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Existence Results for Systems of Nonlinear Caputo Fractional Differential Equations

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Abstract

We aim, in this work, to demonstrate the existence of minimal and maximal coupled quasi-solutions for nonlinear Caputo fractional differential systems with order $q \in (1,2)$. Our approach is based on mixed monotone iterative techniques developed under the concept of lower and upper quasi-solutions. Our results extend those obtained for ordinary differential equations and fractional ones.

Keywords

Mixed Quasi-Monotone Property, Coupled Lower and Upper Solutions, Monotone Method, Nonlinear Fractional Differential System

1. Introduction

Fractional differential equations and systems are appearing in a variety of scientific and engineering branches being mathematical modelling in many fields, namely, in Physics, electromagnetic, acoustic, viscoelasticity, electrochemistry, economics, signal and image processing, control theory, etc. For details, one can see [1]-[8]. Many works treated the problems concerning fractional differential equations or systems by using several methods. Essentially, the authors used the techniques of nonlinear analysis to study these problems. We quote the power series method [9], the compositional method [9], the variational Lyapunov method [10], the Adomian decomposition method [11], the generalized monotone method [12], by the means of fixed point theorems [13]. Recently, the method combining the method of lower and upper solutions, and the monotone iterative techniques were frequently, served for the study of both fractional differential equations or systems involving Caputo or Riemann-Liouville Derivatives, especially for the order q in $(0,1)$, we refer readers to the cited works as examples

[14]-[20]. This restriction is due to that the comparison result has been established only for the order $0<q<1$ until the works of Shi [21] and Al-Refai [22] which treated the case $1<q<2$. In [23], Ramirez and Vatsala were interested in the differential equation involving the Riemann-Liouville fractional derivative

$$D^q u(t) = f(t, u(t)) \text{ in } (a,b)$$

with periodic boundary conditions

$$u(a) = (t-b)^{1-q} u(t)\big|_{t=b}$$

where $q \in (0,1]$, $f \in \mathcal{C}([a,b]\times\mathbb{R},\mathbb{R})$. The authors showed the existence of minimal and maximal solutions. Their method was grounded in developing a monotone method and using lower and upper solutions.

In [24], Al-Refai and Hajji studied the boundary value problem involving the Caputo fractional derivative with order $q \in (1,2)$,

$$D^q u(x) + g(x, u(x)) = 0,\ 0<x<1$$

$$u(0) = \alpha,\ u(1) = \beta,$$

where the nonlinearity g is belonging to $\mathcal{C}([0,1]\times\mathbb{R},\mathbb{R})$. The authors stated existence and uniqueness of solution for the above problem under the assumption g is strictly decreasing with respect to the second variable, and $\frac{\partial g}{\partial u}$ is bounded below in some given sector.

In [25], Denton and Vatsala established new comparison results of the scalar Riemann-Liouville fractional differential equation with order $q \in (0,1)$, and proved that the system

$$D^q u(x) = f(x, u(x)) \text{ in } (a,b)$$

$$u(a) = (x-b)u(x)\big|_{x=b},$$

admits minimal and maximal solutions, where the nonlinearity $f \in \mathcal{C}([a,b]\times\mathbb{R}^n,\mathbb{R}^n)$. The authors suppose that the function f satisfies quasi-monotone property. These results are the generalization of the results of MCRae [26]. Recently, inspired by the works of Cui [27] [28], Toumi, in [29], was concerned with the following finite system of nonlinear fractional differential equations

$$D^q u(x) + f(x, u(x)) = 0,\ x \in (0,1) \tag{1}$$

$$u(0) - \alpha u'(0) = \lambda,\ u(1) + \beta u'(1) = \mu, \tag{2}$$

where D^q is the Caputo fractional derivative with order $q \in (1,2)$. The function $f \in \mathcal{C}([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$, $\lambda, \mu \in \mathbb{R}^n$, $\alpha, \beta \in (\mathbb{R}_+)^n$. The author established the existence of quasi-solutions for the problem (1) and (2). Quasi-solutions are understood in the sense of Definition 7 given below. More precisely, under the hypotheses on the nonlinearity f related to the Green kernel associated with the scalar problem (1) and (2), the author constructs a pair of sequences of coupled upper and lower quasi-solutions converging uniformly to extremal quasi-solutions.

Motivated by the previous papers, we aim in this work to prove the existence of extremal quasi-solutions for (1) and (2) under more general conditions on the nonlinearity *f*.

This paper is organized as follows. Section 2, provides some necessary preliminaries, especially, the new comparison result for the Caputo fractional differential equation. In Section 3, we prove the existence of extremal quasi-solutions of (1) and (2). We end this work with two examples illustrating our results.

2. Preliminary Results

We start this section by recalling definitions and properties related to the Caputo fractional derivatives, then, we state the new positivity result.

Definition 1. A real function *f* is belonging to the space $\mathcal{C}_q, q \in \mathbb{R}$, if there exists $r > q$, such that $f(x) = x^r h(x)$, where the function $h \in \mathcal{C}([0,+\infty),\mathbb{R})$, and *f* is in $\mathcal{C}_q^n$, if $f^{(n)} \in \mathcal{C}_q, n \in \mathbb{N}$.

Definition 2 (See [9] [30]). The fractional integral of order $q > 0$ for a continuous function $u:[0,+\infty) \to \mathbb{R}$ is defined as

$$I^q u(x) = \frac{1}{\Gamma(q)} \int_0^x (x-s)^{q-1} u(s) \mathrm{d}s, \ x > 0,$$

where Γ is the Euler Gamma function.

