-l-j},$$

which in turn is congruent modulo p to:

$$-m\mathcal{BBB}(p-1-2n)$$

$$-\frac{1}{2n+1}\sum_{l=2}^{p-1-2n-4}\binom{p-1-2n}{l}B_l\sum_{j=2}^{p-1-2n-l-2}\binom{p-1-2n-l}{j}\mathcal{B}_j\mathcal{B}_{p-1-2n-l-j}$$

The sum to the right hand side is a multinomial cubic convolution of divided Bernoulli numbers where one of the divided Bernoulli number has been replaced

with a regular Bernoulli number. By considering thrice such a sum, we see that this convolution is nothing else than:

$$\frac{p-1-2n}{3}m\mathcal{B}\mathcal{B}\mathcal{B}\left(p-1-2n\right)$$

And so,

$$S_3 - m\mathcal{B}\mathcal{B}\mathcal{B}\left(p-1-2n\right) = -\frac{2}{3}m\mathcal{B}\mathcal{B}\mathcal{B}\left(p-1-2n\right) \bmod p \tag{4}$$

Next, in order to tackle S_2 we will split it. Indeed, when $i \geq p-1-2n$, we have $p-1 < i+2n+2$, then $2\left(p-1\right)-2n-i-2 < p-1$. Therefore the corresponding sum may be treated modulo p. We thus write:

$$S_2 = \sum_{i=2}^{p-3-2n}\binom{2\left(p-1\right)-2n}{i}\mathcal{B}_i \sum_{j=2}^{2(p-1)-2n-i-2}\binom{2\left(p-1\right)-2n-i}{j}\mathcal{B}_j\mathcal{B}_{2(p-1)-2n-i-j}$$
$$+\sum_{i=p-1-2n}^{p-3}\binom{2\left(p-1\right)-2n}{i}\mathcal{B}_i \sum_{j=2}^{2(p-1)-2n-i-2}\binom{2\left(p-1\right)-2n-i}{j}\mathcal{B}_j\mathcal{B}_{2(p-1)-2n-i-j}$$

Moreover, the sum of the second row is congruent to zero modulo p as p divides the binomial coefficient $\binom{2\left(p-1\right)-2n}{i}$ for this whole range of i. Further, dealing now with the first row, when $i = p-3-2n$, we have $2\left(p-1\right)-2n-i-2 = p-1$ and when $i < p-3-2n$, we have $2\left(p-1\right)-2n-i-2 > p-1$. In the first case, by Staudt's theorem, only the extreme indices contribute to the right hand sum as p divides $\binom{p+1}{j}$ for this whole range of j. In the second case, when $p+1-2n-i \leq j \leq p-3$, for this whole range of j, p divides $\binom{2\left(p-1\right)-2n-i}{j}$. Then, after the adequate reductions modulo p we are left with three main contributions

$$S_2 = S_{2,1} + S_{2,2} + S_{2,3}$$

with:

$$S_{2,1} = \left(p+1\right)p\mathcal{B}_{p-1}\mathcal{B}_2\binom{2\left(p-1\right)-2n}{p-3-2n}\mathcal{B}_{p-3-2n}$$

$$S_{2,2} = 2\sum_{i=2}^{p-5-2n}\binom{2\left(p-1\right)-2n}{i}\mathcal{B}_i\binom{2\left(p-1\right)-2n-i}{p-1}\mathcal{B}_{p-1}\mathcal{B}_{p-1-2n-i}$$

$$S_{2,3} = 2\sum_{i=2}^{p-5-2n}\binom{2\left(p-1\right)-2n}{i}\mathcal{B}_i \sum_{j=2}^{p-3-2n-i}\binom{2\left(p-1\right)-2n-i}{j}\mathcal{B}_j\mathcal{B}_{2(p-1)-2n-i-j}$$

Regarding $S_{2,1}$, we have

$$\binom{2\left(p-1\right)-2n}{p-3-2n} = \frac{2}{p-1-2n}\binom{p-1-2n}{p-3-2n} \bmod p = -\left(2n+2\right) \bmod p$$

Therefore,

$$S_{2,1} = -\frac{1}{6}(n+1)\mathcal{B}_{p-3-2n} \bmod p \tag{5}$$

Dealing with $S_{2,2}$ relies on proving the following lemma.

Lemma 1. *Let i be an integer with $0 \le i \le p-3-2n$.*

$$\binom{2(p-1)-2n-i}{p-1}\mathcal{B}_{p-1} \in \mathbb{Z}_p$$

and

$$\binom{2(p-1)-2n-i}{p-1}\mathcal{B}_{p-1} = -\frac{1}{2n+1+i} \bmod p\mathbb{Z}_p$$

Proof of Lemma 1. For this range of i, we have $2(p-1)-2n-i > p$. Therefore,

$$\binom{2(p-1)-2n-i}{p-1} = p\frac{(p+p-2-2n-i)(p+p-3-2n-i)\cdots(p+1)}{(p-1-2n-i)!}$$

Moreover, $p\mathcal{B}_{p-1} = 1 \bmod p\mathbb{Z}_p$, hence the lemma.

Since, for this same range of non-zero even i's, we also have:

$$\binom{2(p-1)-2n}{i} = \binom{p-1-2n}{i}\frac{2n+1+i}{2n+1} \bmod p,$$

we thus get:

$$S_{2,2} = \frac{n+1}{6}\mathcal{B}_{p-3-2n} - \frac{2}{2n+1}b\mathcal{C}\mathcal{B}(p-1-2n) \bmod p \tag{6}$$

Concerning $S_{2,3}$, for the ranges of even i's and j's that are considered, we have:

$$\binom{2(p-1)-2n-i}{j} = \binom{p-1-2n-i}{j}\frac{2n+1+i+j}{2n+1+i} \bmod p$$

It follows that:

$$S_{2,3} = \sum_{i=2}^{p-5-2n}\binom{p-1-2n}{i}\mathcal{B}_i \sum_{j=2}^{p-3-2n-i}\frac{2n+1+i+j}{2n+1}\binom{p-1-2n-i}{j}\mathcal{B}_j\mathcal{B}_{p-1-2n-i-j} \bmod p$$

Therefore,

$$S_{2,3} = \frac{2}{3}m\mathcal{B}\mathcal{B}\mathcal{B}(p-1-2n) \bmod p \tag{7}$$

It remains to treat S_1. Applying Lemma 1 with $i=0$ immediately yields:

$$S_1 = -\frac{1}{2n+1}b\mathcal{C}\mathcal{B}(p-1-2n) \bmod p \tag{8}$$

By gathering congruences (3)-(7), we obtain, where we also used Miki's identity and the fact that $\mathcal{H}_{p-1-2n} = \mathcal{H}_{2n} \bmod p$ (as $\mathcal{H}_{p-1} = 0 \bmod p$ by Wolstenholme's theorem):

$$p^3 m\Delta = p^3\left(-\frac{3}{2n+1}\mathcal{C}\mathcal{B}(p-1-2n) + \frac{6}{2n+1}\mathcal{H}_{2n}\mathcal{B}_{p-1-2n}\right) \bmod p^4 \tag{9}$$

