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Adomian Decomposition Method for Solving Fractional Time-Klein-Gordon Equations Using Maple

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Abstract

Adomian decomposition is a semi-analytical approach to solving ordinary and partial differential equations. This study aims to apply the Adomian Decomposition Technique to obtain analytic solutions for linear and nonlinear time-fractional Klein-Gordon equations. The fractional derivatives are computed according to Caputo. Examples are provided. The findings show the explicitness, efficacy, and correctness of the used approach. Approximate solutions acquired by the decomposition method have been numerically assessed, given in the form of graphs and tables, and then these answers are compared with the actual solutions. The Adomian decomposition approach, which was used in this study, is a widely used and convergent method for the solutions of linear and non-linear time fractional Klein-Gordon equation.

Keywords

Adomian Decomposition, Klein-Gordon, Fractional Calculus

1. Introduction

Mathematicians have studied explicit and numerical solutions to nonlinear fractional differential equations throughout the past decade [1] [2]. Several methods [3] [4] [5] [6] have been put forward, but the Adomian decomposition method has become very popular and is a good way to solve both explicit and numerical solutions to a large class of differential systems that represent real physical problems. Many fractional calculus applications have been applied in recent years in the fields of science, engineering, and economics [7]. The application and subsequent advances of fractional calculus [8] in these domains have given rise to a significant increase in research on non-linear partial differential equations and linearization methods. The Adomian's decomposition method [9] [10], homotopy perturbation method [11] [12] [13], He's variational iteration method, homotopy analysis method, Galerkin method, collocation method, and other methods [14] [15] [16] have all been used to solve equations of various categories, such as linear or nonlinear, ordinary differential or partial differential equations, integer or fractional, etc. The decomposition method [17] is a good way to get analytic solutions for a large class of dynamical systems without closure approximation, perturbation theory, assumptions of linearization or weak nonlinearity, or restrictive assumptions on stochasticity. Even though perturbation methods can only give approximate solutions to non-linear fractional differential equations, in general, there is no method that gives an exact answer. Adomian's decomposition method also gives an approximation to a solution for nonlinear equations by keeping the problem in its original form. In this paper, we use the Adomian Decomposition Method (ADM) to obtain the solution of the time-fractional Klein-Gordon Equation:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)+bu(x,t)+g\left(u(x,t)\right)=f(x,t) \tag{1}$$

2. Preliminaries

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ is defined as:

$$J_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, t>0, (m-1<\alpha\le m), m\in\mathbb{N}. \tag{2}$$

Definition 2.2. The fractional derivative of $f(x)$ in Caputo sense is defined as:

$$\begin{aligned} D_x^\alpha f(x)&=\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\xi)}{(x-\xi)^{\alpha+1-m}}\mathrm{d}\xi, (m-1<\alpha\le m), x>0, m\in\mathbb{N}, \\ &= J_x^{m-\alpha}D^m f(x) \end{aligned} \tag{3}$$

Here $D^m \equiv \mathrm{d}^m/\mathrm{d}x^m$ and J_x^α stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$ clearly from the definition, we have:

$$D_x^\alpha x^\mu=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha}, \mu>-1 \tag{4}$$

For Riemann-Liouville fractional integral and Caputo fractional derivative, we have the following relation:

$$J_x^\alpha D_x^\alpha f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(a^+)\frac{x^k}{k!} \tag{5}$$

The Caputo fractional derivative is employed in this case because it permits standard initial and boundary conditions to be included in the problem's formulation.

Definition 2.3. In order for m to be the smallest positive integer greater than α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_{*t}^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t(t-\xi)^{m-\alpha-1}\frac{\partial^m u(x,\xi)}{\partial\xi^m}\mathrm{d}\xi, & \text{for } m-1<\alpha<m\\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha=m\in N\end{cases} \tag{6}$$

3. Adomian Decomposition Method for Nonlinear Fractional Partial Differential Equations

We consider the nonlinear fractional partial differential equation written in an operator form as:

$$D_t^{\alpha}u(x,t)+Lu(x,t)+Nu(x,t)=g(x,t),x>0 \tag{7}$$

where D_t^{α} is caputo fractional derivative of order α ($m-1<\alpha\le m$), defined by Equation (3), L is a linear operator which might include other fractional derivatives of order less than α, N is a non-linear operator which also might include other fractional derivatives of order less than α, $g(x,t)$ is source term. We apply the operator J_t^{α} to both sides of Equation (6); use result (5) to obtain:

$$u(x,t)=\sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!}+J_t^{\alpha}g(x,t)-J_t^{\alpha}\left[Lu(x,t)+Nu(x,t)\right]. \tag{8}$$

Next, we decompose the unknown function u into sum of an infinite number of components given by the decomposition series:

$$u=\sum_{n=0}^{\infty}u_n \tag{9}$$

and the nonlinear term is decomposed as follows:

$$Nu=\sum_{n=0}^{\infty}A_n \tag{10}$$

where A_n are the adomian polynomials given by:

$$A_n=\frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}N\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0},n=0,1,2,\cdots \tag{11}$$

