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The Doubly Truncated Generalized Log-Lindley Distribution

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Abstract

The aim of this paper is to present generalized log-Lindely (GLL) distribution, as a new model, and find doubly truncated generalized log-Lindely (DTGLL) distribution, truncation in probability distributions may occur in many studies such as life testing, and reliability. We illustrate the applicability of GLL and DTGLL distributions by Real data application. The GLL distribution can handle the risk rate functions in the form of panich and increase. This property makes GLL useful in survival analysis. Various statistical and reliability measures are obtained for the model, including hazard rate function, moments, moment generating function, mean and variance, quantiles function, Skewness and kurtosis, mean deviations, mean inactivity time and strong mean inactivity time. The estimation of model parameters is justified by the maximum Likelihood method. An application to real data shows that DTGLL distribution can provide better suitability than GLL and some other known distributions.

Keywords

Log-Lindley Distribution, Quantile Function, Maximum Likelihood Estimation, Doubly Truncated

1. Introduction

The Lindley distribution was introduced by Lindley, D.V. [1]. The probability density function (pdf) of a Lindley random variable *X*, with scale parameter σ is given by

$$f(x)=\frac{\sigma^2}{1+\sigma}(1+x)\mathrm{e}^{-\sigma x};\quad x>0,\sigma>0 \tag{1}$$

And cumulative distribution function (cdf)

$$F(x)=1-\left[1+\frac{\sigma x}{1+\sigma}\right]e^{-\sigma x};\quad x>0,\sigma>0 \tag{2}$$

Ghitany, E., *et al.* [2] and Ghitany, M. E., *et al.* [3] studied various properties of Lindley distribution and the two-parameter weighted Lindley distribution with applications to survival data. Bakouch, H. S., *et al.* [4] introduced an extension of the Lindley distribution that offers more flexibility in the modeling of lifetime data. Ghitany, M. E., *et al.* [5] presented results on the two-parameter generalization referred to as the power Lindley distribution. Krishna, H. & Kumar, K. [6] studied reliability estimation of the Lindley distribution with progressive type II censored sample. Teamah, A. M., *et al.* [7] studied Random sum of truncated and randomly truncated Lindley distribution. Hamed D., and Alzaaghal A. [8] introduced new class of Lindley distributions because of having only one parameter, the Lindley distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modeling purposes, it will be useful to consider further generalizations of this distribution.

Zakerzadeh, H. & Dolati, A. [9] have obtained a generalized Lindley distribution and discussed its various properties and applications. Nadarajah, S. *et al.* [10] have recently proposed two parameter extensions of the Lindley distribution named as the generalized Lindley distributions. Arslan, T., *et al.* [11] have obtained generalized Lindley and Power distributions for modeling the wind speed data. Shanker, R. *et al.* [12] have obtained generalization of Two-Parameter Lindley Distribution with properties and applications.

The new distribution called Log-Lindley (LL) distribution has compact expressions for the moments as well as the cdf Gomez-Deniz, E. *et al.* [13] studied its important properties relevant to the insurance and inventory management applications.

The probability density function (pdf) and cumulative distribution function (cdf) of log-lindley distribution defined as

$$f(x)=\frac{\sigma^2}{1+\lambda\sigma}(\lambda-\log x)x^{\sigma-1},\quad 0<x<1,\lambda\ge 0,\sigma>0 \tag{3}$$

and

$$F(x)=\frac{1}{1+\lambda\sigma}x^{\sigma}\left[1+\sigma(\lambda-\log x)\right] \tag{4}$$

2. Generalized Log-Lindley

By taking the cdf of an exponential distribution as cdf of Log-Lindley distribution, we can obtain the following definition.

Definition: A random variable X is said to have a generalized log-lindley (GLL) distribution with three parameters $\underline{\theta}=(\lambda,\sigma,\alpha)$ if its pdf is given by the following form

$$f(x)=\frac{\alpha\sigma^2}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma-1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha-1}(\lambda-\log x),\quad 0<x<1,\lambda\ge 0,\sigma,\alpha>0 \tag{5}$$

the cdf corresponding to (5) is given by

$$F(x)=\frac{1}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma}\left[1+\sigma(\lambda-\log x)\right]^{\alpha} \tag{6}$$

as a result of (5) and (6), the survival function and the hazard rate function of the GLL distribution can be written as

$$S(x)=1-\frac{1}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma}\left[1+\sigma(\lambda-\log x)\right]^{\alpha} \tag{7}$$

and

$$h(x)=\frac{\alpha\sigma^{2}x^{\alpha\sigma-1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha-1}(\lambda-\log x)}{(1+\lambda\sigma)^{\alpha}-x^{\alpha\sigma}\left[1+\sigma(\lambda-\log x)\right]^{\alpha}} \tag{8}$$

the graphical representation of density function and the hazard rate function of the GLL distribution are in Figure 1 and Figure 2, respectively.