We have done only part of the work so far. It remains to study the other differential terms arising in Gessel's identity. By Gessel's identity applied twice and after some simplifications and use of the Kummer congruence, we have:

$$\begin{aligned} p^3\Delta = p^3\Big\{ & m\Delta - \frac{2n+3}{2}\mathcal{B}_{p-3-2n} + 3\mathcal{H}_{p-1-2n}\left(b\mathcal{CB}\left(2\left(p-1\right)-2n\right) - b\mathcal{CB}\left(p-1-2n\right)\right) \\ & + 3\left(\frac{1}{p-2n} + \frac{1}{p+1-2n} + \cdots + \frac{1}{p+p-2-2n}\right) b\mathcal{CB}\left(2\left(p-1\right)-2n\right) \\ & + 6A^{\star}_{2,2(p-1)-2n}\mathcal{B}_{2(p-1)-2n} - 6A^{\star}_{2,p-1-2n}\mathcal{B}_{p-1-2n}\Big\} \bmod p^4 \end{aligned} \tag{10}$$

We need further computations and start with the expression on the third row which we will conveniently denote by R_3. We decompose:

$$A^{\star}_{2,2(p-1)-2n} = A^{\star}_{2,p-1-2n} + \sum_{i=1}^{p-1-2n}\sum_{j=p-2n}^{2(p-1)-2n}\frac{1}{ij} + \sum_{p-2n\le i<j\le 2(p-1)-2n}\frac{1}{ij}$$

$A^{\star}_{2,2(p-1)-2n}$ is the sum of three terms, we will denote the second term by s_2 and the third one by s_3. In s_2, the range of j can be split into $p-2n \le j \le p-1$, $j=p$ and $p+1 \le j \le p+\left(p-2-2n\right)$. When working modulo $p\mathbb{Z}_p$, we thus have:

$$s_2 = -\sum_{i=1}^{p-1-2n}\sum_{j=1}^{2n}\frac{1}{ij} + \frac{1}{p}\mathcal{H}_{p-1-2n} + \sum_{i=1}^{p-1-2n}\sum_{j=1}^{p-2-2n}\frac{1}{ij} \bmod p$$

Moreover, we have (note, these types of congruences get worked out later on):

$$\mathcal{H}_{p-1-2n} = p\mathcal{H}_{2n,2} + \mathcal{H}_{2n} \bmod p^2$$

Then, we get:

$$s_2 = \mathcal{H}_{2n,2} + \mathcal{H}_{2n}\left(\frac{1}{p} + \frac{1}{2n+1}\right) \bmod p$$

We now deal with s_3. Regarding s_3, we split the ranges of i and j into three groups, namely

$$\begin{array}{ccc} p-2n & & p+1 \\ p-\left(2n-1\right) & p & p+2 \\ \vdots & & \vdots \\ p-1 & & p+\left(p-2-2n\right) \end{array}$$

We thus obtain:

$$s_3 = \frac{1}{p}\left(-p\mathcal{H}_{2n,2} - \mathcal{H}_{2n} - p\mathcal{H}_{p-2-2n,2} + \mathcal{H}_{p-2-2n}\right) - \mathcal{H}_{2n}\mathcal{H}_{2n+1} + A^{\star}_{2,2n} + A^{\star}_{2,p-2-2n} \bmod p$$

with

$$\mathcal{H}_{p-2-2n} = p\mathcal{H}_{2n+1,2} + \mathcal{H}_{2n+1} \bmod p^2$$
$$\mathcal{H}_{p-2-2n,2} = -\mathcal{H}_{2n+1,2} \bmod p,$$

s_3 then simplifies to

$$s_3 = \frac{1}{p}\left(\frac{1}{2n+1} + \frac{p}{\left(2n+1\right)^2} + p\mathcal{H}_{2n+1,2}\right) - \mathcal{H}_{2n}\mathcal{H}_{2n+1} + A^{\star}_{2,2n} + A^{\star}_{2,p-2-2n} \bmod p$$

Moreover, we have:

$$A^{\star}_{2,p-2-2n} = A^{\star}_{2,p-1-2n} + \frac{\mathcal{H}_{2n+1}}{2n+1} \bmod p$$

We now need the following intermediate result which gets listed as a lemma.

Lemma 2.

$$A^{\star}_{2,p-1-2n} = -A^{\star}_{2,2n} + \mathcal{H}^2_{2n} \bmod p$$

Proof of Lemma 2. We have:

$$\begin{aligned} A^{\star}_{2,p-1-2n} &= A^{\star}_{2} - \sum_{p-2n\le i<j\le p-1} \frac{1}{ij} - \sum_{i=1}^{p-1-2n} \sum_{j=p-2n}^{p-1} \frac{1}{ij} \\ &= -A^{\star}_{2,2n} + \sum_{i=1}^{p-1-2n} \sum_{j=1}^{2n} \frac{1}{ij} \bmod p \\ &= -A^{\star}_{2,2n} + \mathcal{H}_{2n}\mathcal{H}_{p-1-2n} \bmod p \\ &= -A^{\star}_{2,2n} + \mathcal{H}^2_{2n} \bmod p \end{aligned}$$

□

Then,

$$A^{\star}_{2,p-2-2n} = -A^{\star}_{2,2n} + \mathcal{H}_{2n}\mathcal{H}_{2n+1} + \frac{1}{(2n+1)^2} \bmod p$$

Thus,

$$s_3 = \frac{1}{2n+1}\left(\frac{1}{p} + \frac{2}{2n+1}\right) + \mathcal{H}_{2n+1,2} \bmod p$$

In the end, we have:

Proposition 1.

$$A^{\star}_{2,2(p-1)-2n} = A^{\star}_{2,p-1-2n} + 2\mathcal{H}_{2n+1,2} + \mathcal{H}_{2n+1}\left(\frac{1}{p} + \frac{1}{2n+1}\right) \bmod p$$

It follows that

$$p^3 R_3 = 6\left(\frac{\mathcal{H}_{2n+1}}{2n+1} + 2\mathcal{H}_{2n+1,2}\right) p^3 \mathcal{B}_{p-1-2n} + 6\mathcal{H}_{2n+1} p^2 \mathcal{B}_{2(p-1)-2n} \bmod p^4 \quad (11)$$

We further deal with R_2. Let u denote the p-adic integer:

$$u := \frac{1}{p-2n} + \cdots + \frac{1}{p-1} + \frac{1}{p+1} + \cdots + \frac{1}{p+p-2-2n}$$

We have:

$$R_2 = \frac{3}{p} b\mathcal{CB}\left(2(p-1)-2n\right) + 3ub\mathcal{CB}\left(2(p-1)-2n\right)$$