From this formula, all terms of Adomian polynomials can be derived. Listed below are the first few terms extracted from this formula:

$$\begin{aligned}A_0&=u_0(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_0(x,y)\\ A_1&=u_0(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_1(x,y)+u_1(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_0(x,y)\\ A_2&=u_0(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_2(x,y)+u_1(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_1(x,y)+u_2(x,y)\frac{\partial^{\beta}}{\partial x^{\beta}}u_0(x,y)\end{aligned} \tag{12}$$

The components $u_0,u_1,u_2,\cdots$ are determined recursively by substituting (9), (10) into (8) leads to:

$$\sum_{n=0}^{\infty} u_n = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[L\left(\sum_{n=0}^{\infty} u_n\right) + \sum_{n=0}^{\infty} A_n\right] \tag{13}$$

This can be written as:

$$\begin{aligned} & u_0 + u_1 + u_2 + \cdots \\ & = \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) - J_t^{\alpha}\left[L\left(u_0 + u_1 + u_2^a + \cdots\right) + \left(A_0 + A_1 + A_2 + \cdots\right)\right], \end{aligned} \tag{14}$$

$$\begin{aligned} u_0(x,y) &= \sum_{k=0}^{m-1}\frac{\partial^k}{\partial y^k} u\left(x,0^+\right)\frac{y^k}{k!} + J_y^{\alpha}\left(g(x)\right) \\ u_1(x,y) &= -J_y^{\alpha}\left(A_0\right) \\ u_2(x,y) &= -J_y^{\alpha}\left(A_1\right) \\ u_3(x,y) &= -J_y^{\alpha}\left(A_2\right) \\ &\vdots \\ u_{n+1}(x,y) &= -J_y^{\alpha}\left(A_n\right), n \geq 0 \end{aligned} \tag{15}$$

Adomian method uses formal recursive relations as:

$$\begin{aligned} u_0 &= \sum_{k=0}^{m-1}\left(\frac{\partial^k u}{\partial t^k}\right)_{t=0}\frac{t^k}{k!} + J_t^{\alpha} g(x,t) \\ u_{k+1} &= -J_t^{\alpha}\left[L\left(u_k\right) + \left(A_k\right)\right] \end{aligned} \tag{16}$$

4. Applications

4.1. Example 1

We consider one-dimensional Time Fractional Klein-Gordon Equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} - u \tag{17}$$

with initial condition:

$$\begin{aligned} & u(x,0) = 0, u_t(x,0) = x \\ & 1 < \alpha \leq 2, t > 0, x \in R \\ & \alpha = 1 \end{aligned} \tag{18}$$

The results of ADM's solution to this problem using maple are displayed in Table 1 and Figure 1.

4.2. Example 2

We consider one-dimensional Time Fractional Klein-Gordon Equation:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} - u \tag{19}$$

with initial condition:

$$\begin{aligned} & u(x,0) = 0, u_t(x,0) = x \\ & 1 < \alpha \leq 2, t > 0, x \in R \\ & \alpha = 1.5 \end{aligned} \tag{20}$$

The results of ADM's solution to this problem using maple are displayed in Table 2 and Figure 2.

Table 1. Numerical results and exact solution of time fractional Klein-Gordon equation for example 1.

x	Exact solution	Numerical solution	Error
0.00100000	0.00099900	0.00100000	0.00000100
0.00200000	0.00199600	0.00200000	0.00000400
0.00300000	0.00299101	0.00300000	0.00000899
0.00400000	0.00398402	0.00399999	0.00001597
0.00500000	0.00497504	0.00499998	0.00002494
0.00600000	0.00596407	0.00599996	0.00003589
0.00700000	0.00695111	0.00699994	0.00004883
0.00800000	0.00793617	0.00799991	0.00006374
0.00900000	0.00891924	0.00899988	0.00008064
0.01000000	0.00990033	0.00999983	0.00009950