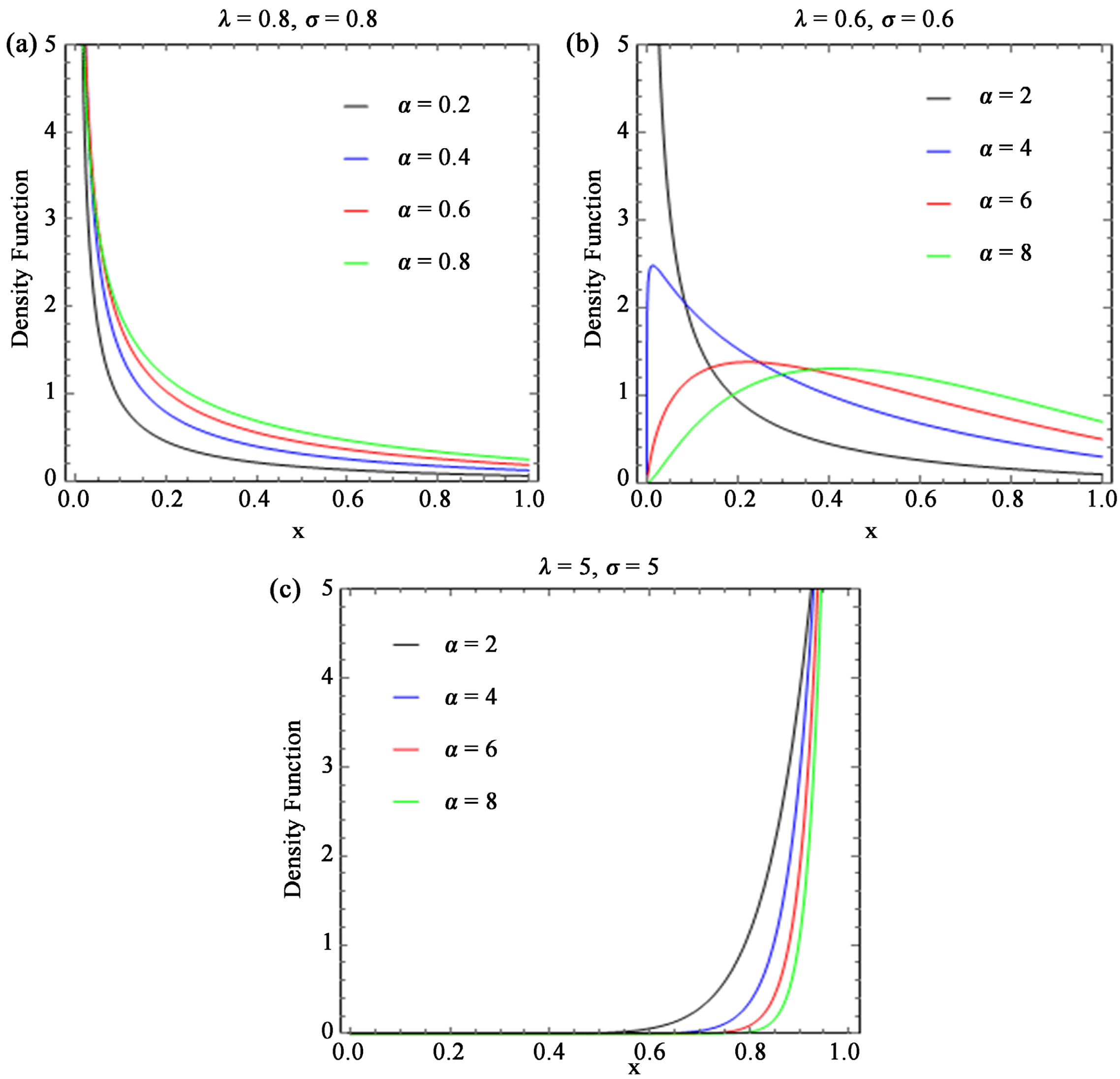

Figure 1. Plots of density function of the GLL distribution. Plots (*a*), (b) and (c) indicate how the parameters affect the GLL density and show the flexibility of density shapes. From these plots it is immediate that the pdf can be (*a*) decreasing, (b) increasing-decreasing and (c) increasing. Hence, the GLL distribution can be very useful in fitting different data sets with various shapes.

