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Adomian Decomposition Method for Solving Boussinesq Equations Using Maple

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Abstract

This paper uses the Adomian Decomposition Method (ADM) to solve Boussinesq equations using Maple. The Boussinesq approximation for water waves is a weakly nonlinear and long-wave approximation in fluid dynamics. The approximation is named after Joseph Boussinesq, who developed it in response to John Scott Russell's observation of a wave of translation (also known as solitary wave or soliton). Bossinesq's article from 1872 introduced the equations that are now known as the Boussinesq equations. Numerical methods are commonly utilized to solve nonlinear equation systems. In this paper, we investigate a nonlinear singly perturbed advection-diffusion problem. Using the usual Adomian Decomposition Method, we formulate an approximate linear advection-diffusion problem and investigate several practical numerical approaches for solving it (ADM). The Adomian Decomposition Method (ADM) is a powerful tool for numerical simulations and approximation analytic solutions. The Adomian Decomposition Method (ADM) is used to solve nonlinear advection differential equations using Maple by illustrating numerous examples. The findings are presented in the form of tables and graphs for several examples. For various examples, the findings are presented in the form of tables and graphs. The difference between the precise and numerical solutions indicates the Maple program solution's efficacy, as well as the ease and speed with which it was acquired.

Keywords

Adomian Decomposition Method, Boussinesq Equations, Maple 18

1. Introduction

Since the beginning of the 1980s, Adomian [1]-[8] has introduced and developed a technique known as the Adomian Decomposition Method (ADM) which is a well-known systematic method for the practical solution of Ordinary Differential Equations (ODEs), Partial Differential Equations (PDEs), integral equations, integro-differential equations, and other operator equations with linear or nonlinear, deterministic, or stochastic operators. The ADM is an effective method that offers quick algorithms for approximating analytical solutions and numerical simulations for practical applications in engineering and the applied sciences. The solution is found as an infinite series that converges rapidly to accurate solutions.

Adomian and co-workers have solved nonlinear differential equations for a wide class of nonlinearities, including product [9], polynomial [10], exponential [11], trigonometric [12], hyperbolic [13], composite [14], negative-power [15], radical [16] and even decimal-power nonlinearities [17]. We find that the ADM solves nonlinear operator equations for any analytic nonlinearity, providing us with an easily computable, readily verifiable, and rapidly convergent sequence of analytic approximate functions.

More recently, Adomian and Rach [18] introduced the phenomena of the so-called "noise terms". The "noise terms" were defined in [3] as identical terms with opposite signs that appear in the first two components of the series solution of $u(x)$.

Recent research by Wazwaz [19] developed a condition that is fundamentally required to guarantee the presence of "noise terms" in inhomogeneous equations. Then, Luo commented on it with a scientific paper [20] and modified this method with a two-step Adomian Decomposition Method. Chen and Lu [21] established a promising algorithm that can be easily programmed in Maple.

The classical Boussinesq equation

$$u_{tt} = \left(u + u^2 + u_{xx}\right)_{xx}, \tag{1}$$

which has been derived in 1872 to describe shallow water waves has the flaw that the Cauchy problem is improperly posed. Therefore, it can't be used for the analysis of numerical wave propagation issues.

It also occurs in various physical applications such as vibrations in a nonlinear string, iron sound waves in plasma, and nonlinear lattice waves. Additionally, it was used to address issues with water percolation in porous subsurface strata. Recently, certain novel approaches to solving nonlinear equations have garnered considerable interest, such as the variational iteration method.

Boussinesq put forth a well-known model of nonlinear dispersive waves in the generalized form.

$$u_{tt} = \left[f(u)\right]_{xx} + u_{xxxx} + h(x,t), \ -\infty < x < \infty, t > 0, \tag{2}$$

with u and h are sufficiently differentiable functions, and $f(0) = 0$. The initial conditions associated with Boussinesq Equation (1) have the form

$$u(x,0) = a(x), u_t(x,0) = b(x), \ -\infty < x < \infty, \tag{3}$$

with $a(x)$ and $b(x)$ given C^{∞}.

This paper aims to solve the Boussinesq equation and compare the exact solution, and the numerical solution obtained using the method of Adomian Decomposition Method by Maple 18 program.

To present a clear overview of the method, we have chosen two examples, to illustrate the Adomian Decomposition Method and the obtained solutions are compared with the exact solutions.