We decompose:

$$\begin{aligned} b\mathcal{CB}\left(2(p-1)-2n\right) &= 2\binom{2(p-1)-2n}{p-1}\mathcal{B}_{p-1}\mathcal{B}_{2(p-1)-2n} \\ &\quad + 2\sum_{i=2}^{p-3-2n}\binom{2(p-1)-2n}{i}\mathcal{B}_i\mathcal{B}_{2(p-1)-2n-i} \\ &\quad + \sum_{i=p+1-2n}^{p-3}\binom{2(p-1)-2n}{i}\mathcal{B}_i\mathcal{B}_{2(p-1)-2n-i} \end{aligned}$$

Then,

$$p^3R_2 = 3p^2b\mathcal{CB}\big(2(p-1)-2n\big)+6\binom{2(p-1)-2n}{p-1}p\mathcal{B}_{p-1}\mathcal{B}_{p-1-2n}p^2u$$
$$+\frac{6}{2n+1}p^3u\sum_{i=2}^{p-3-2n}\binom{p-1-2n}{i}\mathcal{B}_i\mathcal{B}_{p-1-2n-i}(2n+1+i)$$
$$+3p^3u\sum_{i=p+1-2n}^{p-3}\binom{2(p-1)-2n}{i}\mathcal{B}_i\mathcal{B}_{2(p-1)-2n-i} \bmod p^4$$

when $p+1-2n \le i \le p-3$, the binomial coefficient in the last sum is divisible by p, hence the latter sum simply vanishes. Further, after scrutiny, $b\mathcal{CB}\big(2(p-1)-2n\big)$ is wanted modulo p^2 while u is simply desired modulo p. We have:

$$u = \mathcal{H}_{p-2-2n} - \mathcal{H}_{2n} \bmod p$$
$$= H_1 - \sum_{k=p-1-2n}^{p-1}\frac{1}{k} - \mathcal{H}_{2n} \bmod p$$
$$= \frac{1}{2n+1} \bmod p$$

We now rewrite:

$$p^3R_2 = 3p^2b\mathcal{CB}\big(2(p-1)-2n\big) - \frac{6}{(2n+1)^2}p^3\mathcal{B}_{p-1-2n}$$
$$+\frac{3}{2n+1}p^3b\mathcal{CB}(p-1-2n) \bmod p^4$$

We move on to studying the binomial convolution $b\mathcal{CB}\big(2(p-1)-2n\big) \bmod p^2\mathbb{Z}_p$. We have:

$$3p^2b\mathcal{CB}\big(2(p-1)-2n\big) = 3p^2\mathcal{CB}\big(2(p-1)-2n\big) - 6p^2\mathcal{H}_{2(p-1)-2n}\mathcal{B}_{2(p-1)-2n}$$
$$= 3p^2\mathcal{CB}\big(2(p-1)-2n\big) - 6p^2\mathcal{H}_{p-1-2n}\mathcal{B}_{2(p-1)-2n}$$
$$-6p\mathcal{B}_{2(p-1)-2n} - 6p^2u\mathcal{B}_{2(p-1)-2n}$$

The first equality above holds by Miki's identity. This time, we need to know u modulo p^2 instead of simply modulo p. This is routine calculation which we expand right below. We have:

$$\frac{1}{p-2n}+\frac{1}{p+1-2n}+\cdots+\frac{1}{p-1}$$
$$= -\frac{1}{(2n)^2}(p+2n) - \frac{1}{(2n-1)^2}(p+2n-1) - \cdots - \frac{1}{1^2}(p+1) \bmod p^2$$
$$= -p\mathcal{H}_{2n,2} - \mathcal{H}_{2n} \bmod p^2$$

$$\frac{1}{p+1}+\frac{1}{p+2}+\cdots+\frac{1}{p+p-2n-2}$$
$$= -\frac{1}{1^2}(p-1) - \frac{1}{2^2}(p-2) - \cdots - \frac{1}{(p-2n-2)^2}\big(p-(p-2n-2)\big) \bmod p^2$$
$$= -p\mathcal{H}_{p-2n-2,2} + \mathcal{H}_{p-2n-2} \bmod p^2$$

$$= -pH_2 + p\left(\frac{1}{(p-1-2n)^2} + \cdots + \frac{1}{(p-1)^2}\right)$$
$$+ H_1 - \left(\frac{1}{p-1-2n} + \frac{1}{p-2n} + \cdots + \frac{1}{p-1}\right) \bmod p^2$$
$$= 2p\mathcal{H}_{2n+1,2} + \mathcal{H}_{2n+1} \bmod p^2$$

Adding both contributions yields:

$$u = \frac{1}{2n+1} + p\left(\frac{1}{(2n+1)^2} + \mathcal{H}_{2n+1,2}\right) \bmod p^2\mathbb{Z}_p$$

We also have:

$$\mathcal{H}_{p-1-2n} = p\mathcal{H}_{2n,2} + \mathcal{H}_{2n} \bmod p^2\mathbb{Z}_p$$

Therefore,

$$3p^2 b\mathcal{CB}(2(p-1)-2n)$$
$$= 3p^2\mathcal{CB}(2(p-1)-2n) - 12p^3\mathcal{H}_{2n+1,2}\mathcal{B}_{p-1-2n} - 6p(1+p\mathcal{H}_{2n+1})\mathcal{B}_{2(p-1)-2n}$$
$$= 6p^2\sum_{i=2}^{p-3-2n}\mathcal{B}_i\mathcal{B}_{2(p-1)-2n-i} + 6p\mathcal{B}_{p-1}p\mathcal{B}_{p-1-2n} + 3p^2\mathcal{TCB}(p+1-2n, p-3)$$
$$-12p^3\mathcal{H}_{2n+1,2}\mathcal{B}_{p-1-2n} - 6p(1+p\mathcal{H}_{2n+1})\mathcal{B}_{2(p-1)-2n} \bmod p^4$$

In order to apply Ernvall and Metsänkyla's congruence in the first sum above, we must impose $i \le p-5-2n$. Hence we split this sum. After grouping the different terms, we obtain:

$$p^3R_2 = \frac{3}{2n+1}p^3\mathcal{CB}(p-1-2n) + 3p^2\mathcal{CB}(p-1-2n)$$
$$+ 3p^2\left(\mathcal{CB}(p-1-2n) + \mathcal{TCB}(p+1-2n, p-3)\right) - 3p^3\sum_{i=2}^{p-5-2n}\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}}\frac{\mathcal{B}_i}{a^i}$$
$$+ 6p\left[p\mathcal{B}_{p-1} - p^2\left(2\mathcal{H}_{2n+1,2} + \frac{\mathcal{H}_{2n+1}}{2n+1}\right)\right]\mathcal{B}_{p-1-2n} + 6p^3\left(\mathcal{B}_{p+1} - \mathcal{B}_2\right)_1\mathcal{B}_{p-3-2n}$$
$$- 6p(1+p\mathcal{H}_{2n+1})\mathcal{B}_{2(p-1)-2n} \bmod p^4$$