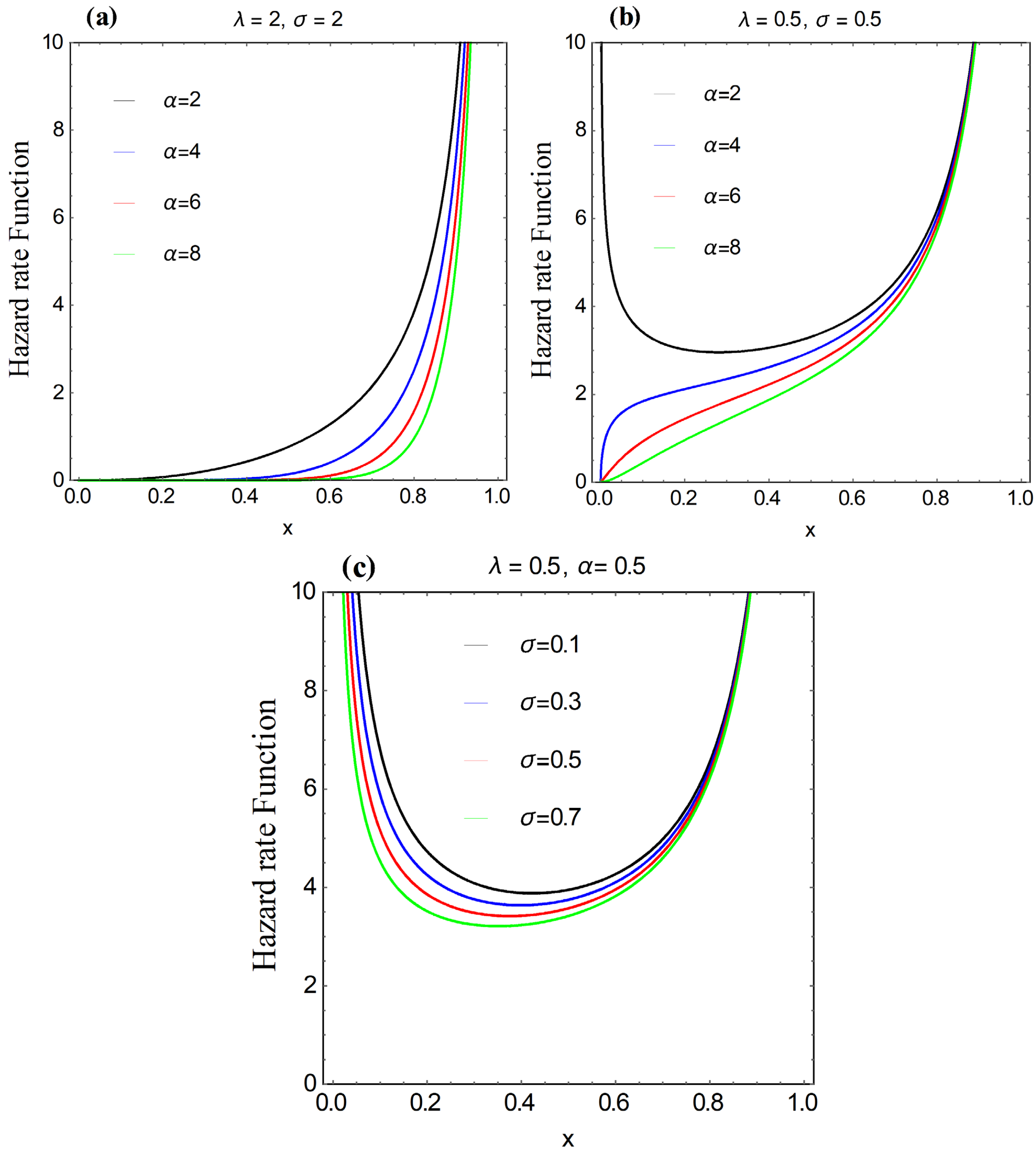

Figure 2. Plots of the GLL distribution displays increasing and bathtub hazard rate shapes.

The rest of the article is organized as follows. In Section 3, introduces the Statistical properties of GLL distribution: Moments and generating function, Mean and variance, Quantile function, Skewness and kurtosis based on quantiles are given. In Section 4, we find the Reliability measures of GLL: Mean inactivity and strong mean inactivity time functions. In Section 5, we introduce the method of likelihood estimation as point estimation and, give the equation used to estimate the parameters, using the maximum product spacing estimates and the least square estimates techniques. In Section 6, we find the p.d.f. of the doubly truncated GLL. Section 7, Finally, we fit the distribution to real data set to examine it.

3. Statistical Properties of GLL

In this section, we obtain some statistical properties of the new model, including the moments, moment generating function, quantile function, skewness, kurtosis, mean deviations.

3.1. Moments and Generating Function

Theorem 1. If X has the GLL $(x;\alpha,\sigma,\lambda)$ distribution with $\lambda \ge 0$, then the *rth* moment of X is given as follows

$$E\left(X^r\right)=\frac{\alpha}{(1+\lambda\sigma)^{\alpha}}\sum_{n}^{\infty}\frac{(\alpha\sigma+r)^n}{n!}\sum_{k=0}^{\infty}\binom{\alpha-1}{k}\sigma^{k+2}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i\lambda^{k-i+1}\frac{u^{n+i+1}}{n+i+1} \quad (9)$$

Proof. From (5), we define the *rth* moment as

$$E\left(X^r\right)=\int_0^{\infty}x^r\frac{\alpha\sigma^2}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma-1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha-1}(\lambda-\log x)\,\mathrm{d}x,$$

so that

$$E\left(X^r\right)=\frac{\alpha\sigma^2}{(1+\lambda\sigma)^{\alpha}}\int_0^{\infty}x^{\alpha\sigma+r-1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha-1}(\lambda-\log x)\,\mathrm{d}x$$

Let $x=\mathrm{e}^u$, the series expansion of $\mathrm{e}^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, $(1+x)^a=\sum_{k=0}^{\infty}\binom{a}{k}x^k$ and after some algebraic manipulation, then the above integral yields the *rth* moment given by (9).

In particular, using (9), the mean of the GLL distribution follows as

$$E(X)=\frac{\alpha}{(1+\lambda\sigma)^{\alpha}}\sum_{n}^{\infty}\frac{(\alpha\sigma+1)^n}{n!}\sum_{k=0}^{\infty}\binom{\alpha-1}{k}\sigma^{k+2}\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i\lambda^{k-i+1}\frac{u^{n+i+1}}{n+i+1}. \quad (10)$$

Theorem 2. If X has the GLL $(x;\alpha,\sigma,\lambda)$ distribution with $\lambda \ge 0$, then the moment generating function (m.g.f) of X is given as follows

$$M(t)=\frac{\alpha}{(1+\lambda\sigma)^{\alpha}}\sum_{n}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(\alpha\sigma+n)^k}{k!}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\sigma^{i+2}\sum_{m=0}^{i+1}\binom{i+1}{m}\lambda^{i-m+1}\frac{u^{k+m+1}}{k+m+1} \quad (11)$$

