Applied Mathematics, 2023, 14, 196-207
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

A Comparative Survey of an Approximate Solution Method for Stochastic Delay Differential Equations

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How to cite this paper: Emenonye, E.C. and Anonwa, D. (2023) A Comparative Survey of an Approximate Solution Method for Stochastic Delay Differential Equations. *Applied Mathematics*, **14**, 196-207.
https://doi.org/10.4236/am.2023.143012

Received: December 24, 2022
Accepted: March 24, 2023
Published: March 28, 2023

Abstract

This study is focused on the approximate solution for the class of stochastic delay differential equations. The techniques applied involve the use of Caratheodory and Euler Maruyama procedures which approximated to stochastic delay differential equations. Based on the Caratheodory approximate procedure, it was proved that stochastic delay differential equations have unique solution and established that the Caratheodory approximate solution converges to the unique solution of stochastic delay differential equations under the Cauchy sequence and initial condition. This Caratheodory approximate procedure and Euler method both converge at the same rate. This is achieved by replacing the present state with past state. The existence and uniqueness of an approximate solution of the stochastic delay differential equation were shown and the approximate solution to the unique solution was also shown.

Keywords

Approximate Solution, Differential Equations, Techniques, Stochastic Differential Equation, Existence, Uniqueness, Approximate Procedure

1. Introduction/Background of the Study

Stochastic differential equations have many applications in science and engineering. One of the important applications is the stochastic representation of solution to ordinary differential equations. A lot of monographs, conference papers and journal articles have been written by various authors on the Caratheodory's, Euler-Maruyama approximate solution to stochastic delay differential equation which is usually represented by means of ordinary differential equation. This paper therefore aims at obtaining the error between the approximate solution and the accurate solution. It would also investigate the effect of Caratheodory's scheme to the stochastic delay differential equation. To achieve this, the work will be among others;

1) Find the error (difference) between approximation solution and accurate solution of stochastic delay differential equation.

2) Establish the difference between approximate solutions and the unique solution to stochastic delay differential, under non uniform Lipschits condition and non linear growth condition.

3) Compare Caratheodory approximation and Euler-Manyama approximate method of convergence.

4) Construct to show that approximate solution converges to a unique solution.

2. Related Literature

Stochastic differential equations have many applications in science and engineering. One of the important applications is the stochastic representation of solution to ordinary differential equations which includes stochastic approximate solution. An approximate solution is one of the essential concepts in the study of stochastic differential equations. A solution is said to be approximated where there is difficulty in finding exact solution or analytical solution. The approximate procedure is known as Caratheodory's approximate procedure. Approximate equations are defined on partitions of time interval and their coefficients are Taylor approximations of the coefficients of the equation. The Euler-Maruyama method was developed as one of the powerful numerical methods for the stochastic differential delay equations with Markovian switching (SDDEWMS).

[1] considered some class of control system governed by the neutral stochastic functional differential equations ith unbounded delay and studied the approximation controllability of the systems. [2] established the difference between an approximate solution and an accurate/exact solution for a stochastic differential delay equation where the approximate solution as were called Caratheodory's, was constructed from successive approximation. [2] obtained the difference between approximate solution and accurate solution for the stochastic differential equations where the approximate solution is called Caratheodory's approximate solution, which has been constructed by successive approximation. [2] obtained the difference between approximate solution and accurate solution for the stochastic differential equations where the approximate solution is called Caratheodory's approximate solution which has been constructed by successive approximations. [3] presented result on an analytic approximate method for the class of stochastic differential equations with coefficients that do not satisfy the Lipschitz and linear growth conditions but behaved like a polynomial. Furthermore, equations from this class have unique solutions with bounded moments and their coefficients satisfy polynomial conditions. [4] considered the stochastic

differential equation and defined the Caratheodory's approximate solution of stochastic differential delay equation.

[5] discussed Caratheodory's and Euler-Maruyama's approximation solutions to stochastic differential delay equation.

To make the theory more understandable, we impose the non-uniform Lipschitz condition and non-linear growth condition. The Euler method discretisation has an optimal strong convergence rate and [5] established Caratheodory's and Euler approximate solutions to stochastic differential delay equation.