Definition 3 (See [9] [30]). For $q > 0, n-1 < q < n, n \in \mathbb{N}$. The Caputo fractional derivative of order *q* for a function $u \in \mathcal{C}_{-1}^n$ is defined as

$$D^q u(x) = \frac{1}{\Gamma(n-q)} \int_0^x (x-s)^{n-q-1} u^{(n)}(s) \mathrm{d}s = I^{n-q} u^{(n)}(x).$$

Next, recall the following

Lemma 1. Let $q > 0$ and *m* be the smallest integer greater than or equal to *q*. Let $u \in \mathcal{C}^m([0,1],\mathbb{R})$. Then, we have

1) $D^q I^q u = u$.

2) $D^q u(x) = 0$, if and only if, $u(x) = \sum_{k=0}^{m-1} c_k x^k$, where $c_k = \frac{u^{(k)}(0^+)}{k!}$ for $k = 0,1,\cdots,m-1$.

3) $D^q x^{m-k} = 0$, for $k = 1,2,\cdots,m$.

For the proof of the above Lemma, and more details concerning the fractional derivative, one can refer to [30] [31].

Now we introduce below the nonlinear fractional differential equation

$$D^q u(x) + F(x,u) = 0, \ x \in (0,1) \tag{3}$$

with boundary value conditions

$$u(0) - \alpha u'(0) = \lambda, \ u(1) + \beta u'(1) = \mu, \tag{4}$$

where $q \in (1,2), \lambda, \mu \in \mathbb{R}, \alpha, \beta \in \mathbb{R}_+$, and the function *F* is in $\mathcal{C}([0,1]\times\mathbb{R},\mathbb{R})$.

Definition 4. A function $v \in \mathcal{C}^2([0,1],\mathbb{R})$ is said a lower solution of (3) and (4) if it satisfies

$$D^q v(x) + F(x,v) \geq 0,\ x \in (0,1) \tag{5}$$

$$v(0) - \alpha v'(0) \leq \lambda,\ v(1) + \beta v'(1) \leq \mu \tag{6}$$

A function $w \in \mathcal{C}^2([0,1],\mathbb{R})$ is said an upper solution of the problem (3) and (4) if w satisfies (5) and (6) with reversed inequalities. In addition, if

$$v(x) \leq w(x),\ \text{ for each } x \in [0,1],$$

then, the lower solution v and the upper solution w are ordered.

Next, we state a comparison result due to Syam and Al-Refai [32]

Lemma 2 (Positivity result). Let $h \in \mathcal{C}^2([0,1],\mathbb{R})$, and $N > 0$. Suppose that h satisfies the following inequalities

$$D^q h(x) - Nh(x) \leq 0,\ x \in (0,1)$$

$$h(0) - \alpha h'(0) \geq 0,\ h(1) + \beta h'(1) \geq 0,$$

where $\alpha, \beta \geq 0$. Then, we have $h(x) \geq 0$ for $x \in [0,1]$, provided that $\alpha(q-1) \geq 1$.

3. Main Results

In the present section, we shall prove the existence of extremal quasi-solutions for the system (1) and (2). For each $i = 1,2,\cdots,n$, let r_i and s_i be nonnegative integers satisfying $r_i + s_i = n-1$. One can split the vector $u \in \mathbb{R}^n$ into $\left(u_i,[u]_{r_i},[u]_{s_i}\right)$. Thus, the Equations (1) and (2) are equivalent to

$$D^q u_i(x) + f_i\left(x, u_i(x), [u(x)]_{r_i}, [u(x)]_{s_i}\right) = 0 \ \text{ in } (0,1) \tag{7}$$

$$u_i(0) - \alpha_i u_i'(0) = \lambda_i,\ u_i(1) + \beta_i u_i'(1) = \mu_i, \tag{8}$$

for each $i = 1,2,\cdots,n$.

Next, we recall that for $v, w \in \mathbb{R}^n$, $v \leq w$ if and only if, $v_i \leq w_i$ for each $i = 1,2,\cdots,n$. Define for $v, w \in \mathcal{C}^2([0,1],\mathbb{R}^n)$ the set

$$[v,w] = \left\{u \in \mathcal{C}^2([0,1],\mathbb{R}^n) : v(x) \leq u(x) \leq w(x), x \in [0,1]\right\}.$$

For the sake of simplicity, we put $\varphi_i(u) = u_i(0) - \alpha_i u_i'(0)$, and $\psi_i(u) = u_i(1) + \beta_i u_i'(1)$, for each $i = 1,2,\cdots,n$.

Now, we introduce a crucial property, known as called quasimonotone.

Definition 5. A function $f \in \mathcal{C}([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$ is said to possess a mixed quasimonotone property if the function $f_i\left(t, u_i, [u]_{r_i}, [u]_{s_i}\right)$ is monotone nondecreasing in $[u]_{r_i}$ and monotone nonincreasing in $[u]_{s_i}$, for each $i = 1,2,\cdots,n$.

Definition 6. Let $v, w \in \mathcal{C}^2([0,1],\mathbb{R}^n)$, the functions v and w are coupled lower and upper quasi-solutions of (7) and (8) if v and w satisfy

$$D^q v_i(x) + f_i\left(x, v_i(x), [v(x)]_{r_i}, [w(x)]_{s_i}\right) \geq 0,\ x \in (0,1) \tag{9}$$

$$\varphi_i(v) \leq \lambda_i,\ \psi_i(v) \leq \mu_i, \tag{10}$$

and

$$D^q w_i(x) + f_i\left(x, w_i(x), [w(x)]_{r_i}, [v(x)]_{s_i}\right) \le 0,\ x \in (0,1) \tag{11}$$

$$\varphi_i(w) \ge \lambda_i,\ \psi_i(w) \ge \mu_i, \tag{12}$$

for each $i = 1, 2, \cdots, n$.