Proof. By definition of the moment generating function, we have

$$M(t)=\int_0^{\infty}\mathrm{e}^{tx}\frac{\alpha\sigma^2}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma-1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha-1}(\lambda-\log x)\,\mathrm{d}x$$

Substitute $x=\mathrm{e}^u$, using the series expansion of e^z and $(1+x)^a$ as $\mathrm{e}^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$, $(1+x)^a=\sum_{k=0}^{\infty}\binom{a}{k}x^k$ and solving the above integral, we have

$$M(t)=\frac{\alpha}{(1+\lambda\sigma)^{\alpha}}\sum_{n}^{\infty}\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{(\alpha\sigma+n)^k}{k!}\sum_{i=0}^{\infty}\binom{\alpha-1}{i}\sigma^{i+2}\sum_{m=0}^{i+1}\binom{i+1}{m}\lambda^{i-m+1}\frac{u^{k+m+1}}{k+m+1},$$

which completes the proof.

Some numerical values for the mean and variance of the GLL distribution are displayed at Table 1 for some arbitrary choices of the distribution parameters.

Table 1. Mean and variance for several arbitrary parameter values. It observed that: the mean is increasing by the increases the values of the parameters, another hand the variance decreasing by the increases the values of α, λ but increases by increases the values of σ.

Parameters	$\sigma = 2$	$\lambda = 0.3$
$\alpha \downarrow$	Mean	Variance
0.4	0.325088	0.074209
0.7	0.446995	0.072601
1.2	0.568398	0.060010
1.8	0.654146	0.046760
2.1	0.684423	0.041591
$\sigma \downarrow$	$\alpha = 1.5$	$\lambda = 0.6$
0.2	0.062018	0.021525
0.4	0.168453	0.050825
0.6	0.270988	0.068320
0.8	0.359273	0.075108
1	0.433025	0.075320
$\lambda \downarrow$	$\alpha = 1.5$	$\sigma = 2$
0.6	0.655938	0.514637
1.1	0.687124	0.048423
1.6	0.702836	0.046272
2.1	0.712282	0.044786
2.6	0.718583	0.043717

3.2. Quantile Function and Random Number Generating

For a non-negative continuous random variable X with cdf $F(x)$ that follows the GLL distribution, the quantile function $Q(u) = F^{-1}(X)$ for $U = u(0,1)$ is given by

$$Q(u) = \mathrm{e}^{-\frac{\mathrm{e}^{\frac{L(u)\alpha - \alpha - \lambda\sigma\alpha + \ln(u) + \alpha\ln(1+\lambda\sigma)}{\alpha}} - 1 - \lambda\sigma}{\sigma}}, \tag{12}$$

where

$$L(u) = -Lambert\left(-\mathrm{e}^{\frac{-\alpha - \lambda\sigma\alpha + \ln(u) + \alpha\ln(1+\lambda\sigma)}{\alpha}}\right).$$

In particular, the distribution median is

$$Q(0.5) = \mathrm{e}^{-\frac{\mathrm{e}^{\frac{L(0.5)\alpha - \alpha - \lambda\sigma\alpha + \ln(0.5) + \alpha\ln(1+\lambda\sigma)}{\alpha}} - 1 - \lambda\sigma}{\sigma}} \tag{13}$$

3.3. Skewness and Kurtosis Based on Quantiles

Skewness measures the degree of the long tail and Kurtosis is a measure of the degree of tail heaviness. Based on quantile function $Q(.)$, Galton, F. [14] and Moors, J.J.A. [15] defined the skewness and kurtosis, respectively, as

$$S_G = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)} \text{ and } K_M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}.$$

Therefore, Galton's skewness and Moors' kurtosis of the quantile function defined by (12) can be get easily. Figure 3 illustrated the graphical representation of the Galton skewness and Moors kurtosis as a function of σ. These plots illustrate the effect of transmuting parameter σ, on skewness and kurtosis.

4. Reliability Measures of GLL Distribution

In this section, we obtain some reliability measures of the GLL distribution, including mean and strong mean inactivity time functions and some measures of residual lifetime and reversed residual lifetime of the GLL distribution, such as density, survival and hazard rate functions with mean and variance.

Mean Inactivity and Strong Mean Inactivity Time Functions

The mean inactivity time (MIT) function, also known as the mean past lifetime and the mean waiting time functions. The MIT function is important characteristic in many applications to describe the time, which had elapsed since the failure. Some recent properties and applications of MIT function can be found in Kayid, M. and Ahmad, I. A. [16], and Kayid, M. and Izadkhah, S [17]. Recently, Block, B. *et al.* [18] introduced a new reliability function called strong mean inactivity time (SMIT) function. This new function lies in the framework of the reversed hazard rate and the MIT functions. Let X be a lifetime random variable