Consider the stochastic delay equation

$$\mathrm{d}x(t) = \alpha\left(x(t), x(t-r)\right)\mathrm{d}t + \beta\left(x(t), x(t-r)\right)\mathrm{d}B(t),$$

for $t_0 \le t \le T$ with initial value

$$x(t_0) = x_0 \in L_1^2 \tag{1}$$

where $\alpha\left(x(t)\right)$ is called the drift coefficient and $\beta\left(x^t\right)$ is the diffusion coefficient.

$\beta(t)$ is a Brownian noise that defines the randomness of the physical system and it is often called the white noise.

The Brownian noise is the simplest intrinsic noise term that adequately model Browian motion. The integral form of (1) is

$$x(t) = x_0 + \int_0^t \alpha\left(x(s), x(t-r)\right)\mathrm{d}s + \int_0^t \beta\left(x(s), x(t-r)\right)\mathrm{d}B(s) \tag{2}$$

The first integral in (1) is a voltera integral equation, and second integral is an **Ito** stochastic integral with respect to the Brownian motion $= \{\beta_t, t \ge 0\}$.

However the Lipschitz condition only guarantees the existence and uniqueness of the solution, then the equation can be solved implicitly. Therefore we often seek the approximate solution rather than the accurate solution.

[5] discussed Caratheodory's and Euler-Maruyama's approximation solutions to stochastic differential delay equation. To make the theory more understandable, we impose the non-uniform Lipschitz condition and non-linear growth condition. [6] considered the class of semi linear stochastic evolution equation with delays and proved that the Caratheodory's approximate solution converges to the solution of stochastic delay evolution equations. [7] obtained the estimate on difference between the Caratheodory approximate solution $x_n^{(t)}$ and the unique solution x^t to the stochastic differential delay equation, and he obtained the estimate on difference between the Caratheodory approximate solution $x_n^{(t)}$ and the unique solution x^t to the stochastic differential delay equation. [8] discussed the Caratheodory approximate solution for the class of doubly perturbed stochastic differential equation. [9] showed that stochastic differential equations with jumps and non-lipschitz coefficients have pair wise unique strong solutions by the Euler approximation method. [10] developed the approximate analytical solution of fractional delay differential equations of the initial value linear and nonlinear boundary problems. [11] considered the Caratheodory's approximate solution of stochastic functional differential Equation (SFDEs) and obtained the existence theorem for stochastic functional differential

equations. Thus, Caratheodory's approximation procedure, extended the Caratheodory's approximate scheme to the case of stochastic differential scheme to the case of stochastic differential delay equations.

[12] studied Euler-Maruyama method for the stochastic differential delay equations with Markovian switching (SDDEWMS). Approximate equations are defined on partitions of time interval and their coefficients are Taylor approximations of the coefficients of the equation. The Euler Maruyama method was developed as one of the powerful numerical methods for the stochastic differential delay equations with Markovian switching (SDDEWMS). [12] worked on an approximate analytical method and introduced new variational iteration method.

3. Methodology

We discuss the following types of approximate solutions, these are

1) Caratheodory approximate method;

2) Euler-Mmaruyama approximate method, together with properties of Brownian noise or wiener processes.

3.1. Caratheodory Approximate Method

Caratheodory's approximate solutions, is said to be an approximate solution, if for every integer $n \geq 1$, with $x_q(t) = x_0$, for on $t_0 \leq t \leq T$, and

$$\begin{aligned} x_q(t) = x_0 &+ \int_{t_0}^{t} \alpha\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-t(s)-\frac{1}{q}\right)\right)\mathrm{d}s \\ &+ \int_{t_0}^{t} \beta\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-\tau(s)-\frac{1}{q}\right)\right)\mathrm{d}b(t) \end{aligned} \tag{3}$$

For all $q \in \mathbb{N}$.