Definition 7. Let $v, w \in \mathcal{C}^2([0,1], \mathbb{R}^n)$. The functions v and w are said a coupled of quasi-solutions of (7) and (8) if v and w satisfy

$$D^q v_i(x) + f_i\left(x, v_i(x), [v(x)]_{r_i}, [w(x)]_{s_i}\right) = 0,\ x \in (0,1)$$

$$\varphi_i(v) = \lambda_i,\ \psi_i(v) = \mu_i,$$

and

$$D^q w_i(x) + f_i\left(x, w_i(x), [w(x)]_{r_i}, [v(x)]_{s_i}\right) = 0,\ x \in (0,1)$$

$$\varphi_i(w) = \lambda_i,\ \psi_i(w) = \mu_i,$$

for each $i = 1, 2, \cdots, n$.

In the sequel, we adopt the following hypotheses

($H1$) $v^0, w^0 \in \mathcal{C}^2([0,1], \mathbb{R}^n)$ are coupled lower and upper quasi-solutions of (7) and (8) satisfying $v^0 \le w^0$ on $[0,1]$.

($H2$) The function f possess the mixed quasimonotone property, and there exists $N \in (\mathbb{R}_+)^n$ such that, for each $i = 1, 2, \cdots, n$, we have

$$f_i\left(x, u_i, [u]_{r_i}, [u]_{s_i}\right) - f_i\left(x, z_i, [u]_{r_i}, [u]_{s_i}\right) \ge -N_i(u_i - z_i) \tag{13}$$

whenever $v^0 \le z \le u \le w^0$ on $[0,1]$.

Theorem 3. Let $\alpha_i(q-1) \ge 1$, for each $i = 1, 2, \cdots, n$. Assume that ($H1$) and ($H2$) are satisfied and suppose that

($H3$) $v^k, w^k, k \ge 1$ is a pair of solutions of

$$D^q v_i^k + f_i\left(x, v_i^{k-1}, [v^{k-1}]_{r_i}, [w^{k-1}]_{s_i}\right) - N_i\left(v_i^k - v_i^{k-1}\right) = 0,\ x \in (0,1) \tag{14}$$

$$\varphi_i(v^{k-1}) \le \varphi_i(v^k) \le \lambda_i,\ \psi_i(v^{k-1}) \le \psi_i(v^k) \le \mu_i, \tag{15}$$

and

$$D^q w_i^k + f_i\left(x, w_i^{k-1}, [w^{k-1}]_{r_i}, [v^{k-1}]_{s_i}\right) - N_i\left(w_i^k - w_i^{k-1}\right) = 0,\ x \in (0,1) \tag{16}$$

$$\varphi_i(w^{k-1}) \ge \varphi_i(w^k) \ge \lambda_i,\ \psi_i(w^{k-1}) \ge \psi_i(w^k) \ge \mu_i, \tag{17}$$

for each $i = 1, 2, \cdots, n$.

Then, we have the following

1) (v^k) and (w^k) are a couple of monotone sequences of ordered coupled of lower and upper quasi-solutions of (7) and (8).

2) The sequences (v^k) and (w^k) converge monotonically and uniformly to the functions v^* and w^*, respectively, with $v^0 \le v^* \le w^* \le w^0$ on $[0,1]$. Moreover, if for $k \ge 0$ and for each $i = 1, 2, \cdots, n$, $\varphi_i(v^k) = \varphi_i(w^k) = \lambda_i$ and $\psi_i(v^k) = \psi_i(w^k) = \mu_i$. Then v^* and w^* are a pair of minimal and maximal quasi-solutions of (7) and (8), in $[v^0, w^0]$.

Proof.

1) First, we show that $\left(v^k\right)_{k\ge1}$ is increasing sequence and $\left(w^k\right)_{k\ge1}$ is decreasing sequence using induction arguments. For $k=1$, we have from (14) and (16),

$$D^q v_i^1 + f_i\left(x, v_i^0, \left[v^0\right]_{r_i}, \left[w^0\right]_{s_i}\right) - N_i\left(v_i^1 - v_i^0\right) = 0 \tag{18}$$

and

$$D^q w_i^1 + f_i\left(x, w_i^0, \left[w^0\right]_{r_i}, \left[v^0\right]_{s_i}\right) - N_i\left(w_i^1 - w_i^0\right) = 0 \tag{19}$$

for each $i = 1, 2, \cdots, n$.

Since v^0 and w^0 are coupled lower and upper quasi-solutions of (7) and (8), we have

$$D^q v_i^0 + f_i\left(x, v_i^0, \left[v^0\right]_{r_i}, \left[w^0\right]_{s_i}\right) \ge 0 \tag{20}$$

and

$$D^q w_i^0 + f_i\left(x, w_i^0, \left[w^0\right]_{r_i}, \left[v^0\right]_{s_i}\right) \le 0 \tag{21}$$

for each $i = 1, 2, \cdots, n$.

Subtracting (18) from (20) and (21) from (19) we obtain

$$D^q\left(v_i^1 - v_i^0\right) - N_i\left(v_i^1 - v_i^0\right) \le 0 \tag{22}$$

and

$$D^q\left(w_i^0 - w_i^1\right) - N_i\left(w_i^0 - w_i^1\right) \le 0. \tag{23}$$

Using the fact that $v^0 \le w^0$ on $[0,1]$, (22) and (23) and the boundary conditions (15) and (17) for $k=1$, we obtain for each $i = 1, 2, \cdots, n$

$$D^q p_i - N_i p_i \le 0 \text{ in } (0,1), \varphi_i(p) = 0, \psi_i(p) = 0 \tag{24}$$

and

$$D^q z_i - N_i z_i \le 0 \text{ in } (0,1), \varphi_i(z) = 0, \psi_i(z) = 0, \tag{25}$$

where $p = v^1 - v^0$ and $z = w^0 - w^1$ on $[0,1]$. Applying Lemma 1 we get $v_i^1 \ge v_i^0$ and $w_i^0 \ge w_i^1$ on $[0,1]$ for each $i = 1, 2, \cdots, n$. So $v^1 \ge v^0$ and $w^1 \le w^0$ on $[0,1]$. Thus, the result is proved for $k=1$. Next, suppose that for $j \in \{1, \cdots, k\}$

$$v^{j-1} \le v^j \text{ and } w^j \le w^{j-1} \text{ on } [0,1]. \tag{26}$$

From (14), we get

$$D^q v_i^{k+1} + f_i\left(x, v_i^k, \left[v^k\right]_{r_i}, \left[w^k\right]_{s_i}\right) - N_i\left(v_i^{k+1} - v_i^k\right) = 0 \tag{27}$$

and

$$D^q v_i^k + f_i\left(x, v_i^{k-1}, \left[v^{k-1}\right]_{r_i}, \left[w^{k-1}\right]_{s_i}\right) - N_i\left(v_i^k - v_i^{k-1}\right) = 0 \tag{28}$$

for each $i = 1, 2, \cdots, n$.

