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# Cosmological BF Theory on Topological Graph Manifold with Seifert Fibered Homology Spheres

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## Abstract

In this article, we show how to build a cosmological model characterized by the hierarchy of coupling constants and a set of Quantum Hall Fluids in BF theory. The resulting field theory is operated on Abelian Gauge fields within Gauge transformations on the $U(1)$ group, which introduces the Chern-Simmons class with topological mass. The mathematical background on which the model is based is a topological graph manifold of Brieskorn Seifert fibered-sphere space-time grid (lower dimensions), through a Kaluza-Klein reduction. This model offers a feasible alternative to the precise calculation of the cosmological constant Λ, much more accurate than the string landscape and baby universe models that have been proposed. Numerical results are given for coupling constants hierarchy. Model predictions may work as an argumental base to justify topological interpretations of space-time.

## Keywords

Topological Fluid Hierarchy, Coupling Constants, Topological Mass

## 1. Introduction

Gauge hierarchy and fine-tuning are relevant problems in particle physics and cosmology. New empirical discoveries may lead us to reconsider these concepts and their interconnections. First, the discovery of accelerated expansion of the universe by measuring luminosity of a distant supernovae [1], with a cosmological positive constant: $\rho_A = (1.35 \pm 0.15) \times 10^{-123}$, [2] [3] which is inconsistent with $\rho_A = 0$. Second, the observation at LHC of Higgs boson as a particle with mass of 126 GeV. This suggests that the SUSY breaking scale is considerably higher than the electroweak scale. This gets complicated since QFT predict $\rho_A$ to be 30 to 120 orders of magnitude larger than the experimental bound [1]. Also, string landscape and baby universes model predictions are not even close to experimental measures [4] [5]. Lately, new conceptions on the subject have been proposed in order to build cosmological models as well as field theories based on a different paradigm such as algebraic and geometric topology. Because of it, new horizons come to sight and bring cosmology based on topological manifolds, taking advantage of its already known properties. In this article, we propose a path to theoretically calculate $\rho_A$ in a precise way compared with late measurements. Starting from mathematical-physics development method found in [6], a topological BF theory is built over Seifert (Brieskorn) fibered homology spheres. The purpose of this effort is to describe physically the topological properties of the cosmology model, and also show how it is possible to obtain a coupling constant hierarchy as a result of it. Mathematics and geometry of the model are given by Brieskorn-Seifert fibered spheres (B-Sfh from now on) connected by a graph manifold, which is going to be the mathematical support and also, the core of the method. This article is structured as follows: Section 2 is about general BF theory and its construction on Manifold *M*. In Section 3, we show how it is possible to attach cosmological properties to a topological BF (field) theory. In Section 4, results and analysis are given, specifically about model predictions. Section 5 presents conclusions, a final recapitulation and suggestions for further work.

## 2. Topological BF Theory on a Manifold *M*

Background field theory is constructed as follows:

$$B_3 = \left( B_I \otimes \kappa^I - \frac{1}{m_0 k} F_D^I \otimes \sigma_I \right) + \left( F_I \otimes \kappa^I + \frac{1}{m_0 k} dH_D^I \otimes \sigma_I \right)$$

$$F_4 = \left( F_I \otimes d\kappa^I + \frac{1}{m_0 k} dH_D^I \otimes d\sigma_I \right) = d\left( A_I \otimes d\kappa^I + \frac{1}{m_0 k} H_D^I \otimes d\sigma_I \right)$$

where $\kappa^I \left( I = 1, \cdots, R \right)$ are $U\left(1\right)^R$ connection 1-forms and $\sigma_I \left( I = 1, \cdots, R \right)$ dual 1-forms. $A_I$ are locally Abelian potentials. Also 2-forms $F_I$, $F_D^I$ y $B_I$ belong to concatenated de Rham cochain $C_{dR}^2\left(X^4\right)$. $\kappa^I$, $\kappa^J$ and $\sigma^I$ will be obtained and defined from mathematical method on further steps in this section. Now we introduce 7-dimensional BF action:

$$S_7 = m_0 k \int_{X^4 \times M_+^3} B_3 \wedge F_4$$

$B_3$ and $F_4$ remain invariant under Gauge transformations $A_I \to A_I + u_I$; $H_D^I \to H_D^I - m_0 k K^{IJ} u_J$; and $B_I \to B_I + v_I$; $F_D^I \to F_D^I + m_0 k K^{IJ} v_J$. Breaking of symmetry is given by the Gauge condition results in:

$$F_D^I = \frac{1}{2} K^{IJ} * F_J; \quad H_D^I = \frac{1}{2} K^{IJ} * H_J$$

then we obtain calibration action [5]:

$$S_{\text{calibr}} = \int_{X^4} K^{IJ} \left( m_0 k B_I \wedge F_J - \frac{1}{2} F_I \wedge * F_J - \frac{1}{2} H_I \wedge * H_J - \frac{1}{4m_0 k} * F_I \wedge d * H_J \right)$$