We compute $x_n(t)$ step by step on the intervals $\left[t_0, t_0+\frac{1}{q}\right]$, $\left[t_0+\frac{1}{q}, \frac{t_0 2}{q}\right]$; for $t_0 \leq t \leq t_0 \frac{1}{q}$, $x_n(t)$ can be compute by

Step I

$$\begin{aligned} x_q(t) = x_0 &+ \int_{t_0}^{t} \forall\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-t(s)-\frac{1}{q}\right)\right)\mathrm{d}s \\ &+ \int_{t_0 \frac{1}{q}}^{t} \beta\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-\tau(s)-\frac{1}{q}\right)\right)\mathrm{d}b(t) \end{aligned} \tag{4}$$

for $n-1$ and $q=1$.

Step II

For $\left[t_0+\frac{2}{q}<t\right]$

$$\begin{aligned} x_q(t) = x_q\left(t_0+\frac{2}{q}\right) &+ \int_{t_0+\frac{2}{q}}^{t} \forall\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-t(s)-\frac{2}{q}\right)\right) \\ &+ \int_{t_0+\frac{1}{q}}^{t} \beta\left(x_q\left(s-\frac{1}{q}\right), x_q\left(s-\tau(s)-\frac{2}{q}\right)\right)\mathrm{d}b(t) \end{aligned} \tag{5}$$

3.2. Euler-Maruyama's Approximate Method

Euler-Maruyama approximate solution is said to be an approximate solution of equation if for every integer $n-1$, $x_n(t_0)=x_0$, for

$$t_0+\frac{k-1}{n}<t\le\left(t_0+\frac{k}{n}\right)\wedge T,\quad k=1,2,\cdots$$

and

$$x_n(t)=x_0+\int_{t_0}^{t}\forall\left(x_n(s),s\right)\mathrm{d}s+\int_{t_0}^{t}\beta\left(x_n(s),s\right)\mathrm{d}b(s) \tag{6}$$

For $t_0+\frac{k-1}{n}$, we compute $x_n(t)=x_0$ as

$$x_n(t)=x_n\left(t_0+\frac{k-1}{n}\right)+\int_{t_0+\left(k-\frac{1}{n}\right)}^{t}x_n\left(t_0+\frac{k-1}{n},s\right)\mathrm{d}s$$
$$+\int_{t_0+\frac{k-1}{n}}^{t}\beta\left(x_n\left(t_0+k-\frac{1}{n},s\right)\right)\mathrm{d}B(s)$$

For $t_0+\frac{k}{n}$

$$x_n(t)=x_n\left(t_0+\frac{k}{n}\right)+\int_{t_0+\frac{k}{n}}^{t}\alpha\left(x_n\left(t_0+\frac{k}{n}\right),s\right)\mathrm{d}s+\beta\left(x_n\left(t_0+\frac{k}{n}\right),s\right)\mathrm{d}B(s) \tag{7}$$

Thus, Caratheodory's and Euler Maruyama approximation procedures converge to stochastic delay deferential equation.

3.3. Properties of Brownian Noise or Weiner Process

1) $B_0=0$,

2) Path of Weiner process is continuous functions of $t\in[0,0]$,

3) Increments of Weiner process on non-overlapping intervals are independent *i.e.* for $(s_1,t_1)\cap(s_2,t_2)=\varnothing$, the random variables $W_{t_2}-W_{s_2},W_{t_1}-W_{s_1}$ are independent,

4) Expectation $EW_t=0$,

5) For any $t_1\cdots t_n$, the random vector $\left(W_{t_1}\cdots W_{t_n}\right)$ is Gaussian.

4. Approximate Solutions

We now discuss Caratheodory's approximate solutions to stochastic delay differential equations and show that the solution $x_n(t)$ of delay equation approximates the solution $x(t)$ of the original equation. The idea is to replace the resent state $x(t)$ with the past $x\left(t-\frac{1}{q}\right)$ to obtain the equation.

$\mathrm{d}x(t)=\alpha\left(x(t)\right)(x-t)\mathrm{d}t+\beta\left(x(t)\right)\mathrm{d}b(t)\quad t\in[t_0,T]$ with initial condition.