Subtracting (28) from (27), we obtain

$$D^q\left(v_i^{k+1}-v_i^k\right)-N_i\left(v_i^{k+1}-v_i^k\right)$$
$$=f_i\left(x,v_i^{k-1},\left[v^{k-1}\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)-f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)-N_i\left(v_i^k-v_i^{k-1}\right).$$

Now using the induction's hypothesis (26), and the mixed monotone property of *f*, we obtain

$$D^q\left(v_i^{k+1}-v_i^k\right)-N_i\left(v_i^{k+1}-v_i^k\right)$$
$$\le f_i\left(x,v_i^{k-1},\left[v^k\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)-f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)-N_i\left(v_i^k-v_i^{k-1}\right).$$

So, hypothesis (*H*2) yields

$$D^q\left(v_i^{k+1}-v_i^k\right)-N_i\left(v_i^{k+1}-v_i^k\right)\le N_i\left(v_i^{k-1}-v_i^k\right)+N_i\left(v_i^k-v_i^{k-1}\right)=0. \tag{29}$$

Therefore, using (29) and the boundary conditions (15) and (17) for $j=k+1$, we obtain

$$D^q\left(v_i^{k+1}-v_i^k\right)-N_i\left(v_i^{k+1}-v_i^k\right)\le 0 \text{ in } (0,1),$$
$$\varphi_i\left(v^{k+1}-v^k\right)\ge 0, \psi_i\left(v^{k+1}-v^k\right)\ge 0.$$

Let $p=v_i^{k+1}-v_i^k$, by Lemma 1 we have $p\ge 0$ and so $v_i^{k+1}\ge v_i^k$ for each $i=1,2,\cdots,n$. Hence $v^{k+1}\ge v^k$ on $[0,1]$. Similarly, we obtain $w^k\le w^{k+1}$ on $[0,1]$. Whence, we verify the result for $j=k+1$.

Now, let us prove that, for each $k\ge 1$, the pair v^k and w^k are an ordered coupled lower and upper quasi-solutions of (7) and (8). By adding $f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)$ to both sides of (14), we get for each $i=1,2,\cdots,n$

$$D^q\left(v_i^k\right)+f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)$$
$$=f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)-f_i\left(x,v_i^{k-1},\left[v^{k-1}\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)+N_i\left(v_i^k-v_i^{k-1}\right)$$

Using the fact that $v_i^k\ge v_i^{k-1}$ and $w_i^{k-1}\ge w_i^k$ and the property of the function *f*, it follows that

$$D^q\left(v_i^k\right)+f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)$$
$$\ge f_i\left(x,v_i^k,\left[v^{k-1}\right]_{r_i},\left[w^k\right]_{s_i}\right)-f_i\left(x,v_i^{k-1},\left[v^{k-1}\right]_{r_i},\left[w^k\right]_{s_i}\right)+N_i\left(v_i^k-v_i^{k-1}\right)$$

So, by hypothesis (*H*2), we conclude

$$D^q\left(v_i^k\right)+f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)\ge 0$$

Similarly, we prove that

$$D^q\left(w_i^k\right)+f_i\left(x,w_i^k,\left[w^k\right]_{r_i},\left[v^k\right]_{s_i}\right)\le 0$$

which together with (15) and (17) prove that $\left(v^k,w^k\right)$ is pair of coupled lower and upper quasi-solutions of (7) and (8).

Now, we shall prove that v^k and w^k are ordered. We use induction arguments. For $k=1$, by subtracting (18) from (19) we get

$$D^q\left(w_i^1-v_i^1\right)-N_i\left(w_i^1-v_i^1\right)$$
$$= f_i\left(x,v_i^0,\left[v^0\right]_{r_i},\left[w^0\right]_{s_i}\right)-f_i\left(x,w_i^0,\left[w^0\right]_{r_i},\left[v^0\right]_{s_i}\right)-N_i\left(w_i^0-v_i^0\right).$$

Using hypothesis ($H1$), and the mixed-monotone property of f, we obtain

$$D^q\left(w_i^1-v_i^1\right)-N_i\left(w_i^1-v_i^1\right)$$
$$\le f_i\left(x,v_i^0,\left[w^0\right]_{r_i},\left[w^0\right]_{s_i}\right)-f_i\left(x,w_i^0,\left[w^0\right]_{r_i},\left[w^0\right]_{s_i}\right)-N_i\left(w_i^0-v_i^0\right),$$

which by hypothesis ($H2$) yields

$$D^q\left(w_i^1-v_i^1\right)-N_i\left(w_i^1-v_i^1\right)\le 0.$$

From (15) and (17), we obtain $\varphi_i\left(w^1-v^1\right)\ge 0$ and $\psi_i\left(w^1-v^1\right)\ge 0$. Let $z=w_i^1-v_i^1$, then by Lemma 1, we obtain $z\ge 0$ for i, $1\le i\le n$, so $w^1\ge v^1$. Next, assume that we have for any j in $\{1,\cdots,k\}$

$$v^j\le w^j. \tag{30}$$

Similarly, by hypothesis ($H2$) and (30) we have

$$D^q\left(w_i^{k+1}-v_i^{k+1}\right)-N_i\left(w_i^{k+1}-v_i^{k+1}\right)$$
$$\le f_i\left(x,v_i^k,\left[v^k\right]_{r_i},\left[w^k\right]_{s_i}\right)-f_i\left(x,w_i^k,\left[w^k\right]_{r_i},\left[v^k\right]_{s_i}\right)-N_i\left(w_i^k-v_i^k\right)$$
$$\le f_i\left(x,v_i^k,\left[w^k\right]_{r_i},\left[w^k\right]_{s_i}\right)-f_i\left(x,w_i^k,\left[w^k\right]_{r_i},\left[w^k\right]_{s_i}\right)-N_i\left(w_i^k-v_i^k\right)\le 0.$$