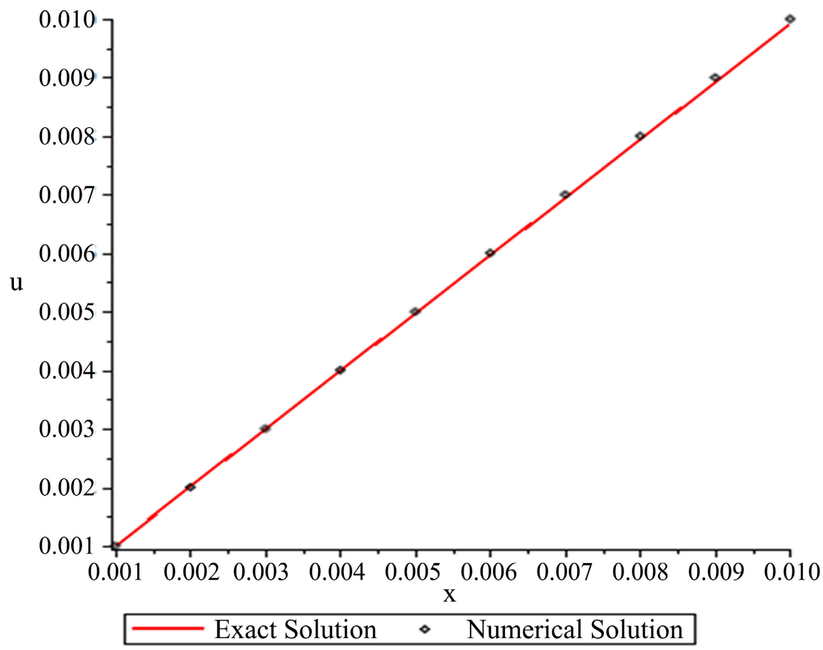

Figure 1. Numerical results and exact solution of time fractional Klein-Gordon equation for example 1.

Table 2. Numerical results and exact solution of time fractional Klein-Gordon equation for example 2.

x	Exact solution	Numerical solution	Error
0.00100000	0.00000100	0.00000099	0.00000001
0.00200000	0.00000400	0.00000395	0.00000005
0.00300000	0.00000900	0.00000885	0.00000015
0.00400000	0.00001600	0.00001570	0.00000030
0.00500000	0.00002500	0.00002447	0.00000053
0.00600000	0.00003600	0.00003516	0.00000084
0.00700000	0.00004900	0.00004777	0.00000123
0.00800000	0.00006400	0.00006228	0.00000172
0.00900000	0.00008100	0.00007869	0.00000231
0.01000000	0.00010000	0.00009699	0.00000301

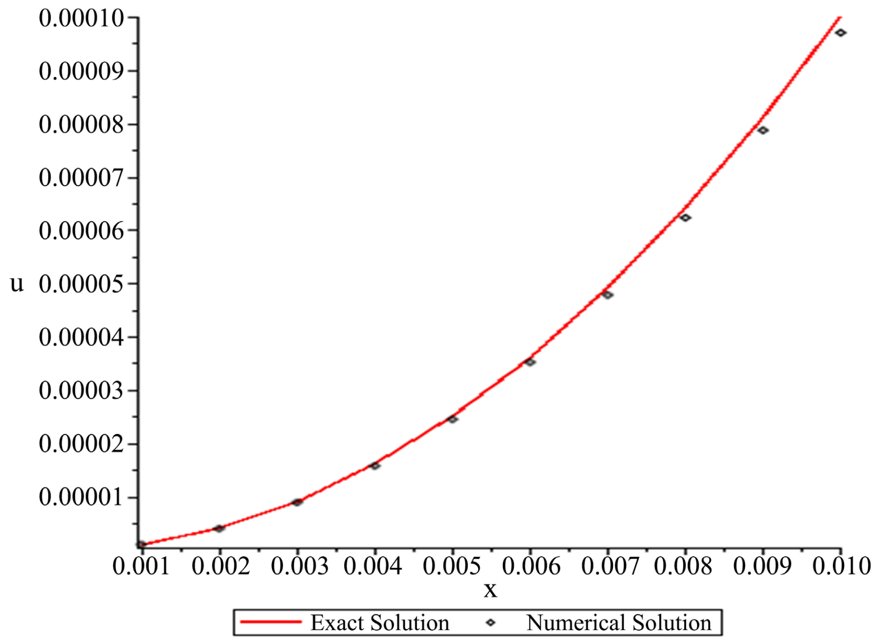

Figure 2. The exact and approximate solutions result of time fractional Klein-Gordon equation for example 2.

5. Conclusion

This study employs the Maple18 software to solve the time-fractional Klein-Gordon equation via the Adomian decomposition technique. In Table 1 and Table 2, the numerical answer and correct solution are displayed. A comparison of numerical results demonstrates that the numerical solution is typically relevant to the exact solution, confirming the efficacy of the procedure and the ability to swiftly and easily obtain the numerical solution correlating to the exact solution. In addition, the outcomes are astoundingly precise. We demonstrated that the decomposition method is quite effective at identifying exact solutions. However, the method provides a straightforward and effective instrument for obtaining the solutions without requiring extensive computations. It is also important to note that one advantage of this method is the rapid convergence of the solutions. In addition, the numerical outcomes of this method indicate a high level of precision and effectiveness in attaining the desired outcomes. By applying the ADM, it is possible to construct approximate solutions to algebraic equations, fractional ordinary differential equations (time-fractional Riccati equations, etc.), fractional partial differential equations (time-fractional Kawahara equations, modified time-fractional Kawahara equations, etc.), integro-differential equations, differential algebraic equations, etc. In practical applications, we can choose a finite sum based on the required precision.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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