Now replace the present state $x(t)$ by its past $x\left(t-\frac{1}{q}\right)$, we have

$$\mathrm{d}x_n(t)=x\left(x_n\left(t-\frac{1}{n}\right),t\right)\mathrm{d}t+\beta\left(x_n\left(t-\frac{1}{n}\right),t\right)\mathrm{d}B(t),\ \text{for}\ t\in[t_0,T] \tag{8}$$

also replace the past $x(t-\tau(t))$ with $x\left(t-\tau(t)-\frac{1}{n}\right)$

$$\begin{aligned}\mathrm{d}x_n(t)=\alpha&\left(x_n\left(t-\frac{1}{n}\right),x_n\left(t-T(t)-\frac{1}{n}\right)\right)\\&+\beta\left(x_n\left(t-\frac{1}{n}\right),x_n\left(t-\tau(t)-\frac{1}{n}\right)\right)\mathrm{d}B(t)\end{aligned}\tag{9}$$

Now R^d-valued stochastic process $x(t)$ on the interval $t_0\le t\le T$ is called a solution of the equation if the following properties are satisfied;

1) It is continuous and $\{x(t)\}_{t_0\le t\le T}$ is F_t-adapted,

2) $\{\alpha(x(t),t)\}\in L^2([t_0,T],R^d)$ and $\{\beta(x(t),t^2)\}\in L^2([t_0,T],R^d\times m)$,

3) $x(t_0)=x_0$ and, for every $t_0\le t\le T$ and $x(t)=x_0+\int_{t_0}^{t}\alpha(x(s),s)\mathrm{d}s+\int_{t_0}^{t}\beta(x(s),s)\mathrm{d}B(s)$.

A solution $x(t)$ is said to be unique if any other solution $x(t)$ is similar (indistinguishable) from it, that is $p\{x(t)=x(t)\text{ for all }t_0\le t\le T\}=1$.

Now we consider the stochastic differential delay Equation (1) with initial details $x(t_0)=\alpha_0$. We define the Caratheodory approximation as follows. Let $n\ge1$ define $x_n(t)=x_0$, for every integer $n\ge1$ on $t_0-1\le t\le t_0$, and $x_n(t)=x_0+\int_{t_0}^{t}\alpha\left(x_n\left(s-\frac{1}{q}\right),x_n(s-t)\right)\mathrm{d}s+\int_{t_0}^{t}\beta\left(x_n\left(s-t(s)-\frac{1}{q}\right),s\right)\mathrm{d}s$, for $t_0\le t\le T$.

We compute $x_n(t)$ step by step on the intervals $\left[t_0,t_0+\frac{1}{m}\right]$, $\left(t_0+\frac{1}{q},t_0+\frac{2}{m}\right]$.

For $t_0\le t\le t_0+\frac{1}{m}$, $x_n(t)$ can be constructed by $x_n(t)=x_0+\int_{t_0}^{t}\alpha(x_0,s)\mathrm{d}s+\int_{t_0}^{t}\beta(x_0,s)\mathrm{d}B(s)$ as follows,

Step I

For $\left[t_0+\frac{1}{q}\right]$, we have

$$\begin{aligned}x_q(t)=x_q\left(t_0+\frac{1}{m}\right)&+\int_{t+\frac{1}{q}}^{t}\alpha\left(x_q\left(s-\frac{1}{q}\right),x_q(s-t)\right)\mathrm{d}s\\&+\int_{t_0+\frac{1}{q}}^{t}\beta\left(x_q\left(s-\tau(s)-\frac{1}{q}\right),s\right)\mathrm{d}B(s)\end{aligned}\tag{10}$$

Step II

For $\left[t_0+\frac{1}{q}\right]$, we have

$$\begin{aligned}x_q(t)=x_q\left(t_0+\frac{2}{q}\right)&+\int_{t+\frac{1}{q}}^{t}\alpha\left(x_q\left(s-\frac{1}{q}\right),x_q(s-t)\right)\mathrm{d}s\\&+\int_{t_0+\frac{1}{q}}^{t}\beta\left(x_q\left(s-t(s)-\frac{2}{q}\right),s\right)\mathrm{d}B(s)\end{aligned}\tag{11}$$