On the other hand, from (15) and (17), we obtain $\varphi_i\left(w^{k+1}-v^{k+1}\right)\ge 0$ and $\psi_i\left(w^{k+1}-v^{k+1}\right)\ge 0$. Let $z=w_i^{k+1}-v_i^{k+1}$, then by Lemma 1, we obtain $z\ge 0$ for each $i=1,2,\cdots,n$. Thus $v^{k+1}\le w^{k+1}$.

2) For each $i=1,2,\cdots,n$, the sequences $\left(v_i^k\right)$ and $\left(w_i^k\right)$ are uniformly bounded and equicontinuous. Thus, using Arzela-Ascoli's Theorem, we deduce that $\lim_{k\to+\infty} v_i^k=v_i^*$ and $\lim_{k\to+\infty} w_i^k=w_i^*$. Since, in this case, the pointwise convergence yields to the uniform one, we deduce that $\left(v_i^k\right)$ and $\left(w_i^k\right)$ converge uniformly on $[0,1]$ and so $\left(v^k\right)$ and $\left(w^k\right)$ converge uniformly to v^* and w^*, respectively.

Now, by using the fact that $v_0\le v^k\le w^k\le w_0$ for each $k\ge 1$ and letting $k\to\infty$ we conclude that $v^*\le w^*$.

Next, let us prove that $\left(v^*,w^*\right)$ is pair a of quasi-solutions of (7) and (8). From (14)

$$D^q v_i^k+f_i\left(x,v_i^{k-1},\left[v^{k-1}\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)-N_i\left(v_i^k-v_i^{k-1}\right)=0$$

Applying the operator I^q and using Lemma 1 (2), we obtain

$$v_i^k-c_0^k-c_1^k x+I^q f_i\left(x,v_i^{k-1},\left[v^{k-1}\right]_{r_i},\left[w^{k-1}\right]_{s_i}\right)-N_i\left(I^q v_i^k-I^q v_i^{k-1}\right)=0.$$

Since $v^k\to v^*$ and $w^k\to w^*$ uniformly when $k\to+\infty$ and the function f is continuous, we obtain, for each $i=1,2,\cdots,n$,

$$v_i^*-c_0^*-c_1^* x+I^q f_i\left(x,v_i^*,\left[v^*\right]_{r_i},\left[w^*\right]_{s_i}\right)=0 \tag{31}$$

where $c_0^* = v_i^*(0) = \lim_{k\to\infty} v_i^k(0)$ and $c_1^* = \lim_{k\to\infty}\left(v_i^k\right)'(0)$. Now, applying D^q to (31) and using Lemma 1 (1) and (3), we obtain,

$$D^q v_i^* + f_i\left(x, v_i^*, \left[v^*\right]_{r_i}, \left[w^*\right]_{s_i}\right) = 0. \tag{32}$$

In addition, it is easy to verify that $\varphi_i\left(v^*\right) = \lambda_i, \psi_i\left(v^*\right) = \mu_i$. So v^* satisfies (9) and (10), in the same manner, we prove that w^* satisfies (11) and (12). Therefore, v^* and w^* are coupled quasi-solutions of (7) and (8) in $\left[v^0, w^0\right]$.

Now, let us prove that the functions v^* and w^* are a pair of minimal-maximal coupled quasi-solutions of the problem (7) and (8) in $\left[v^0, w^0\right]$. Let $v, w \in \left[v^0, w^0\right]$ be a pair of coupled quasi-solutions of (7) and (8). We will proceed by induction. First, it is obvious to see that $v \ge v^0$, and $w \le w^0$. Assume that

$$v \ge v^k \text{ and } w \le w^k \quad \text{on } [0,1] \tag{33}$$

is true. Thus, using (33), we obtain

$$\begin{aligned} &D^q\left(v_i - v_i^{k+1}\right) - N_i\left(v_i - v_i^{k+1}\right) \\ &= f_i\left(x, v_i^k, \left[v^k\right]_{r_i}, \left[w^k\right]_{s_i}\right) - f_i\left(x, v_i, [v]_{r_i}, [w]_{s_i}\right) + N_i\left(v_i^k - v_i\right) \\ &\le f_i\left(x, v_i^k, [v]_{r_i}, [w]_{s_i}\right) - f_i\left(x, v_i, [v]_{r_i}, [w]_{s_i}\right) + N_i\left(v_i^k - v_i\right) \end{aligned}$$

So, by hypothesis ($H2$), we conclude

$$D^q\left(v_i - v_i^{k+1}\right) - N_i\left(v_i - v_i^{k+1}\right) \le 0.$$

From (15), we obtain $\varphi_i\left(v - v^{k+1}\right) \ge 0$ and $\psi_i\left(v - v^{k+1}\right) \ge 0$. Put $z = v_i - v_i^{k+1}$, then by Lemma 1, we have $z \ge 0$ for each $i = 1, 2, \cdots, n$, so $v \ge v^{k+1}$. In the same manner, we prove that $w \le w^{k+1}$. Thus, taking limit $k \to +\infty$, we get

$$v \ge v^* \text{ and } w \le w^*.$$

That is, v^*, w^* are minimal-maximal coupled quasi-solutions of the problem (7) and (8) in $\left[v^0, w^0\right]$. This ends the proof.