Moreover, by Theorem 1 point (i) of [3],

$$\mathcal{TCB}(p+1-2n, p-3) + \mathcal{CB}(p-1-2n)$$
$$= -\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}} + 2\left(\left(pB_{p-1}\right)_1 - 1\right)\mathcal{B}_{p-1-2n} \bmod p$$

Then,

$$p^3R_2 = \frac{3}{2n+1}p^3\mathcal{CB}(p-1-2n) + 3p^2\mathcal{CB}(p-1-2n)$$
$$+ 3p^3\left(\mathcal{CB}(p-1-2n) + \mathcal{TCB}(p+1-2n, p-3)\right)_1$$
$$+ 3p^2\left(2\left(\left(pB_{p-1}\right)_1 - 1\right)\mathcal{B}_{p-1-2n} - \sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}}\right)_0 - 3p^3\sum_{i=2}^{p-5-2n}\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}}\frac{\mathcal{B}_i}{a^i}$$
$$+ 6p\left[p\mathcal{B}_{p-1} - p^2\left(2\mathcal{H}_{2n+1,2} + \frac{\mathcal{H}_{2n+1}}{2n+1}\right)\right]\mathcal{B}_{p-1-2n}$$

$$+6p^3\left(\mathcal{B}_{p+1}-\mathcal{B}_2\right)_1\mathcal{B}_{p-3-2n}-6p\left(1+p\mathcal{H}_{2n+1}\right)\mathcal{B}_{2(p-1)-2n} \bmod p^4 \tag{12}$$

It remains to tackle the third term of the first row of (10). We have:

$$p^3R_1=6\mathcal{H}_{p-1-2n}p\mathcal{B}_{p-1}p^2\mathcal{B}_{p-1-2n}\binom{2(p-1)-2n}{p-1}$$
$$+3\mathcal{H}_{p-1-2n}p^3\sum_{i=2}^{p-3-2n}\left(2\binom{2(p-1)-2n}{i}-\binom{p-1-2n}{i}\right)\mathcal{B}_i\mathcal{B}_{p-1-2n-i} \bmod p^4$$

As seen several times before, for the range of i that is considered,

$$\binom{2(p-1)-2n}{i}=\binom{p-1-2n}{i}\frac{2n+1+i}{2n+1} \bmod p$$

Then,

$$2\binom{2(p-1)-2n}{i}=\binom{p-1-2n}{i}\left(2+\frac{2i}{2n+1}\right) \bmod p$$

It immediately follows that the difference above vanishes. Further, by Lemma 1, we know that:

$$\binom{2(p-1)-2n}{p-1}\mathcal{B}_{p-1}=-\frac{1}{2n+1} \bmod p$$

Then, we have:

$$p^3R_1=-6p^3\frac{\mathcal{H}_{2n}}{2n+1}\mathcal{B}_{p-1-2n} \bmod p^4 \tag{13}$$

By using Result 12 of [3] due to Sun [5] with $k=2$ and $b=p-1-2n$, as well as Result 10 point (i) of [3], the congruence modulo $p^3\mathbb{Z}_p$ arising from (2) and (3) simply reads:

$$p^3\Delta=3p^2\mathcal{CB}(p-1-2n) \bmod p^3\mathbb{Z}_p$$

It is then routine verification that the congruence modulo $p^3\mathbb{Z}_p$ is just trivial. We thus focus our attention on the modulus $p^4\mathbb{Z}_p$. But before moving any further, this is an adequate time to prove the first few theorems of the introduction, as well as their related corollaries, culminating in the proof of Theorem 6.

2.2. Proofs of Theorems 1 - 5

Because the sum of importance in the previous Section 2.1 is the one contained in the left hand side of the congruence of Theorem 2, we start with the proof of Theorem 2.

The proof is inspired from [17] and uses the Stirling numbers. The unsigned Stirling number of the first kind $\begin{bmatrix} p \\ s \end{bmatrix}$ is the unsigned coefficient of x^s in the falling factorial

$$x(x-1)(x-2)\cdots(x-(p-1))=\sum_{s=1}^{p}(-1)^{s-1}\begin{bmatrix} p \\ s \end{bmatrix}x^s$$

By specializing $x=-a$ in the equality above, we obtain:

$$-a(-a-1)(-a-2)\cdots(-a-(p-1))=\sum_{s=1}^{p}(-1)^{s-1}\begin{bmatrix}p\\s\end{bmatrix}(-a)^s$$

Then,

$$\frac{(p-1+a)!}{(a-1)!}=\sum_{s=1}^{p}\begin{bmatrix}p\\s\end{bmatrix}a^s$$

Moreover,

$$(p-1+a)!=(p-1)!p\cdots(p-1+a)=-p(a-1)!\bmod p^2$$

It follows that:

$$-p=\sum_{s=1}^{p}\begin{bmatrix}p\\s\end{bmatrix}a^s \bmod p^2$$

By using Corollary 2 of [15] originally due to Glaisher [2], we get:

$$-p=(pB_{p-1}-p)a+\sum_{s=3}^{p-2}\frac{p}{s}B_{p-s}a^s-\frac{p}{2}a^{p-1}+a^p \bmod p^2$$

It follows that:

Proposition 2.

$$\sum_{i=1}^{p-3}\frac{\mathcal{B}_i}{a^i}=w_p+q_a \bmod p$$

From there, Theorem 2 follows. A rewriting of the theorem is the following.

$$-\frac{1}{2}\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n+1}}+\sum_{a=1}^{p-1}\sum_{i=2}^{p-3}\frac{\mathcal{B}_i}{a^i}\frac{q_a^2}{a^{2n}}=w_p\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}}+\sum_{a=1}^{p-1}\frac{q_a^3}{a^{2n}}\bmod p$$

We now adapt Ernvall and Metsänkyla's proof for the results of Section 1.3 when t is odd. First and foremost, we recall from [5] that

$$S_t=\sum_{a=1}^{p-1}a^t=pB_t+\frac{p^2}{2}tB_{t-1}+\frac{p^3}{6}t(t-1)B_{t-2} \bmod p^3 \tag{14}$$

Suppose t is odd and $t\geq 3$. Then $B_t=0$. If we impose $t\neq 3$, then

$$S_t=\frac{p^2}{2}tB_{t-1} \bmod p^3$$

Again, we have, using the same notations as in Section 1:

$$(a+pv_a)^t=a^t+tpv_aa^{t-1}+p^2v_a^2a^{t-2}\frac{t(t-1)}{2}\bmod p^3$$

Like before, when t is odd and $3\leq t\leq p-2$, we have

$$\sum_{a=1}^{p-1}(a+pv_a)^t=0$$

When excluding $t=3$, reducing modulo p^3 yields:

$$\frac{p^2}{2}tB_{t-1}=-tp\sum_{a=1}^{p-1}v_aa^{t-1}-p^2\frac{t(t-1)}{2}\sum_{a=1}^{p-1}v_a^2a^{t-2} \bmod p^3 \tag{15}$$

We thus have:

$$\frac{p}{2}B_{t-1} = -\sum_{a=1}^{p-1} v_a a^{t-1} - p\frac{t-1}{2}\sum_{a=1}^{p-1} v_a^2 a^{t-2} \bmod p^2$$

And since we also have

$$\sum_{a=1}^{p-1}\left(a + pv_a\right)^{p-1+t} = 0,$$

similar developments and reductions lead to:

$$\frac{p}{2}B_{p-1+t-1} = -\sum_{a=1}^{p-1} v_a a^{p-1+t-1} - p\frac{t-2}{2}\sum_{a=1}^{p-1} v_a^2 a^{t-2} \bmod p^2$$

Then, using also $v_a = aq_a \bmod p$, we obtain Theorem 1.