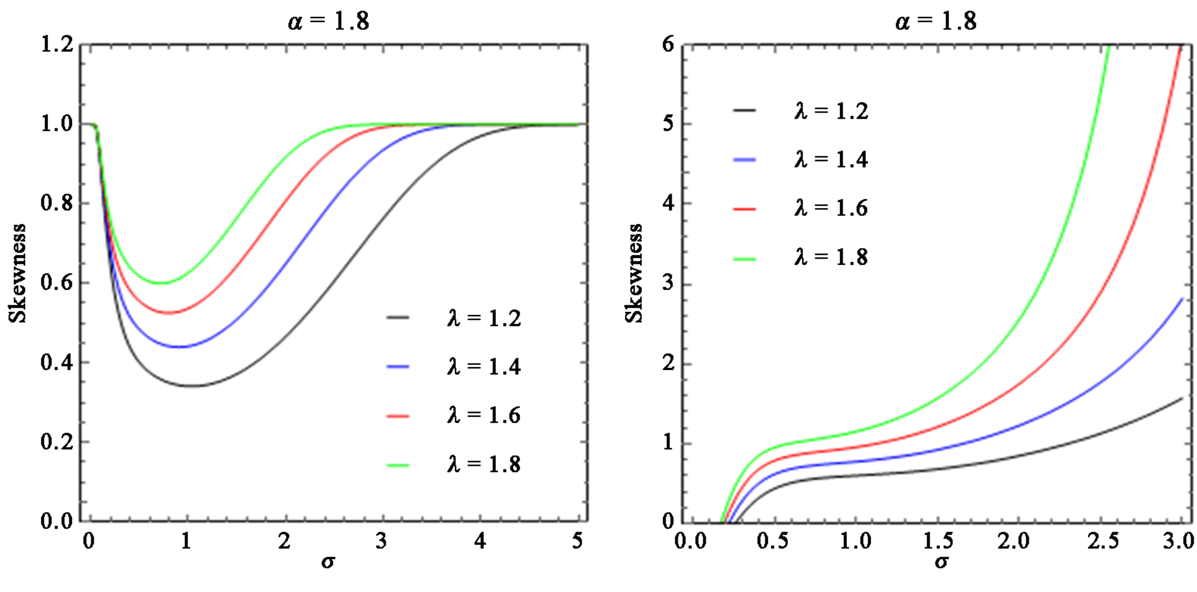

Figure 3. Plots of Galton skewness and Moors kurtosis for the GLL distribution as a function of σ.

with distribution function $F(.)$. Then the MIT and SMIT are defined by

$$\zeta_{MIT}(t)=\frac{1}{F(t)}\int_0^t F(x)\mathrm{d}x,\ t>0. \tag{14}$$

and

$$\vartheta_{SMIT}(t)=\frac{1}{F(t)}\int_0^t 2xF(x)\mathrm{d}x,\ t>0. \tag{15}$$

respectively, The next two propositions give explicit expressions of MIT and SMIT for the GLL distribution.

Proposition 2. The MIT function of a lifetime random variable X with GLL distribution is

$$\zeta_{MIT}(t)=\frac{1}{F(t)(1+\lambda\sigma)^{\alpha}}\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\sigma^{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k}m!\lambda^{k+1}\frac{t^{\alpha\sigma+\frac{1}{\lambda}+1}}{\alpha\sigma+\lambda+1}\right],\quad t>0 \tag{16}$$

where $F(t)$ is defined by (5).

Proof. The MIT function (14) of X with GLL is given by

$$\zeta_{MIT}(t)=\frac{1}{F(t)}\int_0^t\frac{1}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma}\left[1+\sigma(\lambda-\log x)\right]^{\alpha}\mathrm{d}x$$

Using the series expansion of $(1+x)^{a}$, x^{b} and after some simple calculations, then the above integral yields the MIT given by (16).

Proposition 3. The SMIT function of a lifetime random variable X with GLL distribution is

$$\vartheta_{SMIT}(t)=\frac{2}{F(t)(1+\lambda\sigma)^{\alpha}}\left[\sum_{k=0}^{\infty}\binom{\alpha}{k}\sigma^{k}\sum_{m=0}^{k}\binom{k}{m}(-1)^{k}m!\lambda^{k+1}\frac{t^{\alpha\sigma+\frac{1}{\lambda}+2}}{\lambda\alpha\sigma+2\lambda+1}\right],\quad t>0 \tag{17}$$

Proof. By definition (18), we have

$$\vartheta_{SMIT}(t)=\frac{2}{F(t)}\int_0^t\frac{1}{(1+\lambda\sigma)^{\alpha}}x^{\alpha\sigma+1}\left[1+\sigma(\lambda-\log x)\right]^{\alpha}\mathrm{d}x.$$

Using the series expansion of $(1+x)^{a}$, x^{b} and after some algebraic manipulation, then the above integral yields the SMIT given by (17). Figure 4: Plots of (a) MIT and (b) SMIT functions for different choices of α and t. Table 2: displays the MIT and SMIT at the points t, λ,σ and different choices of α.

From Figure 4 and Table 2, it observed that MIT and SMIT of GLL are increasing by the increases the values of α.

5. Maximum Likelihood Estimators of GLL Distribution

In this section, the method of maximum likelihood is considered to estimate the unknown parameters of GLL distribution. Given a random sample, denoted as $X=(x_1,x_2,\cdots,x_n)$, with size n, then using (3) the log-likelihood function can be written as