5. Theorems/Lemma

5.1. Lemma 1 [7]

Suppose that $\left|f(x,t)\right|^2 \vee \left|g(x,t)\right|^2 \le k(1+k)$, for all $n \ge 1$ and $t_0 \le s < t \le T$ with $t - s \le 1$.

$$E\left|x_n(t) - x_n(s)\right|^2 \le C_2(t-s),$$

where $C_2 = 4k(1+C_1)$ and

$$c_1 = c_1 + 3E\left|x_q I^2\right| e^3 k(T-t_0)(T-t_0+1).$$

5.2. Lemma 2 [7]

If $\left|f(x,t)\right|^2 \vee \left|g(x,t)\right|^2 \le k(1+k\alpha I^2)$ for all $n \ge 1$,

$$\sup_{t_0 \le t \le T} E\left|x_n(t)\right|^2 \le C_1 = \left(1+3E\left|x_0\right|^2\right) e^{3k(t-t_0)(T-t_0+1)} \quad \text{for all } t_0 \le t \le T$$

Proof

Let $n \ge 1$ and if $x_n(t)$ with condition such that $\left|f(x,t)\right|^2 \vee \left|g(x,t)\right|^2 \le k\left(1+|x|^2\right)$.

Then $\left\{x_n(t)\right\}_{t_0 \le t \le T} \in M^2\left([t_0,T], R^2\right)$.

By the Caratheodory approximate solution we have

$$\left|x_n(t)\right|^2 \le 3\left|x_0\right|^2 + 3\left|\int_{t_0}^t f\left(x_n\left(s-\frac{1}{m}\right), x_n(s-r)\right)\mathrm{d}s\right|^2 + 3\left|\int_{t_0}^t g\left(x_n\left(s-\frac{1}{m}\right), x_n(s-r)\right)\mathrm{d}B(s)\right|^2$$

Using the Holder inequality, we have

$$\begin{aligned} E\left|x_n(t)\right|^2 &\le 3E\left|x_0\right|^2 + 3(t-t_0)E\int_{t_0}^t \left|f\left(x_n\left(s-\frac{1}{m}\right), x_n(s-r)\right)\right|^2 \mathrm{d}s \\ &\quad + 3E\int_{t_0}^t \left|g\left(x_m\left(s-\frac{1}{m}\right), x_m(s-r)\right)\right|^2 \mathrm{d}s \\ &\le 3E\left|x_0\right|^2 + 3k(T-t_0+1)\int_{t_0}^t \left[1 + E\left|x_n\left(s-\frac{1}{m}\right)\right|^2\right]\mathrm{d}s \\ &\le 3E\left|x_0\right|^2 + 3k(T-t_0+1)\int_{t_0}^t \left[1 + \sup_{t_0 \le t \le T} E\left|x_n(r)\right|^2\right]\mathrm{d}s \end{aligned}$$

for all $t_0 \le t \le T$.

Then

$$1 + \sup_{t_0 \le t \le T} E\left|x_n(r)\right|^2 \le 1 + 3E\left|x_0\right|^2 + 3k(T-t_0+1)\int_{t_0}^t \left[1 + \sup_{t_0 \le t \le T} E\left|x_n(r)^2\right|\right]\mathrm{d}s$$

By the Gronwall inequality

$$\Rightarrow 1 + \sup_{t_0 \le r \le t} E\left|x_n(r)\right|^2 \le \left(1+3E\left|x_0\right|^2\right) e^{3k(t-t_0)(T-t_0+1)}$$

for all $t_0 \le t \le T$.