Transformations as a generalization for Gauge symmetry are written as:

$$A_I \to A_I + \xi_I, \quad B_I \to B_I + \eta_I$$

with $\xi_I$ (topological superconductivity) and $\eta_I$, both closed 1-forms. This is a generalisation for Gauge symmetries of topological action on matter [7]. Variation on $S_{\text{calibr}}$ gives Klein Gordon equations:

$$-m^2 F_I = 2md * H_I, \quad -m^2 H_I = -2md * F_I$$

where $m = 2m_0 k$ is topological mass. Topology is involved in Yang-Mills gap mass phenomenon since mass is characterized from base state of topological fluids and determines empty low energy density. Therefore, we get a topological field theory with a Chern-Simmons class [8]:

$$c_1\left(\hat{\mathscr{L}}\right) = \int_M \kappa \wedge \mathrm{d}\kappa = n_0 + \sum_{j=1}^{N} \frac{\beta_i}{\alpha_i}$$

Chern-Simmons classes over orbifold $\hat{\Sigma}$, in space (manifold) $M$ are associated to $U(1)$ V-bundle. Due to Chern-Simons gauge theory it is possible to connect Seifert fibered manifolds possessing $U(1)$ -invariant contact structures (in general a $U(1)$ -action) with Seifert invariants too [5] [9].

## 2.1. Graph Manifold over Seifert Fibered Space

We define a Seifert fibered homology sphere ([10]) as:
$\Sigma(a_1, \cdots, a_n) = \left\{ \sum_{k=1}^{n} c^{ik} z_k^{a_k}, i = 0 \right\} \cap S^{2n-1}(0)$ [11]. Let $S_j$ be Seifert-Brieskorn fibered homology spheres (B-Sfh) with primary sequence [5]:

$$\left\{ \Sigma\left( p_{i+1}, p_{i+2}, q_i \right) \mid i \in N \right\}$$

where $p_i$ is the $i$-th prime number (*i.e.* $p_1 = 2, p_2 = 3, p_3 = 5, \cdots, p_i$), and choosing numbers [5] as fiber parameters ($S_j = \Sigma\left(a_{j,1}, a_{j,2}, a_{j,3}\right)$). Prime number sequence has been chosen in [5]. Then B-Sfh's may be generated by:

$$\Sigma^{(1)}\left(a_1, a_2, a_3\right) := \Sigma\left(a_1, \sigma_1, ak+1\right) = \Sigma\left(a_1^{(1)}, a_2^{(2)}, a_3^{(3)}\right)$$

which is named in [5] as "*K*" derivative, and it generates successive B-Sfh's and set them into a tree graph (**Figure 1**) as it describes in [6]. *k*-derivative sequence is built with fibered Seifert space generation [10]. Seifert invariants may be written of the form $a_{ni}^{(l)}$. Applying plumbing as a topological operation among B-Sfh's we write the condition:

$$a_2^I a_3^{I+1} > a_1^I a_3^I a_1^{I+1} a_2^{I+1}$$

which secures further graph-matrix elements to be positive definite. In the other hand, proposed graph $\Gamma_p$ has a tree open structure with plumbed B-Sfh's contained in it [5] [6]. For plumbed graph $\Gamma_p$ we may extract other topological structures such as lens space [12] which is a 3-D invariant structure emergent from plumbed graph. It may be defined as $L(p;q)$ which is the set of quotient