It follows from Theorem 2 and Corollary 2 with $t+1 = p-2-2n$ that

$$\sum_{a=1}^{p-1}\sum_{i=2}^{p-3}\frac{\mathcal{B}_i}{a^i}\frac{q_a^2}{a^{2n}} = w_p\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}} + \sum_{a=1}^{p-1}\frac{q_a^3}{a^{2n}} + \frac{1}{2}\mathcal{B}_{p-3-2n} \bmod p \tag{16}$$

This result can also be derived from a weaker version of Lemma 1 of [6]. Namely, as part of Miki's congruence stated in his Lemma 1 of [6], we have:

$$aq_a = -1 + \left(1 - \left(pB_{p-1}\right)_1\right)a - \sum_{k=1}^{p-3}\frac{1}{p}\binom{p}{k}B_k a^{p-k} \bmod p$$

It comes:

$$\sum_{a=1}^{p-1} aq_a\frac{q_a^2}{a^{2n+1}} = -\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n+1}} + \left(1 - \left(pB_{p-1}\right)_1\right)\sum_{a=1}^{p-1}\frac{q_a^2}{a^{2n}} - \sum_{k=1}^{p-3}\frac{1}{p}\binom{p}{k}B_k\sum_{a=1}^{p-1} q_a^2 a^{p-1-2n-k} \bmod p$$

From there, by using again Corollary 2 with $t+1 = p-2-2n$, we retrieve (16).

After several adequate applications of the Ernvall-Metsänkyla congruence in (16), we further get:

$$\begin{aligned}\sum_{a=1}^{p-1}\frac{q_a^3}{a^{2n}} = &-2\sum_{i=2}^{p-5-2n}\mathcal{B}_i\left(\mathcal{B}_{2(p-1)-2n-i} - \mathcal{B}_{p-1-2n-i}\right)_1 + \left(\sum_{a=1}^{p-1} q_a^2 a^2 - \frac{1}{2}\right)\mathcal{B}_{p-3-2n}\\ &+\left(\sum_{a=1}^{p-1} q_a^2\right)\mathcal{B}_{p-1-2n} + 2w_p\left(\mathcal{B}_{2(p-1)-2n} - \mathcal{B}_{p-1-2n}\right)_1\\ &-2\sum_{i=p+1-2n}^{p-3}\mathcal{B}_i\left(\mathcal{B}_{3(p-1)-2n-i} - \mathcal{B}_{2(p-1)-2n-i}\right)_1 \bmod p\end{aligned} \tag{17}$$

In order to conclude to Theorem 5, we first need to prove Theorems 3 and 4 of Section 1.3. These theorems are built out of two perspectives. The first perspective relies on Miki's result providing the residue of aq_a for a integer with $1 \le a \le p-1$. The second perspective relies on Sun's congruence for the sums of powers of integers issued from [5]. We first discuss the first perspective. It treats the first two congruences in each theorem.

By a reformulation of Proposition 0.2 of Section 1.3 due to Miki, we have:

$$aq_a = -\frac{1}{2} + \left(1 - \left(pB_{p-1}\right)_1\right)a + \sum_{k=2}^{p-3}\mathcal{B}_k a^{p-k} \bmod p$$

Then,

$$\sum_{a=1}^{p-1} q_a^2 = \sum_{a=1}^{p-1} aq_a \frac{q_a}{a} = -\frac{1}{2}\sum_{a=1}^{p-1} q_a a^{p-2} + \left(1-\left(pB_{p-1}\right)_1\right) w_p + \sum_{k=2}^{p-3}\sum_{a=1}^{p-1} \frac{\mathcal{B}_k}{a^k} q_a \bmod p$$

Moreover, Lehmer's congruence for odd powers implies that

$$\sum_{a=1}^{p-1} q_a a^{p-2} = -pB_{p-3} \bmod p^2$$

Then, the residue of the first sum is simply zero.

Further, by the original Friedmann-Tamarkine congruence, we have:

$$\sum_{a=1}^{p-1} q_a a^{p-1-k} = -\mathcal{B}_{p-1-k} \bmod p$$

Then, up to sign, the last sum is the convolution $\mathcal{CB}(p-1)$. Hence the first congruence of Theorem 3. The second congruence follows from [3] which provides the residue of $\mathcal{CB}(p-1)$ (see Result 2.11 of [4]).

As for Theorem 4, we have:

$$\begin{aligned}\sum_{a=1}^{p-1} q_a^2 a^2 &= \sum_{a=1}^{p-1} (aq_a)(aq_a) \\ &= -\frac{1}{2}\sum_{a=1}^{p-1} q_a a + \left(1-\left(pB_{p-1}\right)_1\right)\sum_{a=1}^{p-1} q_a a^2 + \sum_{k=2}^{p-3}\sum_{a=1}^{p-1} \frac{\mathcal{B}_k}{a^k} q_a a^2 \bmod p\end{aligned}$$

when $4 \le k \le p-3$, the Friedmann-Tamarkine congruence applies and yields:

$$\sum_{a=1}^{p-1} q_a a^{p+1-k} = -\mathcal{B}_{p+1-k} \bmod p$$

Moreover, the Friedmann-Tamarkine congruence also applies to the first two terms. We thus get the first congruence of Theorem 4. The second congruence will follow immediately from forthcoming Lemma 3 of Section 2.3.

The proof for the last congruence in each theorem relies on Congruence (14) by Z-H. Sun in [5] and is left to the reader.

From there, Theorem 5 follows.

2.3. Proof of Theorem 6

This part closes the proof for Theorem 6. The rest of the computations will be based on the following series of lemmas.

Lemma 3.

$$\mathcal{TCB}(4, p-3) = 2\left(\mathcal{B}_{p+1} - \mathcal{B}_2\right)_1 + \frac{1}{12} + \frac{1}{6}\left(pB_{p-1}\right)_1 \bmod p$$

Proof. Immediately follows from Theorem 1 point (ii) of [3].

Lemma 4.

$$\mathcal{TCB}(6, p-3) = \frac{7}{720} + 2\left(\mathcal{B}_{p+3} - \mathcal{B}_4\right)_1 - \frac{1}{60}\left(pB_{p-1}\right)_1 \bmod p$$

Proof. Again, this is an immediate consequence of Theorem 1 point (iii) of [3].