5.3. Theorem [7]

Suppose $f(x,t)$ and $g(x,t)$ are continuous. Let x_0 be a bounded R^d-valued, F-measurable random variable. Suppose further that there exists a continuous increasing concave function $C: R_+ \to R_+$ such that

$$\int_0^t \frac{\mathrm{d}u}{c(u)} = \infty \tag{12}$$

for all $x, y \in R^d$, $t_0 \le t \le T$.

$$\left|F(x,t) - \bar{f}(y,t)\right|^2 \vee \left|g(x,t) - \bar{g}(g,t)\right|^2 \le c\left(\left|x-y\right|^2\right) \tag{13}$$

Then $\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}B(t)$, has a unique solution $x(t)$ and Caratheodory approximate solutions $x_n(t)$ converges to $x(t)$. In the sense of $E\left(\sup_{t_0 \le t \le T}\left|x_m(t) - x(t)\right|^2\right) = 0$.

Proof

We divide the proof into two parts

1) Uniqueness

Let $x(t)$ and $\hat{x}(t)$ be two solution of $\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}B(t)$. From equation, we show that

$$E\left(\sup_{t_0 \le t \le T}\left|x(r) - \hat{x}(r)\right|^2\right) \le 2(T - t_0 + 4)\int_{t_0}^t Ek\left(\left|x(s) - \hat{x}(s)\right|^2\right)\mathrm{d}s \tag{14}$$

Since $k(*)$ is concave, by Jensen inequality we have

$$Ek\left(\left|x(s) - \hat{x}(s)\right|^2\right) \le k\left(E\left|x(s) - \hat{x}(s)\right|^2\right) \le k\left[E\left(\sup_{t_0 \le t \le T}\left|x(r) - \hat{x}(r)\right|^2\right)\right]$$

For $\varepsilon > 0$,

$$E\left(\sup_{t_0 \le t \le T}\left|x(r) - \hat{x}(r)\right|^2\right) \le \varepsilon + 2(T - t_0 + 4)\int_{t_0}^t k\left[E\left(\sup_{t_0 \le t \le T}\left|x(r) - \hat{x}(r)\right|^2\right)\right]\mathrm{d}s$$

for all $t_0 \le t \le T$.

Let $G(r) = \int_1^r \frac{\mathrm{d}u}{c(u)}$ on $r > 0$,

And let $\frac{1}{G}(*)$ be the inverse function of $G(*)$

From Equation (1) we have

$$\lim_{e \to 0} G(\varepsilon) = \varpi \quad \text{and} \quad DOM\left(\frac{1}{G}\right) = \left(-\varpi, G(\varpi)\right).$$

for $\varepsilon > 0$,

$$E\left(\sup_{t_0 \le r \le T}\left|x(r) - \hat{x}(r)\right|^2\right) \le \frac{1}{g}\left[G(\varepsilon) + 2(T - t_0 + 4)(T - t_0)\right]$$

Letting $\varepsilon > 0$, we have

$$E\left(\sup_{t_0 \le r \le T}\left|x(r)-\hat{x}(r)\right|^2\right)=0 \tag{15}$$

We have $x(t)=\hat{x}(t)$ for all $t_0 \le t \le T$.

2) Convergence

To show the existence then convergence,

Let $x(t_0)=x_0$, and $n=1,2,3,\cdots$, we define a sequence

$$x_n(t)=x_0+\int_{t_0}^{t}\alpha\left(x_{n-1}(s),s\right)\mathrm{d}s+\int_{t_0}^{t}\left(x_{n-1}(s),s\right)\mathrm{d}B(s) \tag{16}$$

for $t\in[t_0,T]$; since $x_0(t_0)\in M^2\left([t_0,T],R^d\right)$.

By induction, we have that

$$x_n(t_0)\in M^2\left([t_0,T],R^d\right),$$

such that

$$E\left|x_n(t)\right|^2 \le C_1+3k(T+1)\int_{t_0}^{t}E\left|x_{n-1}(s)\right|\mathrm{d}s(s)$$

where $C_1=3E\left|x_0\right|^2+3kT(T+1)$; that is for any $k\ge 1$,

$$\begin{aligned}\max_{1\le n\le k}E\left|x_n(t)\right|^2 &\le C_1+3k(T+1)\int_{t_0}^{t}\max_{1\le n\le k}E\left|x_{n-1}(s)\right|^2\mathrm{d}s\\ &\le C_1+3k(T+1)\int_{t_0}^{t}\left(E\left|x_n\right|^2\mathrm{d}s(s)\max_{1\le n\le k}E\left|x_n(s)\right|^2\mathrm{d}s\right)\\ &\le C_2+3k(T+1)\int_{t_0}^{t}\max_{1\le n\le k}E\left|x_n(s)\right|^2\mathrm{d}s.\end{aligned}$$

where $C_2=C_1+3kT(T+1)E\left|x_0\right|^2$.