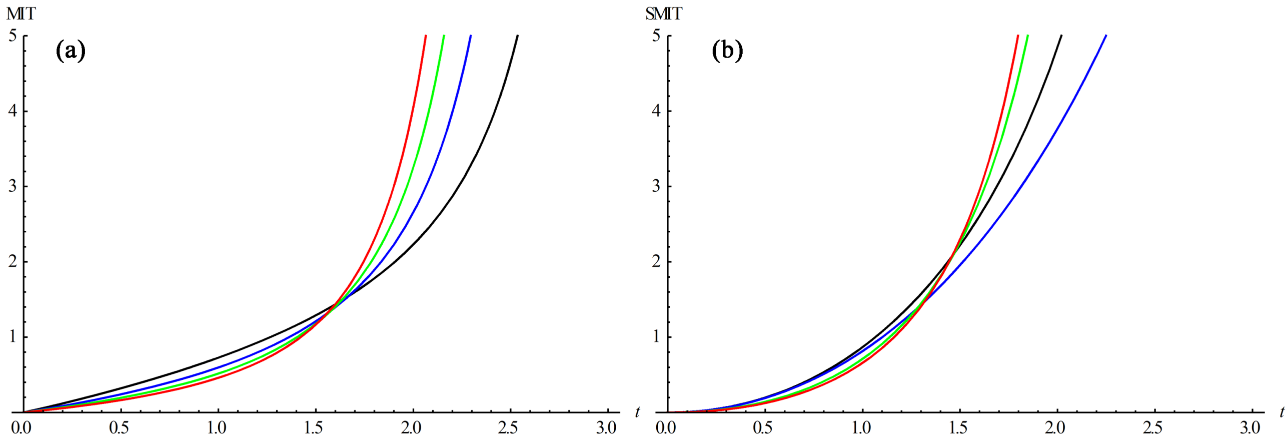

Figure 4. Plots of (a) MIT and (b) SMIT functions.(a) $\lambda=0.1,\sigma=1$ and $\alpha=1$ (black), $\alpha=2$ (blue), $\alpha=3$ (green), $\alpha=4$ (red). (b) $\lambda=0.1,\sigma=1$ and $\alpha=1$ (black), $\alpha=2$ (blue), $\alpha=3$ (green), $\alpha=4$ (red).

Table 2. Displays the MIT and SMIT at the point $t=2$ for GLL$(\lambda=0.3,\sigma=1.5)$ and different choices of α.

Parameters $\alpha\downarrow$	λ = 0.3, σ = 1.5, t = 2	
	MIT	SMIT
0.3	1.926	4.16754
0.9	1.95324	4.59202
1.5	2.08689	5.12413
2.3	2.36624	6.00555
3.5	2.97695	7.75782

$$\begin{aligned}\ell &= n\log(\alpha)+2n\log(\sigma)-n\alpha\log(1+\lambda\sigma)+(\alpha\sigma-1)\sum_{i=1}^{n}\log(x_i)\\ &+(\alpha-1)\sum_{i=1}^{n}\log\left(1+\sigma\left(\lambda-\log(x_i)\right)\right)+\sum_{i=1}^{n}\log\left(\lambda-\log(x_i)\right).\end{aligned} \tag{18}$$

Differentiating (21) with respect to α, σ and λ, respectively, we have

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-n\log(1+\lambda\sigma)+\sigma\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}\log\left(1+\sigma\left(\lambda-\log(x_i)\right)\right), \tag{19}$$

$$\frac{\partial\ell}{\partial\sigma}=\frac{2n}{\sigma}-\frac{n\alpha\lambda}{1+\lambda\sigma}+\alpha\sum_{i=1}^{n}\log(x_i)+\sum_{i=1}^{n}\frac{(\alpha-1)\left(\lambda-\log(x_i)\right)}{\left(1+\sigma\left(\lambda-\log(x_i)\right)\right)}, \tag{20}$$

and

$$\frac{\partial\ell}{\partial\lambda}=-\frac{n\alpha\sigma}{1+\lambda\sigma}+\sum_{i=1}^{n}\frac{(\alpha-1)\sigma}{1+\sigma\left(\lambda-\log(x_i)\right)}+\sum_{i=1}^{n}\frac{1}{\lambda-\log(x_i)}. \tag{21}$$

Setting $\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\sigma}$ and $\frac{\partial\ell}{\partial\lambda}$ equal to zero and solving these equations, For the difficulty of finding the solution analytically, we solve these equations numerically using the statistical software package Mathematica, yields the maximum likelihood estimators (MLEs) $\hat{\Theta}=\left(\hat{\alpha},\hat{\sigma},\hat{\lambda}\right)$ of $\Theta=(\alpha,\sigma,\lambda)$. For interval estimation and testing hypotheses on the model parameters, we require the observed information matrix. The corresponding 3 × 3 observed information matrix $I_n=I_n(\alpha,\sigma,\lambda)$ is

$$I_n=-\begin{pmatrix} I_{\alpha\alpha} & I_{\alpha\sigma} & I_{\alpha\lambda} \\ I_{\sigma\alpha} & I_{\sigma\sigma} & I_{\sigma\lambda} \\ I_{\lambda\alpha} & I_{\lambda\sigma} & I_{\lambda\lambda} \end{pmatrix},$$

whose elements are given in the Appendix.