Lemma 5

$$\sum_{a=1}^{p-1}\sum_{i=2}^{p-5-2n} \frac{B_i}{a^i}\frac{q_a^2}{a^{2n}} = -2\sum_{i=2}^{p-5-2n} B_i\left(\mathcal{B}_{2(p-1)-2n-i} - \mathcal{B}_{p-1-2n-i}\right)_1 \bmod p$$

$$\sum_{a=1}^{p-1}\sum_{i=p+1-2n}^{p-3}\frac{B_i}{a^i}\frac{q_a^2}{a^{2n}}=-2\sum_{i=p+1-2n}^{p-3}B_i\left(\mathcal{B}_{3(p-1)-2n-i}-\mathcal{B}_{2(p-1)-2n-i}\right)_1 \bmod p$$

Proof. For the first range of i, we have $4\le p-1-2n-i\le p-3$. For the second range of i, we have $4\le 2(p-1)-2n-i\le p-3$. In both cases the Ernvall-Metsänkyla congruence applies and yields the result.

Lemma 6

$$\sum_{a=1}^{p-1}\sum_{i=2}^{p-5-2n}\frac{\mathcal{B}_i}{a^i}\frac{q_a^2}{a^{2n}}=-2\sum_{i=2}^{p-5-2n}\mathcal{B}_i\left(\mathcal{B}_{2(p-1)-2n-i}-\mathcal{B}_{p-1-2n-i}\right)_1 \bmod p$$

Proof. This range for i satisfies to the conditions of application of the Ernvall Metsänkyla's congruence.

More computational efforts using Lemma 3 - 6 lead to Theorem 6 which is thus entirely proven. Corollary 6 points (i) and (ii) are then respectively obtained by specializing $2n=4$ and $2n=p-7$ in the congruence of Theorem 6. In the first case, the truncated convolution to the left hand side reduces to only one term. Moreover, from Corollary 4 applied with $2n=4$, the convolution $\mathcal{CBD}(p-5)$ gets fully determined. We then obtain the residue for $\mathcal{CBD}(p-5)$ in terms of $\left(\mathcal{CB}(p-5)+\mathcal{B}_{p-3}^2\right)_1$.

Remark 6. *Theorem* 0.3 *of* [3] *establishes a congruence for* $\left(\mathcal{CB}(p-5)+\mathcal{B}_{p-3}^2\right)_0$. *Finding the next residue in the p-adic expansion remains an open question. Likewise, Theorem* 1 *point* (i) *of* [3] *applied with* $2n=p-7$ *provides the residue for* $\left(\mathcal{TCB}(8,p-3)+\mathcal{CB}(6)\right)_0$ *while the next residue in the p-adic expansion remains unknown.*

The next part deals again with the techniques exposed in the introduction.

2.4. Proof of Theorem 7

In the case when t is even, point (i) of Theorem 7 follows from a joint application of Proposition 0.1 and Theorem 0.1 of Section 1.3, both due to Ernvall and Metsänkyla.

In the case when t is odd, the sum of powers S_t verifies

$$S_t=\frac{p^2}{2}tB_{t-1} \bmod p^3 \tag{18}$$

And we have seen in Section 2.2 (cf. Congruence (15)) that

$$p\sum_{a=1}^{p-1}a^{t-1}v_a=-\frac{p^2}{2}B_{t-1}-\frac{p^2}{2}(t-1)\sum_{a=1}^{p-1}a^t q_a^2 \bmod p^3$$

By summing both contributions and using our Corollary 2, we obtain the result of (i) which is thus entirely proven.

Regarding point (ii), we will study the polynomial

$$X^{p-1}+(p-1)!\in\mathbb{Z}_p[X]$$

in the same way as Ernvall and Metsänkyla had used the polynomial

$$X^{p-1}-1\in\mathbb{Z}_p[X]$$

in order to prove their congruences. The first polynomial had played a key role

in [15] in order to recover Glaisher's result that $(p-1)! = pB_{p-1} - p \bmod p^2$ and in order to find a formula for $(p-1)!$ modulo p^3 by a different method than the one of Sun [5], see Corollary 6 in Section 1 of [15]. The analogs of the Teichmüller characters are the $(p-1)$ p-adic integer roots of $X^{p-1}+(p-1)!$ which we denote by Ω_a, $1 \le a \le p-1$. Moreover, we define the w_a's such that:

$$\Omega_a = a + pw_a$$

It is shown in [15] that

$$pw_a = a\left(1+(p-1)!+pq_a\right) \bmod p^2$$

This is the purpose of Lemma 1 in Section 2 of [15]. Thus, we have:

$$w_a = a\left(w_p + q_a\right) \bmod p$$

By the same argument as in Section 1, we have:

$$\sum_{a=1}^{p-1}\left(a+pw_a\right)^t = 0$$

We assume $t \neq 2$. Expanding modulo p^3 leads to an analog of the Ernvall and Metsänkyla's congruence, namely

$$\mathcal{B}_t = -\sum_{a=1}^{p-1} w_a a^{t-1} - p\frac{t-1}{2}\sum_{a=1}^{p-1} w_a^2 a^{t-2} \bmod p^2,$$

where v_a was replaced with w_a. By substituting:

$$w_a^2 = a^2 w_p^2 + q_a^2 a^2 + 2a^2 q_a w_p \bmod p,$$

we then obtain for even t:

$$\mathcal{B}_t + \sum_{a=1}^{p-1} w_a a^{t-1} = -p\frac{t-1}{2}\sum_{a=1}^{p-1} q_a^2 a^t - p(t-1)w_p \sum_{a=1}^{p-1} q_a a^t \bmod p^2$$

By the Lehmer congruence applied with even powers modulo p and by the Ernvall-Metsänkyla's congruence, we further obtain:

$$\sum_{a=1}^{p-1} w_a a^{t-1} = (t-1)\mathcal{B}_{p-1+t} - t\mathcal{B}_t + p(t-1)w_p\mathcal{B}_t \bmod p^2 \tag{19}$$

The case t is even of Theorem 7 point (ii) follows. When t is odd, the congruence reads instead:

$$\sum_{a=1}^{p-1} w_a a^{t-1} = -\frac{p}{2}B_{t-1} - p\frac{t-1}{2}\sum_{a=1}^{p-1} q_a^2 a^t - p(t-1)w_p\sum_{a=1}^{p-1} q_a a^t \bmod p^2$$

The first sum to the right hand side of the congruence is known modulo p by our Corollary 2. As for the second sum it is known modulo p^2 by Lehmer's result applied with odd powers. The latter sum thus vanishes from the congruence since $t \neq 1 \bmod (p-1)$. We get:

$$\sum_{a=1}^{p-1} w_a a^{t-1} = -pB_{t-1} \bmod p^2 \tag{20}$$

We deduce point (ii) of Theorem 7 in the case when t is odd.