By the Gronwall inequality, we have

$$\max_{1\le n\le k}E\left|x_n(t)\right|^2 \le C_2\mathrm{e}^{3kT(T+1)}$$

Since k is constant, we have

$$E\left|x_n(t)\right|^2 \le C_2\mathrm{e}^{3kT(T+1)},\ \text{for all}\ t_0\le t\le T\ \text{ and }\ n\ge 1$$

Since

$$\left|x_1(t)-x(t_0)\right|^2=\left|x_1(t)-x_0\right|^2\le 2\left|\int_{t_0}^{t}\alpha(x_0,s)\mathrm{d}s^2\right|=2\left|\int_{t_0}^{t}\beta(x_0,s)\mathrm{d}\beta(s)^2\right|$$

Taking the expectation we have

$$E\left|x_1(t)-x(t_0)\right|^2\le 2k(t-t_0)^2\left(1+E\left|x_0\right|^2\right)+2k(t-t_0)\left(1+E\left|x_0\right|^2\right)\le C \tag{17}$$

where $C=2kc(t-t_0+1)(t-t_0)\left(1+E\left|x_0\right|^2\right)$ for $n\ge 0$

$$E\left|x_{n+1}(t)-x_n(t)\right|^2\le\frac{C\left[M(t-t_0)\right]}{n!},\ \text{for}\ t_0\le t\le T, \tag{18}$$

where $M=2\bar{k}(T-t_0+1)$.

By (17) we see that (18) holds when $n=0$

We now show (18) for $(n+1)$

$$\begin{aligned}\left|x_{n+2}(t)-x_{n+1}(t)\right|^2 &\le 2\left|\int_{t_0}^{t}\left[\alpha\left(x_{n+1}(s),s\right)-\alpha\left(x_n(s),s\right)\right]\mathrm{d}s\right|\\ &\quad+2\left|\int_{t_0}^{t}\left[\beta\left(x_{n+1}(s),s\right)-\beta\left(x_n(s),s\right)\right]\mathrm{d}B(s)\right|^2\end{aligned} \tag{19}$$

$$E\left|x_{n+2}(t)-x_{n-1}(t)\right|^2 \le 2\bar{k}(t-t_0+1)E\int_{t_0}^{t}\left|x_{n+1}(s)-x_n(s)\right|^2 \mathrm{d}s$$
$$\le M\int_{t_0}^{t}E\left|x_{n+1}(s)-x_n(s)\right|^2 \mathrm{d}s$$
$$\le M\int_{t_0}^{t}\frac{C\left[M(s-t_0)\right]^n}{n!}\mathrm{d}s$$
$$=\frac{C\left[M(t-t_0)\right]^{n+1}}{(n+1)!}$$

so (18) holds for $n+1$. Hence, by induction, (18) holds for all $n \ge 0$, by replacing "*n*" with $n-1$ we have

$$\sup_{t_0=t\le T}\left|x_{n+1}(t)-x_n(t)\right|^2 \le 2R(T-t_0)\int_{t_0}^{T}\left|x_n(s)-x_{n-1}(s)\right|^2 \mathrm{d}s + 2\sup_{t_0\le t\le T}\left|\int_{t_0}^{T}\left[g(x_n(s),s)-g(x_{n-1}(s),s)\right]\mathrm{d}B(s)\right|^2 \tag{20}$$

By taking the expectation we have

$$E\left(\sup_{t_0=t\le T}\left|x_{n+1}(t)-x_n(t)\right|^2\right) \le 2\hat{k}(T-t_0+4)\int_{t_0}^{T}E\left|x_n(s)-x_{n-1}(s)\right|^2 \mathrm{d}s$$
$$\le 4M\int_{t_0}^{T}\frac{C\left[M(s-t_0)\right]^{n-1}}{(n+1)!}\mathrm{d}s$$
$$=\frac{4C\left[M(T-t_0)\right]^n}{n!}$$