Remark 1. We remark that if $n = 1$ then $s_i = r_i = 0$, and so (7) and (8) is reduced to the scalar boundary value problem. In this case, Theorem 2 improves the result in [24].

Remark 2. Note that, if, for each $i = 1, \cdots, n$, $r_i = n - 1$, $s_i = 0$. Then, we obtain minimal and maximal solutions of (7) and (8). Hence, Theorem 1 covers the case of quasimonotone nondecreasing nonlinearity.

We state uniqueness result in the following

Theorem 4. Assume that assumptions ($H1$) - ($H3$) hold. Moreover, suppose that, for each $i = 1, 2, \cdots, n$, $L_i > 0$ and

$$f_i\left(x, u_i, [u]_{r_i}, [v]_{s_i}\right) - f_i\left(x, v_i, [v]_{r_i}, [u]_{s_i}\right) \le -L_i\left(u_i - v_i\right)$$

on $[0,1]$, for any v, $v^0 \le v \le u \le w^0$. Then, the problem (7) and (8) admits a unique solution in $\left[v^0, w^0\right]$.

Proof. Since $v^* \le w^*$, we need only to prove that $w^* \le v^*$ on $[0,1]$. Let

$z_i = v_i^* - w_i^*$ on $[0,1]$ for each $i = 1,\cdots,n$. Then we get

$$D^q z_i = f_i\left(x, w_i^*, \left[w^*\right]_{r_i}, \left[v^*\right]_{s_i}\right) - f_i\left(x, v_i^*, \left[v^*\right]_{r_i}, \left[w^*\right]_{s_i}\right) \le L_i\left(v_i^* - w_i^*\right).$$

So, we conclude

$$D^q z_i - L_i z_i \le 0.$$

Since $\varphi_i\left(v^*\right) = \varphi_i\left(w^*\right) = \lambda_i$, and $\psi_i\left(v^*\right) = \psi_i\left(w^*\right) = \mu_i$. Then $\varphi_i(z) = 0$ and $\psi_i(z) = 0$. Therefore, by Lemma 1, we have $z_i \ge 0$ for each $i = 1,2,\cdots,n$, which implies $w^* \le v^*$ on $[0,1]$. So, $v^* = w^*$ is the unique solution of (7) and (8) in $\left[v^0, w^0\right]$, which ends the proof.

Remark 3. It is worth mentioning that one can generate a numerical approximation of v^*, w^* the minimal-maximal coupled quasi-solutions of the problem (7) and (8) in $\left[v^0, w^0\right]$ for a given coupled lower and upper quasi-solutions.

4. Examples

This section is devoted to some examples to illustrate our results.

Example 1. We consider the following nonlinear problem

$$\begin{cases} D^{\frac{3}{2}} u(x) - u^2(x) = 0,\ x \in (0,1) \\ u(0) - 2u'(0) = 0,\ u(1) = 1 \end{cases} \tag{34}$$

So, $n = 1, q = \frac{3}{2}, \alpha = 2, \beta = \lambda = 0, \mu = 1$ and $f(t,u) = -u^2$. First, condition $\alpha(q-1) \ge 1$ is satisfied. The pair $v^0(x) = 0$ and $w^0(x) = 1$ are ordered coupled lower and upper quasi-solutions of (34), so ($H1$) is verified. Moreover, we have for $N = 2$, the hypothesis ($H2$) holds. Next, define for each $k \ge 1$, the sequences $v^k(x)$ and $w^k(x)$, respectively, by

$$v^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(2v^{k-1}(y) - \left(v^{k-1}(y)\right)^2\right) - 2v^k(y)\mathrm{d}y$$

and

$$w^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(2w^{k-1}(y) - \left(w^{k-1}(y)\right)^2\right) - 2w^k(y)\mathrm{d}y,$$

where

$$k(x,y) = \frac{1}{3\Gamma\left(\frac{3}{2}\right)} \begin{cases} (2+x)\sqrt{1-y},\ 0 \le x \le y \\ (2+x)\sqrt{1-y} - 3\sqrt{x-y},\ y < x \le 1 \end{cases} \tag{35}$$

By Lemma 2 in [29], the sequences $v^k(x)$ and $w^k(x)$ satisfy the linear problems

$$\begin{cases} D^{\frac{3}{2}} v^k + \left(2v^{k-1} - \left(v^{k-1}\right)^2\right) - 2v^k = 0 \text{ in } (0,1) \\ v^k(0) - 2\left(v^k\right)'(0) = 0,\ v^k(1) = 1 \end{cases}$$

and

$$\begin{cases} D^{\frac{3}{2}}w^k + \left(2w^{k-1} - \left(w^{k-1}\right)^2\right) - 2w^k = 0 \text{ in } (0,1) \\ w^k(0) - 2\left(w^k\right)'(0) = 0,\ w^k(1) = 1 \end{cases}$$

Thus, the hypothesis (*H*3) is satisfied. Hence, Theorem 1 ensures the existence of minimal and maximal solutions of (34) in $[0,1]$.