We now present a different proof for Congruences (19) and (20), one which does not refer to Ernvall and Metsänkyla's original proof of [7]. We use an expansion of w_a one p-power further. This is achieved in [15]. Our $(w_a)_1$ is the $t_a^{(1)}$ of [15]. Also, $\delta_i(a)$ of [15] is $(q_a)_i$. Using the current notations, Lemma 3 of [15] asserts that:

$$(w_a)_1 = a\left((q_a)_0 + (q_a)_1 + \left(\sum_{i=1}^{p-1}(q_i)_0\right)^2 + \left(1+(q_a)_0\right)\sum_{i=1}^{p-1}(q_i)_0\right) \bmod p$$

Recall also from Theorem 1 of [15] that

$$\sum_{a=1}^{p-1} q_a = w_p \bmod p$$

Then, using again

$$\mathcal{B}_t = -\sum_{a=1}^{p-1} a^t q_a \bmod p,$$

we have:

$$\sum_{a=1}^{p-1} w_a a^{t-1} = p w_p (t-1)\mathcal{B}_t - p\mathcal{B}_t + \sum_{a=1}^{p-1} a^t q_a \bmod p^2$$

But by the Lehmer congruence applied with t even,

$$\sum_{a=1}^{p-1} a^t q_a = p\mathcal{B}_t + (t-1)\mathcal{B}_{p-1+t} - t\mathcal{B}_t \bmod p^2$$

Combining both congruences yields (19).

When t is odd, we rather apply the Lehmer congruence with odd powers and it leads to (20).

It remains to deal with (iii). We define in turn the p-adic integers z_a's such that for each integer a with $1 \le a \le p-1$,

$$\gamma_a = a + pz_a$$

By Hensel's lifting algorithm, the residue z_a gets determined by

$$(a+pz_a)^{p-1} + pB_{p-1} \in p^2\mathbb{Z}_p$$

We thus have:

$$z_a = a\left(q_a + \left(pB_{p-1}\right)_1\right) \bmod p$$

Then the sketch of the proof with respect to the Ernvall-Metsänkyla type of proof is identical as in (ii) with w_p being replaced with $\left(pB_{p-1}\right)_1$. It yields the result of (iii) straight away. Again, another proof consists of pushing the expansion for z_a one p-power further as in Proposition 3 below whose proof relies on Hensel's lifting algorithm.

Proposition 3.

$$z_a = a\left(q_a + \left(pB_{p-1}\right)_1\right) + ap\left(1+\left(pB_{p-1}\right)_1\right)\left(q_a + \left(pB_{p-1}\right)_1\right) \bmod p^2$$

Our next and last part is concerned with proving Theorem 0.

2.5. Convolutions of Divided Bernoulli Numbers and Products of Those with Harmonic Numbers

The work of [4] had allowed to find an expression for $A_{p-1}=(p-1)!$ modulo p^4. The formula expressed in terms of Bernoulli numbers was satisfactory in that it involved only a constant number of terms, independently from the prime p. This expression had also allowed to write the residue of a cubic convolution of order $(p-1)$ like follows.

$$\mathcal{BBB}(p-1)=\Big\{-6A_{p-1}+3p^3\left(\mathcal{CB}(p-1)\right)_1+3p^2\left(\mathcal{CB}(p-1)\right)_0 -6p\mathcal{B}_{p-1}-\frac{15}{4}p^3\mathcal{B}_{p-3}\Big\}_3 \bmod p$$

Contrary to the generic case for $\mathcal{BBB}(p-1-2n)$ studied in Section 2.1 and Section 2.3, Gessel's identity does not provide any information on $\mathcal{BBB}(p-1)$. However interestingly, combined with Miki's identity it fully determines a convolution of divided Bernoulli numbers with a product of divided Bernoulli numbers and harmonic numbers. Indeed, setting $n=p-1$ in the Gessel identity (Theorem 0.4 of Section 1.3) yields (after using Wolstenholme's theorem imposing $\mathcal{H}_{p-1}=0 \bmod p^2$):

$$\mathcal{BBB}(p-1)=\sum_{i=2}^{p-5}\mathcal{B}_i b\mathcal{CB}(p-1-j)+6\mathcal{B}_{p-1}A^{\star}_{2,p-1}-\frac{9}{4}\mathcal{B}_{p-3} \bmod p \qquad (21)$$

When applying Miki's identity, the terms $\mathcal{BBB}(p-1)$ cancel each other, leaving only one convolution. Moreover, a result by Jianqiang Zhao from [41] claims that

$$A^{\star}_{2,p-1}=\frac{\mathcal{H}_{p-1}}{p} \bmod p^3$$

In [3], combining our work and Zhao's result, we showed that

$$A^{\star}_{2,p-1}=-p\left(\mathcal{B}_{2p-4}-2\mathcal{B}_{p-3}\right) \bmod p^3$$

Plugging back into (21) with $p\mathcal{B}_{p-1}=1 \bmod p$ yields Theorem 0 of Section 2.1. It is interesting to note that by an application of Wolstenholme's theorem we have $\mathcal{H}_{p-1-i}=\mathcal{H}_i \bmod p$. Then, by considering twice the convolutions of Corollary 0, we get by a usual trick used many times in [4] the result of the corollary.

Acknowledgements

The author is indebted to the anonymous referee for his or her thoughtful comments which have allowed to better the manuscript.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Dunne, G.V. and Schubert, C. (2004) Bernoulli Number Identities from Quantum

Field Theory. IHES Preprint P/04/31.

[2] Glaisher, J.W.L. (1900) On the Residues of the Sums of Products of the First p-1 Numbers and Their Powers, to Modulus p^2 or p^3. *The Quarterly Journal of Mathematics*, **31**, 321-353.

[3] Levaillant, C. (2020) Congruences Related to Miki's Identity.

[4] Levaillant, C. (2022) Multiple Harmonic Sums Modulo p^4 and Applications. *Journal of Combinatorics and Number Theory*, **12**, 79-114.

[5] Sun, Z.-H. (2000) Congruences Concerning Bernoulli Numbers and Bernoulli Polynomials. *Discrete Applied Mathematics*, **105**, 193-223. https://doi.org/10.1016/S0166-218X(00)00184-0

[6] Miki, H. (1978) A Relation between Bernoulli Numbers. *Journal of Number Theory*, **10**, 297-302. https://doi.org/10.1016/0022-314X(78)90026-4

[7] Ernvall, R. and Metsänkylä, T. (1991) Cyclotomic Invariants for Primes between 125000 and 150000. *Mathematics of Computation*, **56**, 851-858. https://doi.org/10.2307/2008413

[8] Friedmann, A. and Tamarkine, J. (1909) Quelques formules concernant la théorie de la fonction [x] et des nombres de Bernoulli. *Journal für die reine und angewandte Mathematik*, **135**, 146-156. https://doi.org/10.1515/crll.1909.135.146

[9] Lehmer, E. (1938) On Congruences Involving Bernoulli Numbers and the Quotients of Fermat and Wilson. *Annals of Mathematics*, **39**, 350-360. https://doi.org/10.2307/1968791

[10] Faulhaber, J. (1631) Academia algebrae, darinnen die miraculosische Inventiones zu den höchsten Cossen weiters continuirt und prolifiert warden.