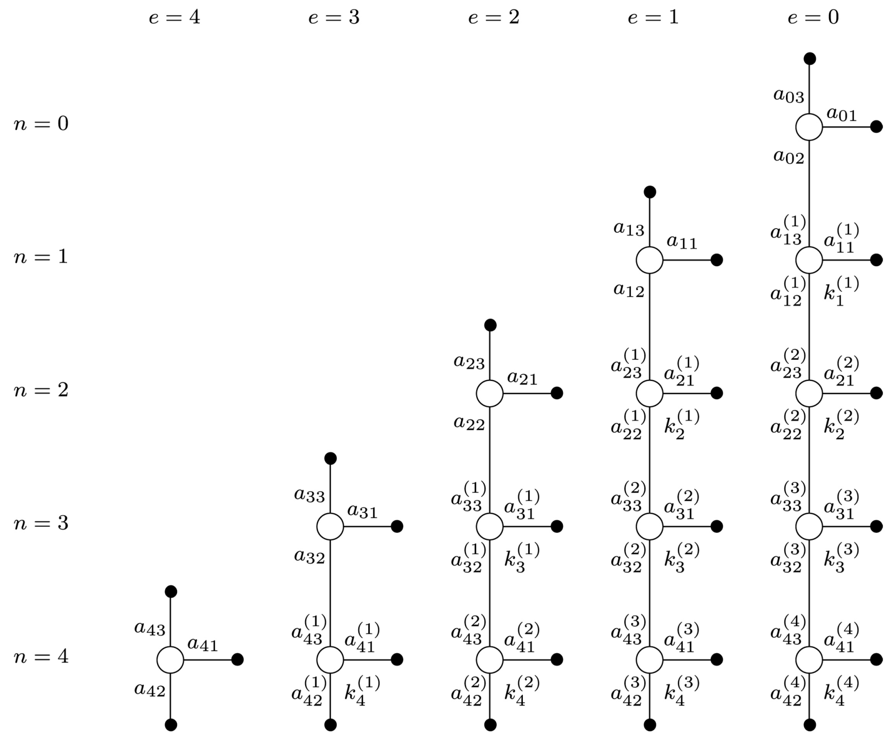


**Figure 1.** Graph manifold plumbing graph.

spaces of $S^3$ by actions $\mathbb{Z}/p$. *i.e.*, let $p$ y $q$ co-prime integers and consider $S^3$ as unitary sphere en $\mathbb{C}^2$. Then action $\mathbb{Z}/p$ in $S^3$ is generated by $(z_1, z_2) := \left(e^{2\pi i/p} \cdot z_1, e^{2\pi i q/p} \cdot z_2\right)$. Relation between orbital Seifert invariants with Euler numbers and Lens Spaces parameters $p$ and $q$ comes from $(p^I, q^I)$, $I = 1, \cdots, R-1$ which characterized a solid tori $TT(p^I, q^I) \cong T^2 \times [0,1]$ generated by plumbing among graph $\Gamma_p$ nodes $N^I$ y $N^{I+1}$.

$$p^I = a_2^I a_3^{I+1} - a_1^I a_3^I a_1^{I+1} a_2^{I+1}, q^I = b_2^I a_3^{I+1} + a_1^{I+1} a_2^{I+1} \left(b_1^I a_3^I + a_1^I b_3^I\right)$$

$e^I$ are Euler integer numbers defined for internal chain, and they are related to Seifert invariants, with Euler's formula:

$$e(\Sigma) = \sum_{k=1}^{n} b_k / a_k - b = -1/a,$$

plumbing operation on graph $\Gamma_p$ may connect B-Sfh's as we look in **Figure 2**.

Besides, invariants identify new lens spaces additional $L(p^I, q^I)$ and are a bound component of a 4D V-Cobordism with available formulas for $p^I$ y $q^I$ [13]. Now a single graph manifold $M(\Gamma_p)$ may be constructed over lens spaces glued in disjoint union thru JSJ (Jacko-Shalen-Johansen) decomposition [14] as it is proposed in [5]. Therefore $M(\Gamma_p)$ would have a JSJ covering conformed by $R$ pieces $\hat{M}^I$ with Seifert defined fibrations characterized by 1-forms $\kappa^I$. Then $\mathscr{T}_{JSJ} \subset \mathscr{T}_W$ is a subcollection of the Waldhausen graph structure [14] over Seifert fibered bundles $M_{JSJ}(N^I)$. Edges $\Gamma_p$ contained in a chain $C \in \mathscr{C}(p)$ belongs to the set of parallel tori in en $M(\Gamma_p)$. Choosing a tori $T_C^2$ we define:

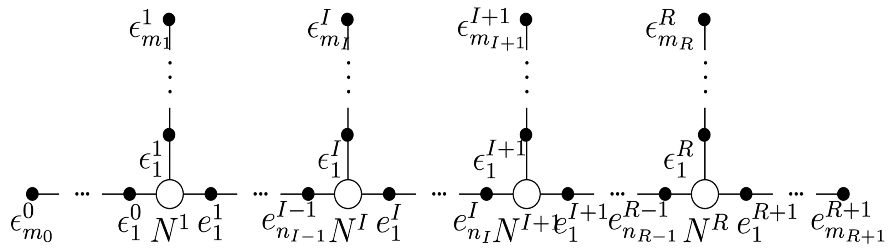


**Figure 2.** General plumbing diagram.

$$\mathscr{T}_{JSJ} = \bigsqcup_{C \in \mathscr{C}_i(p)} T_C^2$$

It is still possible to extend isotopy of natural Seifert structure to the whole Seifert fibration in $M^I_{JSJ}$ with exceptional fibers $M^I_W$. Indeterminations are removed as is shown in [5]. Thus, $M\left(\Gamma_p\right)$ with JSJ has to be:

$$M\left(\Gamma_p\right) = \bigcup_{I=1}^{R} \bar{M}^I_{JSJ}$$

lens spaces $L(p;q)$ may be rewrite in terms of JSJ covering for $M\left(\Gamma_p\right) = \bigcup_{I=1}^{R} \bar{M}^I_{JSJ}$ obtained in [5] as follows:

$$L\left(p^I, q^I\right) = ST_{\text{in}} \cup_J TT\left(p^I, q^I\right) \cup_J ST_{\text{fin}}$$

$$L\left(p^I, q^{*I}\right) = ST_{\text{in}} \cup_J TT\left(p^I, q^{*I}\right) \cup_J ST_{\text{fin}}$$

Expressions for $p$ and $q$ are already given in terms of Seifert invariants. Disjoint union of lens spaces gives us a new topological layer defined as V-cobordism [5]. Furthermore, there is a topological graph manifold as a result.