Thus

$$\left\{\sup_{t_0=t\le T}\left|x_{n+1}(t)-x_n(t)\right| > \frac{1}{2n}\right\} \le \frac{4C\left[4M(T-t_0)\right]^n}{n!} \tag{21}$$

Since

$$\sum_{n=0}^{\varpi}\frac{4C\left[4M(T-t_0)\right]^n}{n!} < +\infty$$

By the Borel Cantelli lemma yields that for almost all $w \in \mathcal{R}$, there exists a positive integer $n_0 = n_0(\varpi)$ such that

$$\sup_{t_0\le t\le T}\left|x_{n+1}(t)-x_n(t)\right| \le \frac{1}{2^n} \tag{22}$$

For *n*, Z *n*, with the probability 1, the partial sums

$$x(t_0)+\sum_{i=0}^{n-1}\left[x_{i+1}(t)-x_i(t)\right] = x_n(t)$$

i.e.

$$x_n(t) = x_0 + \sum_{i=0}^{n-1}\left[x_{i+1}(t)-x_i(t)\right] \tag{23}$$

Since $x_n(t_0) = x_0$ the uniformly convergent in *E E* [0, *T*], since $x(t)$ is continuous and f_t adapted and also a Cauchy sequence is L^2, as $x_n(t) \to x(t)$ in L^2, letting $n \to \varpi$ in above equation, gives

$$E\left|x(t)\right|^2 \le C_2 e^{ekT(T+1)}$$

for all $t_0 \le t \le T$, for $x(t) \in M^2\left([t_0,T],R^d\right)$.

Since $\in(\mathrm{u})$ is concave and increasing, there exist a positive number "*b*" such that

$$C(u) \le b(1+u), \text{ on } u \ge 0,$$

Let $a = \sup_{t_0 \le t \le T}\left(\left|f(0,t)\right|^2 \vee \left|g(0,t)\right|^2\right) < \infty$, then

$$\begin{aligned}&\left(\left|f(x,t)\right|^2 \vee \left|g(x,t)\right|^2\right)\\ &\le 2\left(\left|f(0,t)\right|^2 \vee \left|g(0,t)\right|^2\right) + 2\left(\left|f(x,t) - f(0,t)\right|^2 \vee \left|g(x,t) - g(0,t)\right|^2\right)\\ &\le 2a + 2c\left(\left|x\right|^2\right) \le 2a + 2b\left(1+\left|x\right|^2\right) \le 2(a+b)\left(1+\left|x\right|^2\right).\end{aligned}$$

By the linear growth condition and $c = 2(a+b)$, we conclude that the approximate solution $x_n(t)$ converges to $x(t)$ in the sense of stochastic delay differential equation, *i.e.* Equation (1) hence the proof is completed.

6. Summary/Conclusions

The concept of approximations has been extended to stochastic delay differential equation. This was achieved by replacing the present state with past state through Caratheodory's scheme. The Caratheodory's and Euler-Maruyama approximation procedures were compared and the difference and error between them were obtained. The two procedures were shown to converge to stochastic delay differential equation.

The work has proved the existence and uniqueness of an approximate solution of the stochastic delay differential equation using Brownian white noise of the Ito-type. Relevant lemma and theorem associated with the approximate solution method are included. By linear growth condition it was shown that the approximate solution converges to the unique solution in the sense of stochastic delay equation.

Acknowledgements

We sincerely acknowledge the ingenuity of all the researchers who have worked in this area' this paper "A comparative survey of an approximate solution method for stochastic delay differential equations" was done by Donatus Anonwa and Christian E Emenonye. The two did the work separately and met to combine their ideas. Donatus Anonwa prepared the final manuscript which was edited and later proof read by Christian Emenonye. We sincerely acknowledge the reviewers and the editorial team of this Journal for their competence in the discharge of their work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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