[11] Jacobi, C.G.J. (1834) De usu legitimo formulae summatoriae Maclaurinianae. *Journal für die reine und angewandte Mathematik*, **12**, 263-272. https://doi.org/10.1515/crll.1834.12.263

[12] Knuth, D.E. (1993) Johann Faulhaber and Sums of Powers. *Mathematics of Computation*, **61**, 277-294. https://doi.org/10.1090/S0025-5718-1993-1197512-7

[13] Voronoi, G.F. (1890) On Bernoulli Numbers. *Communications of the Kharkiv Mathematical Society*, **2**, 129-148. (In Russian)

[14] Kummer, E.E. (1850) Allgemeiner Beweis des Fermatschen Satzes, daß die Gleichung $X^\lambda + Y^\lambda = Z^\lambda$ durch ganze Zahlen unlösbar ist, für alle diejenigen Potenz-Exponenten λ, welche ungerade Primzahlen sind und in den Zählern der ersten $\frac{\lambda-3}{2}$ Bernoullischen Zahlen als Faktoren nicht vorkommen. *Journal für die reine und angewandte Mathematik*, **40**, 131-138. https://doi.org/10.1515/crll.1850.40.130

[15] Levaillant, C. (2019) Wilson's Theorem Modulo p^2 Derived from Faulhaber Polynomials.

[16] Gouvea, F. (1993) p-Adic Numbers, an Introduction. 2nd Edition, Springer, Berlin. https://doi.org/10.1007/978-3-662-22278-2_1

[17] Levaillant, C. (2020) Powers of Two Weighted Sum of the First p Divided Bernoulli Numbers Modulo p.

[18] Gessel, I.M. (2005) On Miki's Identity for Bernoulli Numbers. *Journal of Number Theory*, **110**, 75-82. https://doi.org/10.1016/j.jnt.2003.08.010

[19] Johnson, W. (1975) p-Adic Proofs of Congruences for the Bernoulli Numbers. *Journal of Number Theory*, **7**, 251-265. https://doi.org/10.1016/0022-314X(75)90020-7

[20] Clausen, T. (1840) Theorem. *Astronomische Nachrichten*, **17**, 351-352. https://doi.org/10.1002/asna.18400172204

[21] Von Staudt, C. (1840) Beweis eines Lehrsatzes, die Bernoullischen Zahlen betreffend. *Journal für die reine und angewandte Mathematik*, **21**, 372-374. https://doi.org/10.1515/crll.1840.21.372

[22] Agoh, T. (1995) On Giuga's Conjecture. *Manuscripta Mathematica*, **87**, 501-510. https://doi.org/10.1007/BF02570490

[23] Giuga, G. (1950) Su una presumibile proprieta caratterista dei numeri primi. *Istituto Lombardo Accademia di Scienze e Lettere. Rendiconti. Classe di Scienze Matematiche e Naturali. Serie A*, **83**, 511-528.

[24] Wolstenholme, J. (1862) On Certain Properties of Prime Numbers. *The Quarterly Journal of Pure and Applied Mathematics*, **5**, 35-39.

[25] Buhler, J., Crandall, R., Ernvall, R. and Metsänkyla, T. (1993) Irregular Primes and Cyclotomic Invariants to Four Million. *Mathematics of Computation*, **61**, 151-153. https://doi.org/10.1090/S0025-5718-1993-1197511-5

[26] Selfridge, J.L. and Pollack, B.W. (1964) Fermat's Last Theorem Is True for Any Exponent up to 25,000. *Notices of the AMS*, **11**, 97.

[27] Wagstaff, S.S. (1978) The Irregular Primes to 125000. *Mathematics of Computation*, **32**, 583-591. https://doi.org/10.1090/S0025-5718-1978-0491465-4

[28] Johnson, W. (1974) Irregular Prime Divisors of the Bernoulli Numbers. *Mathematics of Computation*, **28**, 653-657. https://doi.org/10.1090/S0025-5718-1974-0347727-0

[29] Ireland, K. and Rosen, M. (1982) A Classical Introduction to Modern Number Theory. Springer, New York, 239-248. https://doi.org/10.1007/978-1-4757-1779-2

[30] Shiratani, K. and Yokoyama, S. (1982) An Application of *p*-Adic Convolutions. *Memoirs of the Faculty of Science*, *Kyushu University. Series A*, *Mathematics*, **36**, 73-83. https://doi.org/10.2206/kyushumfs.36.73

[31] Faber, C. and Pandharipande, R. (2000) Hodge Integrals and Gromov-Witten Theory. *Inventiones Mathematicae*, **139**, 137-199. https://doi.org/10.1007/s002229900028

[32] Artamkin, I.V. (2007) An Elementary Proof of the Miki-Zagier-Gessel Identity. *Russian Mathematical Surveys*, **62**, 1194-1196. https://doi.org/10.1070/RM2007v062n06ABEH004482

[33] Crabb, M.C. (2005) The Miki-Gessel Bernoulli Number Identity. *Glasgow Mathematical Journal*, **47**, 327-328. https://doi.org/10.1017/S0017089505002545

[34] Sitaramachandrarao, R. and Davis, B. (1986) Some Identities Involving the Riemann Zeta Function II. *Indian Journal of Pure and Applied Mathematics*, **17**, 1175-1186.

[35] Sankaranaryanan, A. (1987) An Identity Involving Riemann Zeta Function. *Indian Journal of Pure and Applied Mathematics*, **18**, 794-800.

[36] Zhang, W.P. (1991) On the Several Identities of Riemann Zeta Function. *Chinese Science Bulletin*, **22**, 1852-1856.

[37] Petojević, A. and Srivastava, H.M. (2009) Computation of Euler's Type Sums of the Products of Bernoulli Numbers. *Applied Mathematics Letters*, **22**, 796-801. https://doi.org/10.1016/j.aml.2008.06.040

[38] Dilcher, K. (1996) Sums of Products of Bernoulli Numbers. *Journal of Number Theory*, **60**, 23-41. https://doi.org/10.1006/jnth.1996.0110

[39] Gorsky, A. and Zhiboedov, A. (20106) Aspects of the $N = 4$ SYM Amplitude—Wilson Polygon Duality. *Nuclear Physics B*, **835**, 343-363. https://doi.org/10.1016/j.nuclphysb.2010.04.003

[40] Agoh, T. (2016) On Miki's Identity for Bernoulli Numbers. *Integers*, **16**, 1-12.

[41] Zhao, J. (2007) Bernoulli Numbers, Wolstenholme's Theorem and p^5 Variations of Luca's Theorem. *Journal of Number Theory*, **123**, 18-26. https://doi.org/10.1016/j.jnt.2006.05.005

[42] Sun, Z.-H. A Note on Wilson's Theorem and Wolstenholme's Theorem.

[43] Sun, Z.-H. (1997) Congruences for Bernoulli Numbers and Bernoulli Polynomials. *Discrete Mathematics*, **163**, 153-163. https://doi.org/10.1016/S0012-365X(97)81050-3