## 2.2. Adjacency Matrix with Continued Fractions

We can extract an adjacency matrix of plumbing graph manifold using continued fractions as it is explained in [6]. Beginning with $p$ and $q$ we obtain:

$$-\frac{p^I}{q^I} = \left[e^I_1, \cdots, e^I_{n_I}\right]$$

which are continued fractions related to Euler numbers shown in **Figure 2**, its mechanism may be found in [15]. Now we take plumbing graph representation where it is clear that for $I = 2, \cdots, R-1$ the set $\hat{M}^I$ has the form:

$$\hat{M}^I = \bar{M}^I_W \cup TT\left(p^{I-1}, q^{*I-1}\right) \cup TT\left(p^I, q^I\right) \cup ST\left(a^I, b^I\right),$$

where $ST\left(a^I, b^I\right)$ is a Seifert fibered tori with Seifert invariants $\left(a^I, b^I\right)$. We have also

$$\hat{M}^R = \bar{M}^R_W \cup TT\left(p^{R-1}, q^{*R-1}\right) \cup ST\left(a^R, b^R\right) \cup ST\left(a^{R+1}, b^{R+1}\right),$$

and

$$\hat{M}^I \cap \hat{M}^{I+1} = TT\left(p^I, q^I\right) \cong^* TT\left(p^I, q^{*I}\right),\ I = 2, \cdots, R-1$$

Following expression representing the basis of the 1-form [16]

$$\left(\sigma_I^2, \kappa^I\right), \left(\sigma_{I+1}^3, \kappa^{I+1}\right)$$

then integrals may be calculated along fibers [5] as:

$$\int_{s_2^I} \sigma_I^2 = \int_{s_3^{I+1}} \sigma_{I+1}^3 = \int_{f_I} \kappa^I = 1;$$

$$\int_{f_I} \sigma_I^2 = \int_{f_{I+1}} \sigma_{I+1}^3 = \int_{s_2^I} \kappa^I = \int_{s_3^{I+1}} \kappa^{I+1} = 0$$

Thus we get respective transforms among dual 1-forms:

$$\sigma_I^2 = -q^{*I} \sigma_{I+1}^3 + p^I \kappa^{I+1}; \quad \kappa^I = p^{*I} \sigma_{I+1}^3 + q^I \kappa^{I+1}. \tag{1}$$

Therefore, in expression:

$$\Lambda^{I+1,I+1} = \int_{TT\left(p^I, q^{*I}\right)} \kappa^{I+1} \wedge \mathrm{d}\kappa^{I+1} = \int_{L\left(p^I, q^{*I}\right)} \kappa^{I+1} \wedge \mathrm{d}\kappa^{I+1}, \quad I = 1, \cdots, R-1,$$

integral gathered linking intersection numbers of fibered structures $\kappa^I$ and $\kappa^{I+1}$ define in solid tori $TT\left(p^I, q^I\right) \cong^* TT\left(p^I, q^{*I}\right)$ associated to lens space $L\left(p^I, q^I\right) \cong^* L\left(p^I, q^{*I}\right)$ (homeomorphism between this two lens spaces is a consequence of $q^I q^{*I} + p^I p^{*I} = 1$ [5]). If we apply duality conditions [16]:

$$\Lambda^{I,I+1} = \Lambda^{I,I+1} = \frac{1}{p^I}, \quad \Lambda^{I,I} = \frac{q^I}{p^I}; \quad \Lambda^{I+1,I+1} = \frac{q^{*I}}{p^I}.$$

rational numbers $\Lambda^{I,I}$ and $\Lambda^{I+1,I+1}$, are known as Chern classes with V-*bundle* associated with Seifert fibrations where invariant $U(1)$ is connected with forms $\kappa^I$ and $\kappa^{I+1}$ on lens spaces $L\left(p^I, q^I\right)$ y $L\left(p^I, q^{*I}\right)$ respectively.