6. The Doubly Truncated GLL Distribution

The probability distribution function (pdf): Let X be a random variable having the doubly truncated GLL distribution in the interval [a, b]. The truncated pdf of any variable takes the form: Block, B., *et al.* [18].

$$\begin{aligned} & f\left(x;\alpha,\sigma,\lambda/a<X<b\right) \\ & =\frac{g\left(x;\alpha,\sigma,\lambda\right)}{\int_a^b g\left(x;\alpha,\sigma,\lambda\right)\mathrm{d}x},\quad 0<a<x<b<1,\ \alpha,\sigma>0,\ \lambda\ge 0 \\ & =\frac{g\left(x;\alpha,\sigma,\lambda\right)}{G\left(x;\alpha,\sigma,\lambda\right)\big|_a^b} \end{aligned} \tag{22}$$

$$f\left(x;\alpha,\sigma,\lambda,a,b\right)=\frac{g\left(x;\alpha,\sigma,\lambda\right)}{G\left(b;\alpha,\sigma,\lambda\right)-G\left(a;\alpha,\sigma,\lambda\right)} \tag{23}$$

The pdf of DTGLL distribution takes the form:

$$=\frac{\alpha\sigma^2x^{\alpha\sigma-1}\left[1+\sigma\left(\lambda-\log(x)\right)\right]^{\alpha-1}\left(\lambda-\log(x)\right)}{b^{\alpha\sigma}\left[1+\sigma\left(\lambda-\log(b)\right)\right]^{\alpha}-a^{\alpha\sigma}\left[1+\sigma\left(\lambda-\log(a)\right)\right]^{\alpha}} \tag{24}$$

and can be expressed as:

$$f\left(x;\alpha,\sigma,\lambda,a,b\right)=\frac{\alpha\sigma^2}{k}x^{\alpha\sigma-1}\left[1+\sigma\left(\lambda-\log(x)\right)\right]^{\alpha-1}\left(\lambda-\log(x)\right) \tag{25}$$

where the constant k is

$$k=b^{\alpha\sigma}\left[1+\sigma\left(\lambda-\log(b)\right)\right]^{\alpha}-a^{\alpha\sigma}\left[1+\sigma\left(\lambda-\log(a)\right)\right]^{\alpha}$$

and have likelihood function can be written as

$$\begin{aligned} l= & n\log(\alpha)+2n\log(\sigma)-n\log(k)+(\alpha\sigma-1)\sum_{i=1}^{n}(x_i) \\ & +(\alpha-1)\sum_{i=1}^{n}\log\left(1+\sigma\left(\lambda-\log(x_i)\right)\right)+\sum_{i=1}^{n}\left(\lambda-\log(x_i)\right) \end{aligned} \tag{26}$$

In **Figure 5** presents the shape of the pdf of DTGLL distribution function using Equation (25) with different values of left truncation points ($a=0.4$, 0.35, 0.3) and a fixed right truncation point at ($b=0.9$) together with the original GLL distribution. However, **Figure 6** presents the shape of the pdf of DTGLL distribution

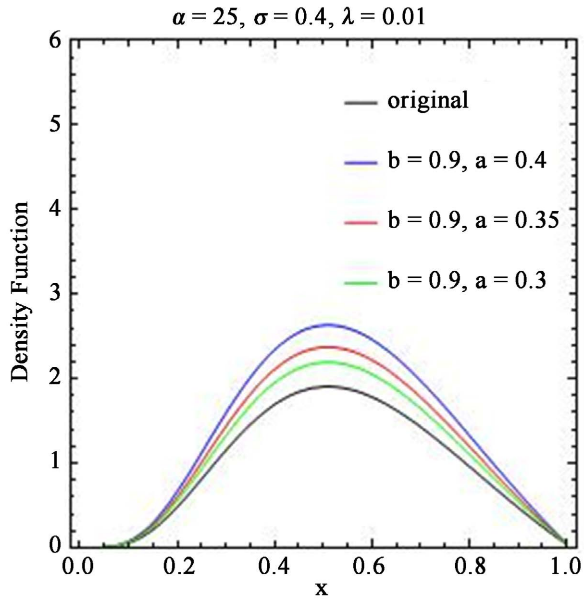

Figure 5. The pdf of DTGLL distribution, $\alpha = 25, \sigma = 0.4, \lambda = 0.01$ with different left truncation points ($a = 0.4$, 0.35, 0.3) and fixed right truncation point at ($b = 0.9$) together with the original distribution.

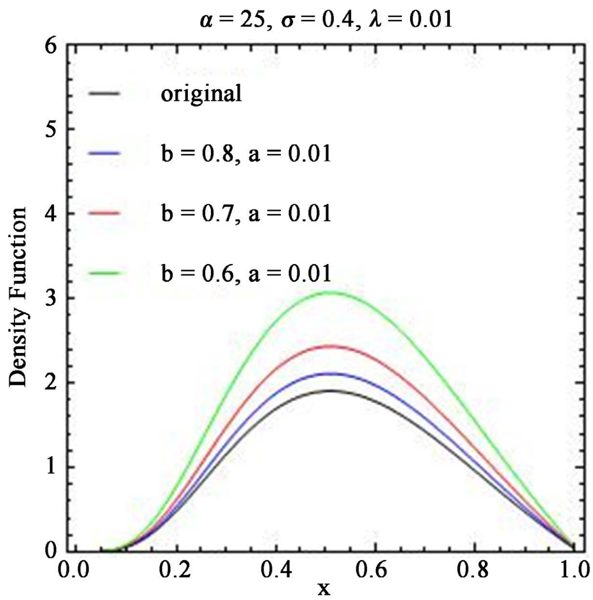

Figure 6. The pdf of the DTGLL distribution, $\alpha = 25, \sigma = 0.4, \lambda = 0.01$ with different right truncation points ($b = 0.8$, 0.7, 0.6) and fixed left truncation point at ($a = 0.01$) together with the original GLL distribution.

function with different values of right truncation points ($b = 0.8$, 0.7, 0.6) and a fixed left truncation point at ($a = 0.01$) together with the original GLL distribution.