Example 2. Let f be in $\mathcal{C}\left([0,1]\times\mathbb{R}^3,\mathbb{R}^3\right)$, and defined by $f = (f_1, f_2, f_3)$, where

$$f_1(x,u_1,u_2,u_3) = \mathrm{e}^{-u_3} - 3u_1 + u_2^2$$
$$f_2(x,u_1,u_2,u_3) = u_2 - 4u_1 + u_3^2$$
$$f_3(x,u_1,u_2,u_3) = \mathrm{e}^{-u_3} - u_1 - u_2$$

In this example, we deal with the following nonlinear problem

$$\begin{cases} D^{\frac{3}{2}}u(x) + f(x,u(x)) = 0 \text{ in } (0,1) \\ u(0) - \alpha u'(0) = \lambda,\ u(1) = \mu, \end{cases} \tag{36}$$

where, $n = 3, q = \frac{3}{2}, \alpha = (2,2,2), \lambda = (0,0,0)$ and $\mu = (1,1,1)$. For each $i = 1,2,3$, the condition $\alpha_i(q-1) \ge 1$ is satisfied. It is easy to verify that the pair $v^0(x) = (0,0,0)$ and $w^0(x) = (1,1,1)$ are ordered coupled lower and upper quasi-solutions of (36). Now, let us verify (*H*2). For $i = 1$ we take $r_1 = 1, s_1 = 1$. So the function $f_1(x,u_1,u_2,u_3)$ is nondecreasing in u_2 and nonincreasing in u_3. Moreover, for $N_1 = 3$, condition (13) holds. For $i = 2$ we take $r_2 = 2, s_1 = 0$. So the function $f_2(x,u_2,u_1,u_3)$ is nondecreasing in (u_1,u_3) and for $N_2 = 4$, condition (13) holds. For $i = 3$ we take $r_3 = 0, s_3 = 2$. So the function $f_3(x,u_1,u_2,u_3)$ is nonincreasing in (u_2,u_3) and for $N_3 = 1$, the condition (13) holds. Thus (*H*2) is satisfied. Now, let us define for $k \ge 1$, the sequences $v^k(x)$ and $w^k(x)$, by

$$v_1^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_1\left(y, v_1^{k-1}, v_2^{k-1}, w_3^{k-1}\right) + 3v_1^{k-1}\right) - 3v_1^k \mathrm{d}y,$$

$$v_2^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_2\left(y, v_2^{k-1}, v_1^{k-1}, v_3^{k-1}\right) + 4v_2^{k-1}\right) - 4v_2^k \mathrm{d}y,$$

$$v_3^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_3\left(y, v_3^{k-1}, w_1^{k-1}, w_3^{k-1}\right) + v_3^{k-1}\right) - v_3^k \mathrm{d}y$$

and

$$w_1^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_1\left(y, w_1^{k-1}, w_2^{k-1}, v_3^{k-1}\right) + 3w_1^{k-1}\right) - 3w_1^k \mathrm{d}y,$$

$$w_2^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_2\left(y, w_2^{k-1}, w_1^{k-1}, w_3^{k-1}\right) + 4w_2^{k-1}\right) - 4w_2^k \mathrm{d}y,$$

$$w_3^k(x) = \frac{2}{3}(1+x) + \int_0^1 k(x,y)\left(f_3\left(y, w_3^{k-1}, v_1^{k-1}, v_3^{k-1}\right) + w_3^{k-1}\right) - w_3^k \mathrm{d}y,$$

where $k(x, y)$ is defined by (35).

Using Lemma 2 in [29], the sequences $v^k(x)$ and $w^k(y)$ satisfy (14) - (17). Therefore, the hypothesis ($H3$) is satisfied. Hence, Theorem 1 assures the existence of minimal-maximal coupled quasi-solutions of (36) in $[v^0, w^0]$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Diethelm, K. and Freed, A.D. (1999) On the Solution of Nonlinear Fractional-Order Differential Equations Used in the Modelling of Viscoplasticity. In: Keil, F., Mackens, W., Voss, H. and Werther, J., Eds., *Scientific Computing in Chemical Engineering II*: *Computational Fluid Dynamics*, *Reaction Engineering and Molecular Properties*, Springer, Berlin, 217-224. https://doi.org/10.1007/978-3-642-60185-9_24

[2] Glockle, W.G. and Nonnenmacher, T.F. (1995) A Fractional Calculus Approach to Self-Similar Protein Dynamics. *Biophysical Journal*, **68**, 46-53. https://doi.org/10.1016/S0006-3495(95)80157-8

[3] Ladaci, S., Loiseau, J.L. and Charef, A. (2008) Fractional Order Adaptive High-Gain Controllers for a Class of Linear Systems. *Communications in Nonlinear Science and Numerical Simulation*, **13**, 707-714. https://doi.org/10.1016/j.cnsns.2006.06.009

[4] Mainardi, F. (1995) Fractional Diffusive Waves in Viscoelastic Solids. In: Wegner, J.L. and Norwood, F.R., Eds., *Nonlinear Waves in Solids*, ASME Book No. AMR 137, Fairfield, 93-97.

[5] Mainardi, F. (1997) Fractional Calculus: Some Basic Problems in Continuum and Statical Mechanics. In: Carpinteri, A. and Mainardi, F., Eds., *Fractals and Fractional Calculus in Continuum Calculus Mechanics*, *Vol.* 378, *CISM International Centre for Mechanical Sciences*, Springer, Vienna, 291-348.

[6] Metzler, R. and Klafter, J. (2000) Boundary Value Problems for Fractional Diffusion Equations. *Physica A*: *Statistical Mechanics and its Applications*, **278**, 107-125. https://doi.org/10.1016/S0378-4371(99)00503-8

[7] Scher, H. and Montroll, E. (1975) Anomalous Transit-Time Dispersion in Amorphous Solids. *Physical Review B*, **12**, 2455-2477. https://doi.org/10.1103/PhysRevB.12.2455

[8] Lu, Z., Li, Y., and Shi, X., (2021) Monotone Iterative Technique for Nonlinear Differential Equation of Fractional Order. Authorea, Inc., Brooklyn. https://doi.org/10.22541/au.162201485.54208914/v1

[9] Podlubny, I. (1993) Fractional Differential Equations Academic Press, New York.