Let $M_+^3 = -M\left(\Gamma_p\right)$ be our graph manifold, we integrate it as:

$$K^{IJ} = \int_{M_+^3} \kappa^I \wedge d\kappa^J = -\int_{M\left(\Gamma_p\right)} \kappa^I \wedge d\kappa^J.$$

Following condition assure that rational linking matrix is positive definite:

$$p^I = a_2^I a_3^{I+1} - a_1^I a_3^I a_1^{I+1} a_2^{I+1}, \quad q^I = b_2^I a_3^{I+1} + a_1^{I+1} a_2^{I+1}\left(b_1^I a_3^I + a_1^I b_3^I\right)$$

From [5] we have rational matrix elements:

$$K^{RR} = -\left(\frac{q^{*R-1}}{p^{R-1}} + \frac{b^R}{a^R} + \frac{b^{R+1}}{a^{R+1}}\right). \tag{2}$$

$$K^{II} = -\left(\frac{q^{*I-1}}{p^{I-1}} + \frac{q^I}{p^I} + \frac{b^I}{a^I}\right), I = 1, \cdots, R-1$$

Notice that these expressions are consistent with Chern-Simmons class formula, defined in Section 2. Now it is possible to write a general expression to represent rational linking matrix in terms of Euler numbers $e_A$ belonging each one to a graph vertex $v_A$. This would be:

$$Q^{AB}\left(\Gamma_p\right) = \begin{cases} e_A, & \text{if } A = B; \\ -1, & \text{if } A \neq B \text{ and } v_A \text{ is connected a } v_B \text{ by a bundle;} \\ 0, & \text{other case.} \end{cases}$$

Rational linking may be structured as a block matrix such as it is illustrated in **Figure 3**.

Matrix $A^{AB}$ and its formulas help us to obtain a topological fluid filling factor

$$Q^{AB}(\Gamma_p) = \begin{bmatrix} \ddots & -1 & & & & & & & & & & & \\ -1 & e^{I-1}_{n_{I-1}} & 0 & 0 & \cdots & 0 & -1 & & & & & & \\ & 0 & \epsilon^I_{m_I} & -1 & & & 0 & & & & & & \\ & 0 & -1 & \ddots & & & \vdots & & & & & & \\ & \vdots & & & \ddots & -1 & 0 & & & & & & \\ & 0 & & & -1 & \epsilon^I_1 & -1 & & & & & & \\ & -1 & 0 & \cdots & 0 & -1 & 0^I & -1 & & & & & \\ & & & & & & -1 & e^I_1 & -1 & & & & \\ & & & & & & & -1 & \ddots & & & & \\ & & & & & & & & & \ddots & -1 & & \\ & & & & & & & & & -1 & e^I_{n_I} & -1 & \\ & & & & & & & & & & -1 & 0^{I+1} & -1 \\ & & & & & & & & & & & -1 & \ddots \end{bmatrix}$$

**Figure 3.** Matrix $Q^{AB}\left(\Gamma_p\right)$.

hierarchy. This matrix and its (partial) diagonalized version $D^{MN}$ may be obtained by Gauss-Neumann method [6]

$$Q^{AB}_{\text{part.diag}}\left(\Gamma_p\right) = K^{IJ}_{\text{reduced}} \oplus D^{MN}$$

## 3. Cosmological Approach

In BF theory, a simple model of Gauge interactions is a non trivial topological ensemble that modeled a 7-D space-time manifold. For $M^3_+$ this is given by the disjoint union of lens spaces as a result of plumbing operations on graph:

$$M^3_+ \bigsqcup_{K,s} L\left(p^K_s, q^K_s\right) \bigsqcup_{L,t} L\left(a^L_t, b^L_t\right)$$

Full space-time manifold is the cross product of a regular 4-dimensional manifold $X^4$ with $M^3_+$, $X^4 \times M^3_+$, where internal 3-manifold belong to the family $M^3_+ = -M\left(\Gamma_p\right)$ with Waldhausen topology $\tau_W$ [17]. Now, action $S_7$ presented in Section 2 has different variants since rational linking matrix results. Different block matrices in adjacency matrix give us a boundary action

$$S_{\text{boundary}} = \int_{\partial X^4}\left(m_0 k K^{IJ} A_I \wedge dA_J + 2H^I_D \wedge dA_I + \frac{1}{m_0 k} K_{IJ} H^I_D \wedge dH^J_D\right),$$

with action QHF (Quantum Hall fluid) related to matrix computing and its results (Section 4), and represents an action for topological fluids

$$S_{\text{QHF}} = m_0 \int_{\partial X^4}\left(\tilde{K}^{IJ} A_I \wedge dA_J + 2t^I A_{\text{ext}} \wedge dA_I + \tilde{K}_{IJ} t^I t^J A_{\text{ext}} \wedge dA_{\text{ext}}\right),$$

where Hall states came from generalized states hierarchy [5] [18].
$\tilde{K}^{IJ} = \det\left(K_{IJ}\right) K^{IJ}$ is tridiagonal integer matrix and $t^I$, both characterized generalised hierarchy of Hall states. Also $\tilde{K}_{IJ} = K_{IJ} / \det\left(K_{IJ}\right)$ is rational inverse matrix of $\tilde{K}^{IJ}$. Filling factor $\nu = \tilde{K}_{IJ} t^I t^J$ describes conductivity $\sigma = \nu/2\pi$ on topological fluid. Currents $J_I$ are defined as $J_I = (1/2\pi) * dA_I$ and motion equations given by:

$$\tilde{K}^{IJ} dA_J + t^I dA_{\text{ext}} = 0$$

and

$$J_I = -(1/2\pi) \tilde{K}_{IJ} t^J * dA_{\text{ext}}$$

where Hodge operator $*$ is defined with respect to boundary $\partial X^4$. Then we have

$$t^I * J_I = (\nu/2\pi) dA_{\text{ext}} = \sigma F_{\text{ext}}$$

which is the equation of QH effect. Fractional topological fluids or *insulators* [18] [19] [20] in this model has an analogue structure to a quasi-particle hierarchy system in 4-D space-time described by *bulk* action [5]

$$S_4^{\text{bulk}} = \int_{X^4} \left( m_0 k K^{IJ} B_I \wedge F_J - F_I \wedge F_D^I + B_I \wedge dH_D^I - \frac{1}{m_0 k} K_{IJ} F_D^I \wedge dH_D^J \right)$$

Bulk action contains the mechanism for mass generation (topological order) effect [5] [21]. Now it is possible to rewrite BF action equations as

$$S_{\text{BF}} = \int_{X^4} \left( m_0 k K^{IJ} B_I \wedge F_J - F_I \wedge F_D^I - H_I \wedge H_D^I - \frac{1}{m_0 k} K_{IJ} F_D^I \wedge dH_D^J \right)$$

where $H_I = dB_I$ is the rational linking matrix $K^{IJ}$ and BF model may be interpreted as a consistent matrix set of coupling constants $K^{IJ}$ which describes a Gauge kind hierarchy of coupling constants with respect to a dimensional scale factors *k*.

### 3.1. Kaluza-Klein Winding

Field equations formulated in this section and the precedent are the building blocks for a mutidimensional cosmological model of Kaluza-Klein kind. Therefore, a global expression of our model in terms of rolled dimensions, and it may be expressed thru graph manifold (Seifert) spaces [5]

$$X^4 \times \left( M_+^3 \bigsqcup_{K,s} L\left(p_s^K, q_s^K\right) \bigsqcup_{L,t} L\left(a_t^L, b_t^L\right) \right)$$

multidimensional space on time-space's is formed by *baby universes* [5], where they fill the set of disjoint homeomorphic manifolds $X_s^{4,K} \times L\left(p_s^K, q_s^K\right)$ and $X_t^{4,L} \times L\left(a_t^L, b_t^L\right)$. These 4D space-time manifolds $X_s^{4,K}$ and $X_t^{4,L}$ are homeomorphic to $X^4$ but have a different scale. Thus it is necessary to reduce dimension due integration over lens spaces where coupling constants set emerge $\left[e_s^K, \cdots, e_{n_K}^K\right]$ y $\left[\varepsilon_t^L, \cdots, \varepsilon_{m_L}^L\right]$. Continued fractions in this particular case possess an absolute value higher than 1, therefore scales in 3D space belong to Planck suborder.

### 3.2. Generalized States Hierarchy

Following expressions describe low energy physics of topological fluids hierarchy with $n_I$ and $m_I$ respectively [5]

$$K^{ab}(\underline{e}, I) = \begin{pmatrix} e_1^I & -1 & & \\ -1 & \ddots & & \\ & & \ddots & -1 \\ & & -1 & e_{n_I}^I \end{pmatrix} \text{ y } K^{\alpha\beta}(\underline{\varepsilon}, I) = \begin{pmatrix} \varepsilon_{m_I}^I & -1 & & \\ -1 & \ddots & & \\ & & \ddots & -1 \\ & & -1 & \varepsilon_1^I \end{pmatrix}$$

Topological fluids characterized charge vectors $t^a(\underline{e},I)$ and $t^b(\underline{\varepsilon},I)$, along with *K*-matrices define *filling* factors $\nu(\underline{e},I)=t^a(\underline{e},I)K_{ab}(\underline{e},I)t^b(\underline{e},I)$; $\nu(\underline{\varepsilon},I)=t^\alpha(\underline{\varepsilon},I)K_{\alpha\beta}(\underline{\varepsilon},I)t^\beta(\underline{\varepsilon},I)$, where $K_{ab}(\underline{e},I)$ y $K_{\alpha\beta}(\underline{\varepsilon},I)$ are inverse matrices of $K^{ab}(\underline{e},I)$ and $K^{\alpha\beta}(\underline{\varepsilon},I)$ respectively. If we pick $t^a(\underline{e},I)=\delta_1^a$, then $\nu_1(\underline{e},I)=K_{11}(\underline{e},I)=\dfrac{\operatorname{adj}K^{11}(\underline{e},I)}{\det K^{ab}(\underline{e},I)}=\dfrac{1}{\left[e_1^I,\cdots,e_{n_I}^I\right]}=-\dfrac{q^I}{p^I}$; and choosing