7. Real Data Application

Here, we illustrate the applicability of DTGLL distribution by considering the following data setlisted in Table 3. We fitted the following distributions to data set:

Log-Lindley (LL) distribution, Generalized Log-Lindley (GLL) distribution. For this data set, we estimate the unknown parameters of each distribution by the maximum-likelihood method, and with these obtained estimates, we obtain the values of Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion (CAIC) and Hannan-Quinn Information Criterion (HQIC). Additionally, to compare the models, we used four criterions.

The MLEs of the parameters for some models fitted to the Arrest dataset, the values of AIC, BIC, CAIC and HQIC statistics for some models fitted to the Arrest data set and the values of K-S, p-value, -Log L and W^* statistics for some models fitted to the Arrest data set are in **Tables 4-6**, respectively.

Table 3. Arrest data set Fonseca, M. B., and França, M. G. C. (2007) [19] studied the soil fertility influence and the characterization of the biologic fixation of N2 for the Dimorphandrawilsoniirizz growth. For 128 plants, they made measures of the phosphorus concentration in the leaves.

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23	0.24	0.20	0.08
0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	016	0.09	0.10	0.12	0.12	0.10	0.09	0.17
0.19	0.21	0.18	0.26	0.19	0.17	0.18	0.20	0.24	0.19	0.21	0.22	0.17	0.08	0.08	0.06
0.09	0.22	0.23	0.22	0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12	0.15	0.08	0.12	0.09
0.14	0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13	0.11	0.11	0.11	0.08	0.22
0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15	0.17	0.14	0.12	0.18	0.14	0.18
0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11	0.11	0.14	0.07	0.07	0.19	0.17	0.18	0.16
0.19	0.15	0.07	0.09	0.17	0.10	0.08	0.15	0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13

Table 4. The MLEs of the parameters for some models fitted to the Arrest data set.

Distributions	Estimates		
LL (σ, λ)	0.981337	9.99×10^{-10}	
GLL (α, σ, λ)	1.074963	0.0000206	0.0067559
DTGLL (α, σ, λ, $b = 0.28$, $a = 0.05$)	0.922999	0.999999	0.439999

Table 5. The values of AIC, BIC, CAIC and HQIC statistics for some models fitted to the Arrest data set.

Distribution	AIC	BIC	CAIC	HQIC
LL (σ, λ)	−173.324	−167.62	−173.228	−171.006
GLL (α, σ, λ)	−417.483	−408.927	−417.289	−414.006
DTGLL (α, σ, λ, $b = 0.6$, $a = 0.04$)	−446.764	−438.208	−446.571	−443.288

Table 6. The values of K-S, p-value, -Log L and W^* statistics for some models fitted to the Arrest data set

Distribution	K-S	p-value	−Log L	W^*	A^*
LL (σ, λ)	0.370991	1.33×10^{-15}	−88.6619	5.29643	26.9503
GLL (α, σ, λ)	0.339921	2.84×10^{-13}	−211.741	4.39431	22.3743
DTGLL (α, σ, λ, $b = 0.9$, $a = 0.1$)	0.284672	1.95×10^{-9}	−226.382	3.25215	18.4389

Table 7. Descriptive statistics of the GLL distribution for the Arrest data set.

Mean	Median	SD	MD-mean	MD-median
0.140781	0.13	0.05440	0.0458789	0.0453125
Skewness	Kurtosis	S. Entropy	Min.	Max.
0.452605	−0.663184	1.4024417	0.5	0.28

MD = Mean deviation, S = Shannon.

Descriptive statistics of the GLL distribution for the Arrest data set are in Table 7.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

The elements of the 3 × 3 observed information matrix $I_n = I_n(\alpha,\sigma,\lambda)$ are:

$$I_{\alpha\alpha} = -\frac{n}{\alpha^2},$$

$$I_{\sigma\sigma} = -\frac{2n}{\sigma^2} + \frac{n\alpha\lambda^2}{(1+\lambda\sigma)^2} - \sum_{i=1}^{n}\frac{(\alpha-1)(\lambda-\log(x_i))^2}{(1+\sigma(\lambda-\log(x_i)))^2},$$

$$I_{\lambda\lambda} = -\frac{n\alpha\sigma^2}{(1+\lambda\sigma)^2} - \sum_{i=1}^{n}\frac{(\alpha-1)\sigma^2}{(1+\sigma(\lambda-\log(x_i)))^2} - \sum_{i=1}^{n}\frac{1}{(\lambda-\log(x_i))^2},$$

$$I_{\alpha\sigma} = I_{\sigma\alpha} = -\frac{n\lambda}{1+\lambda\sigma} + \sum_{i=1}^{n}\frac{\lambda-\log(x_i)}{1+\sigma(\lambda-\log(x_i))} + \sum_{i=1}^{n}\log(x_i),$$

$$I_{\alpha\lambda} = I_{\lambda\alpha} = -\frac{n\sigma}{1+\lambda\sigma} + \sum_{i=1}^{n}\frac{\sigma}{1+\sigma(\lambda-\log(x_i))},$$

and

$$I_{\sigma\lambda} = I_{\lambda\sigma} = -\frac{n\alpha\lambda\sigma}{(1+\lambda\sigma)^2} - \frac{n\alpha}{1+\lambda\sigma} + (\alpha-1)\sum_{i=1}^{n}\left(\frac{1}{1+\sigma(\lambda-\log(x_i))} - \frac{\sigma(\lambda-\log(x_i))}{(1+\sigma(\lambda-\log(x_i)))^2}\right).$$