[10] Devi, J.V., McRae, F.A. and Drici, Z. (2012) Variational Lyapunov Method for Fractional Differential Equations. *Computers and Mathematics with Applications*, **64**, 2982-2989. https://doi.org/10.1016/j.camwa.2012.01.070

[11] Dhaigude, D.B. and Birajdar, G.A. (2012) Numerical Solution of System of Fractional Partial Differential Equations by Adomian Decomposition Method. *Journal*

of Fractional Calculus and Applications, **3**, 1-11.

[12] Deekshitulu, G.V.S.R. (2009) Generalized Monotone Iterative Technique for Fractional R-L Differential Equations. *Nonlinear Studies*, **16**, 89-98.

[13] Bai, Z. and Lü, H. (2005) Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation. *Journal of Mathematical Analysis and Applications*, **311**, 495-505. https://doi.org/10.1016/j.jmaa.2005.02.052

[14] Benchohra, M. and Hamani, S. (2009) The Method of Upper and Lower Solutions and Impulsive Fractional Differential Inclusions. *Nonlinear Analysis: Hybrid Systems*, **34**, 433-440. https://doi.org/10.1016/j.nahs.2009.02.009

[15] Cabada, A. (1994) The Method of Lower and Upper Solutions for Second, Third, Fourth and Higher Order Boundary Value Problems. *Journal of Mathematical Analysis and Applications*, **185**, 302-320. https://doi.org/10.1006/jmaa.1994.1250

[16] Dhaigude, D.B., Nanware, J.A. and Nikmam, V.R. (2012) Monotone Technique for System of Caputo Fractional Differential Equations with Periodic Boundary Conditions, Dynamics of Continuous. *Discrete and Impulsive Systems*, **19**, 575-584.

[17] Jankowski, T. (2004) Monotone Method for Second-Order Delayed Differential Equations with Boundary Value Conditions. *Applied Mathematics and Computation*, **149**, 589-598. https://doi.org/10.1016/S0096-3003(03)00166-8

[18] Wang, P. and Hou, Y. (2013) Generalized Quazilinearization for the System of Fractional Differential Equations. *Journal of Function Spaces*, **2013**, Article ID: 793263. https://doi.org/10.1155/2013/793263

[19] Yakar, A. and Koksal, M.E. (2012) Existence Results for Solutions of Nonlinear Fractional Differential Equations. *Abstract and Applied Analysis*, **2012**, Article ID: 267108. https://doi.org/10.1155/2012/267108

[20] Denton, Z. and Ramírez, J.D. (2017) Existence of Minimal and Maximal Solutions to RL Fractional Integro-Differential Initial Value Problems. *Opuscula Mathematica*, **37**, 705-724. https://doi.org/10.7494/OpMath.2017.37.5.705

[21] Shi, A. and Zhang, S. (2009) Upper and Lower Solutions Method and a Fractional Differential Equation Boundary Value Problem. *Electronic Journal of Qualitative Theory of Differential Equations*, **30**, 1-13. https://doi.org/10.14232/ejqtde.2009.1.30

[22] Al-Refai, M. (2012) On the Fractional Derivative at Extreme Points. *Electronic Journal of Qualitative Theory of Differential Equations*, **55**, 1-5. https://doi.org/10.14232/ejqtde.2012.1.55

[23] Ramírez, J.D. and Vatsala, A.S. (2009) Monotone Iterative Technique for fractional Differential Equations with Periodic Boundary Conditions. *Opuscula Mathematica*, **29**, 289-304. https://doi.org/10.7494/OpMath.2009.29.3.289

[24] Al-Refai, M. and Hajji, M. (2011) Monotone Iterative Sequences for Nonlinear Boundary Value Problems of Fractional Order. *Nonlinear Analysis: Theory, Methods & Applications*, **74**, 3531-3539. https://doi.org/10.1016/j.na.2011.03.006

[25] Denton, Z. and Vatsala, A.S. (2011) Monotone Iterative Technique for Finite Systems of Nonlinear Riemann-Liouville Fractional Differential Equations. *Opuscula Mathematica*, **31**, 327-339. https://doi.org/10.7494/OpMath.2011.31.3.327

[26] McRae, F.A. (2009) Monotone Iterative Technique for Existence Results for Fractional Differential Equations. *Nonlinear Analysis: Theory, Methods & Applications*, **71**, 6093-6096. https://doi.org/10.1016/j.na.2009.05.074

[27] Cui, Y. and Zou Y. (2014) Existence Results and the Monotone Iterative Technique for Nonlinear Fractional Differential Systems with Coupled Four-Point Boundary Value Problems. *Abstract and Applied Analysis*, **2014**, Article ID: 242591.

https://doi.org/10.1155/2014/242591

[28] Hu, C., Liu, B. and Xie, S. (2013) Monotone Iterative Solutions for Nonlinear Boundary Value Problems of Fractional Differential Equation. *Abstract and Applied Analysis*, **2013**, Article ID: 493164. https://doi.org/10.1155/2013/493164

[29] Toumi, F. (2016) Monotone Iterative Technique for Systems of Nonlinear Caputo Fractional Differential Equations. In: Pinelas, S., Došlá, Z., Došlý, O. and Kloeden, P.E., Eds., *Differential and Difference Equations with Applications*, *Series Springer Proceedings in Mathematics and Statistics*, Vol. 164, Springer, Cham, 99-107. https://doi.org/10.1007/978-3-319-32857-7_10

[30] Kilbas, A., Srivastava, H. and Trujillo, J. (2006) Theory and Applications of Fractional Differential Equations. In: *North-Holland Mathematics Studies*, Vol. 204, Elsevier Science, BV.

[31] Samko, S., Kilbas, A. and Marichev, O. (1993) Fractional Integrals and derivatIve. Theory and Applications. Gordon and Breach Science Publishers, Switzerland.

[32] Syam, M. and Al-Refai, M. (2013) Positive Solutions and Monotone Iterative Sequences for a Class of Higher Order Boundary Value Problems. *Journal of Fractional Calculus and Applications*, **14**, 1-13.