$t^a(\underline{e},I)=\delta_{n_I}^a$ we have that

$\nu_n(\underline{e},I)=K_{n_I,n_I}(\underline{e},I)=\dfrac{\operatorname{adj}K^{n_I,n_I}(\underline{e},I)}{\det K^{ab}(\underline{e},I)}=\dfrac{1}{\left[e_{n_I}^I,\cdots,e_1^I\right]}=-\dfrac{q^{*I}}{p^I}$. Finally, to select

$t^\alpha(\underline{\varepsilon},I)=\delta_1^\alpha$ the *filling* factor is given by

$\nu_1(\underline{\varepsilon},I)=K_{11}(\underline{\varepsilon},I)=\dfrac{\operatorname{adj}K^{11}(\underline{\varepsilon},I)}{\det K^{\alpha\beta}(\underline{\varepsilon},I)}=\dfrac{1}{\left[\varepsilon_1^I,\cdots,\varepsilon_{m_I}^I\right]}=-\dfrac{b^I}{a^I}$. Diagonal elements

from rational linking matrix $K^{II}$ may be rewritten thru its respective filling factors $K^{II}=\nu_n(\underline{e},I-1)+\nu_1(\underline{e},I)+\nu_1(\underline{\varepsilon},I)$, for $I=2,\cdots,R-1$ besides $K^{11}=\nu_1(\underline{\varepsilon},0)+\nu_1(\underline{\varepsilon},1)+\nu_1(\underline{e},1)$; $K^{RR}=\nu_n(\underline{e},R-1)+\nu_1(\underline{\varepsilon},R)+\nu_1(\underline{\varepsilon},R+1)$. $K^{IJ}$ is the Gauge coupling constant matrix. Furthermore, for graph manifolds such as homology $\mathbb{Z}$-spheres next conditions are satisfied:

$$\left|K^{II}\right|\ll\min\left\{\left|\nu_n(\underline{e},I-1)\right|,\left|\nu_1(\underline{e},I)\right|,\left|\nu_1(\underline{\varepsilon},I)\right|\right\},\ \text{for}\ I=2,\cdots,R-1;$$

$$\left|K^{RR}\right|\ll\min\left\{\left|\nu_n(\underline{e},R-1)\right|,\left|\nu_1(\underline{\varepsilon},R)\right|,\left|\nu_1(\underline{\varepsilon},R+1)\right|\right\}$$

which is fine-tuning for coupling constants, and its universal for all of them [5].

### 3.3. Cosmological Constant

$K^{II}(e)$ is associated with a specific interaction depending on the discrete energy scale parameter *e*. For example, when *e* runs from 0 to 4 the *cosmological constant* $K^{RR}(e)$ changes according to the following sequence: $2.66\times10^{-134}\to1.63\times10^{-67}\to4.04\times10^{-34}\to2.22\times10^{-17}\to4.48\times10^{-9}$. If we correspond the last value of the cosmological constant to the unit Planck scale, then the running cosmological constant is reduced to the Planck units and reads $5.94\times10^{-126}\to3.64\times10^{-59}\to9.02\times10^{-26}\to4.96\times10^{-9}\to1$ [5]. The cosmological constant, understood as vacuum energy density, depends on the discrete energy parameter *e*.

## 4. Analysis and Results

The running cosmological constants acquire the sense of the vacuum energy scales associated with the topology changes of the extra dimensional space, which induce the unification of gauge interactions (**Figure 4**), given as a result a rational linking matrix. The diagonal elements of the matrix $K^{IJ}(0)$ (**Figure 5**) have a hierarchy very closed to the one of the dimensionless low energy coupling (DLEC) constants [22]. It is natural to suppose that diagonal elements of the other rational linking matrices $K^{IJ}(e), e=1,2,3,4$ (as well as their eigenvalues, see [23]) simulate hierarchy of the vacuum-level coupling constants

| Inter | e/n | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|
| strong | 0 | 9.68 x $10^{-1}$ | | | | |
| Electr. | 1 | 7.21 x $10^{-3}$ | 8.33 x $10^{-2}$ | | | |
| Weak. | 2 | 1.76 x $10^{-12}$ | 9.62 x $10^{-6}$ | 2.99 x $10^{-3}$ | | |
| Grav. | 3 | 3.68 x $10^{-44}$ | 2.50 x $10^{-21}$ | 6. 58 x $10^{-10}$ | 2.17 x $10^{-6}$ | |
| Cosm. | 4 | 2.66 x $10^{-134}$ | 1.63 x $10^{-67}$ | 4.04 x $10^{-34}$ | 2.22 x $10^{-17}$ | 4.48 x $10^{-9}$ |

**Figure 4.** Coupling constant hierarchy table.

$$K^{IJ}(0)=\begin{pmatrix} \mathbf{9.69\times10^{-1}} & -3.13\times10^{-2} & 0 & 0 & 0 \\ -3.13\times10^{-2} & \mathbf{7.21\times10^{-3}} & -1.44\times10^{-8} & 0 & 0 \\ 0 & -1.44\times10^{-8} & \mathbf{1.76\times10^{-12}} & -1.93\times10^{-29} & 0 \\ 0 & 0 & -1.93\times10^{-29} & \mathbf{3.68\times10^{-44}} & -3.12\times10^{-89} \\ 0 & 0 & 0 & -3.12\times10^{-89} & \mathbf{2.66\times10^{-134}}. \end{pmatrix}$$

**Figure 5.** Coupling constant values.

of the fundamental interactions acting in the states characterized by higher densities of vacuum energy (according to this hypothesis we refer to $e$ as a discrete energy scale parameter). The cosmological constant problem strongly suggest the existence of fine-tuning mechanism, since the empirical energy density of cosmological vacuum is at least 60 orders of magnitude smaller than several theoretical contributions to it. Recall that in quantum field theory some contributions to the vacuum density are evaluated as follows (see [24] [25]): from the standard theory $(200\text{ GeV})^4 \approx 10^{67}$; from the low energy super-symmetry breaking scale $(10^3\text{ GeV})^4 \approx 10^{64}$; from grand unification schemes $(10^{13}\text{ GeV})^4$ - $(10^{16}\text{ GeV})^4 \approx 10^{24}$ - $10^{12}$ (depending on a model); from quantum gravity $(10^{19}\text{ GeV})^4 \approx 1$. Within our framework an enormous fine-tuning for the cosmological constant is modeled owing to the topological properties of graph manifolds under consideration in [5]. Therefore, coupling constants hierarchy from the model, satisfies energy scale predicted by lately experimental recovered evidence such as mentioned at the intro. Note that in our model the "running cosmological constant" (or the sequence of vacuum energy scales) is associated with the last diagonal elements $K^{RR}(e)$ rational linking matrices of the graph manifolds $M_3^+(e)\,|\,e=0,\cdots,4$ and thus undergoes a change when the topology of extra-dimensional space is transformed. Therefore the cosmological constant, understood as vacuum energy density, depends on the discrete energy parameter $e$. This problem strongly suggests the existence of a fine-tuning mechanism, since the empirical energy density of cosmological vacuum is at least 60 orders of magnitude smaller than several theoretic contributions to it.

## 5. Conclusions

The hierarchy and fine-tuning of the gauge coupling constants are described on the basis of topological invariants (Chern classes interpreted as filling factors) characterizing a collection of fractional topological fluids emerging from three-dimensional graph manifolds, which play the role of internal spaces in the Kaluza-Klein approach to the topological BF theory [6]. Due to the method results, it is strongly argued that topological invariants in general (also in graph manifolds) may be closely related to BF theory and field physics of space-time. There are other interesting topological invariants, such as Casson invariant, Floer and De Rham cohomology as well as homotopy groups, which in case to be calculated or interpreted, may be helpful in the analysis of BF models, in order to extract cosmological constants to enrich model and predictions. It is also remarkable that a basic prime numbers sequence has been used to fill the Seifert fibered parameters. Other configurations on prime numbers sequence may be given to building a similar model with closely tied results. One of the most remarkable ideas given by this model is that mass gap problem may be solved by topological fluids hierarchy and its interpretations, starting with topological mass. This would be possible thanks to Seifert-Riemann surface research on Yang-Mills problem [22]. Theoretical description of the hierarchical systems such as fractional quantum Hall states and fractional topological insulators are based on the existence of a new state of matter characterized by a new type of order: topological order. These methods are applied not only to the description of three-dimensional phenomena such as quantum Hall effect, but also to four-dimensional systems connected with topological superconductivity or a topological confinement, without any spontaneous symmetry breaking pattern. As future work, this model can be quantized. It could be convenient to establish a defined metric for model to, cosmologically speaking, research the physical implications of black holes and space-time expansion starting from this space-time setup. Besides, the study of topological insulators and fine-tuning of Chern-Simmon classes on Gauge fields is interesting. Therefore, Yang-Mills mass gap problem can be addressed from fine tuning topological fluids (Quantum Hall states) system. As further work, the authors suggest trying different combinations of fibers for homology spheres in order to get wider and more precise results and predictions for our model. Also, it is possible to search for more physical quantities obtained from computing other topological invariants from those calculated in this paper, such as fundamental homology and homotopy groups from topological manifold $M$ to interpret values in terms of physical constants.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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