2. Adomian Decomposition Method

To illustrate the methodology of the proposed method, using the Adomian Decomposition Method, we consider

$$Lu + R(u) + F(u) = g(x), \tag{4}$$

where L is the highest order derivative in the equation, R is the remainder of the differential operator, $F(u)$ expresses the nonlinear terms, and $g(x)$ is an inhomogeneous term. If the operator L is a first-order operator, characterized by

$$L = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{5}$$

The inverse operator L^{-1} L is given by assuming that L is invertible.

$$L^{-1}(\cdot) = \int_0^x (\cdot)\,\mathrm{d}x. \tag{6}$$

So that

$$L^{-1}Lu(\cdot) = u(x) - u(0). \tag{7}$$

If, on the other hand, L is a second-order differential operator defined by

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2}, \tag{8}$$

so that the inverse operator L^{-1} is regarded a two-fold integration operator defined by

$$L^{-1}(\cdot) = \int_0^x \int_0^x (\cdot)\,\mathrm{d}x\mathrm{d}x. \tag{9}$$

So that

$$L^{-1}Lu = u(x) - u(0) - xu'(0). \tag{10}$$

In a parallel manner, if L is a third-order differential operator, we can easily show that

$$L^{-1}Lu = u(x) - u(0) - xu'(0) - \frac{1}{2!}x^2u''(0), \tag{11}$$

For higher-order operators we can easily similarly define the related inverse operators. Applying L^{-1} to both sides of (4) gives

$$u(x) = \varphi_0 - L^{-1}(g(x)) - L^{-1}Ru - LF(u), \tag{12}$$

$$u(0), \quad \text{for } L = \frac{\mathrm{d}}{\mathrm{d}x},$$

$$u(0)+xu'(0), \quad \text{for } L=\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

$$u(0)+xu'(0)+\frac{1}{2!}x^2u''(0), \quad \text{for } L=\frac{\mathrm{d}^3}{\mathrm{d}x^3},$$

and so on. The Adomian Decomposition Method admits the decomposition of u into an infinite series of components.

$$u(x)=\sum_{n=0}^{\infty}u_n, \tag{13}$$

$$F(u)=\sum_{n=0}^{\infty}A_n, \tag{14}$$

where A_n are the Adomian polynomials. Substituting (13) and (14) into (12) gives

$$\sum_{n=0}^{\infty}u_n=\varphi_0-L^{-1}\left(g(x)\right)-L^{-1}\left(g\sum_{n=0}^{\infty}u_n\right)-L^{-1}\left(\sum_{n=0}^{\infty}A_n\right). \tag{15}$$

The various components u_n of the solution u can be easily determined by using the recursive relation.

$$u_0=\varphi_0-L^{-1}\left(g(x)\right), \tag{16}$$

$$u_{k+1}=-L^{-1}\left(Ru_k\right)-L^{-1}\left(A_k\right),\ k\geq 0. \tag{17}$$

Consequently, the first few components can be written as

$$u_0=\varphi_0-L^{-1}g(x),$$

$$u_1=-L^{-1}\left(Ru_0\right)-L^{-1}\left(A_0\right),$$

$$u_2=-L^{-1}\left(Ru_1\right)-L^{-1}\left(A_1\right),$$

$$u_3=-L^{-1}\left(Ru_2\right)-L^{-1}\left(A_2\right),$$

$$u_4=-L^{-1}\left(Ru_3\right)-L^{-1}\left(A_3\right).$$

Having determined the components $u_1, n\geq 0$, the solution u in a series form follows immediately. As stated before, the series may be summed to provide the solution in a closed form. However, for concrete problems, the n-term partial sum

$$\varphi_0=\sum_{k=0}^{n-1}u_k, \tag{18}$$

may be used to give the approximate solution. In this section, we solve some examples, and we can compare the numerical results with the exact solution.

3. Application

Two examples are given in this section to illustrate the effects of the proposed method.

3.1. Example 1

Consider the Boussinesq equation

$$u_{tt} = u_{xx} + 3\left(u^2\right)_{xx} + u_{xxxx}, \tag{19}$$

$$u(x,0) = 2\frac{ak^2 e^{kx}}{\left(1+ae^{kx}\right)^2},\ u_t(x,0) = -2\frac{ak^3\sqrt{1+k^2}e^{kx}\left(ae^{kx}-1\right)}{\left(1+ae^{kx}\right)^3},$$

with the exact solution $u(x) = 2\frac{e^{kx+\sqrt{1+k^2}t}}{\left(1+e^{kx+k\sqrt{1+k^2}t}\right)^2}$.

3.2. Example 2

Consider the Boussinesq equation

$$u_{tt} = u_{xx} + 3\left(u^2\right)_{xx} + u_{xxxx}, \tag{20}$$

$$u(x,0) = \frac{-3k^2}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right),\ u_t(x,0) = \frac{3k^3\sqrt{1-k^2}}{2}\operatorname{sech}^2\left(\frac{kx}{2}\right)\tanh\left(\frac{kx}{2}\right),$$

with the exact solution $u(x) = -\frac{3}{2}k^2\operatorname{sech}\left(\frac{1}{2}k\left(x+\sqrt{-k^2+1}t\right)\right)^2$.

Figure 1 and Figure 2 show the exact and approximate solutions. This problem was solved by ADM and their results are shown in Table 1 and Table 2 using maple.

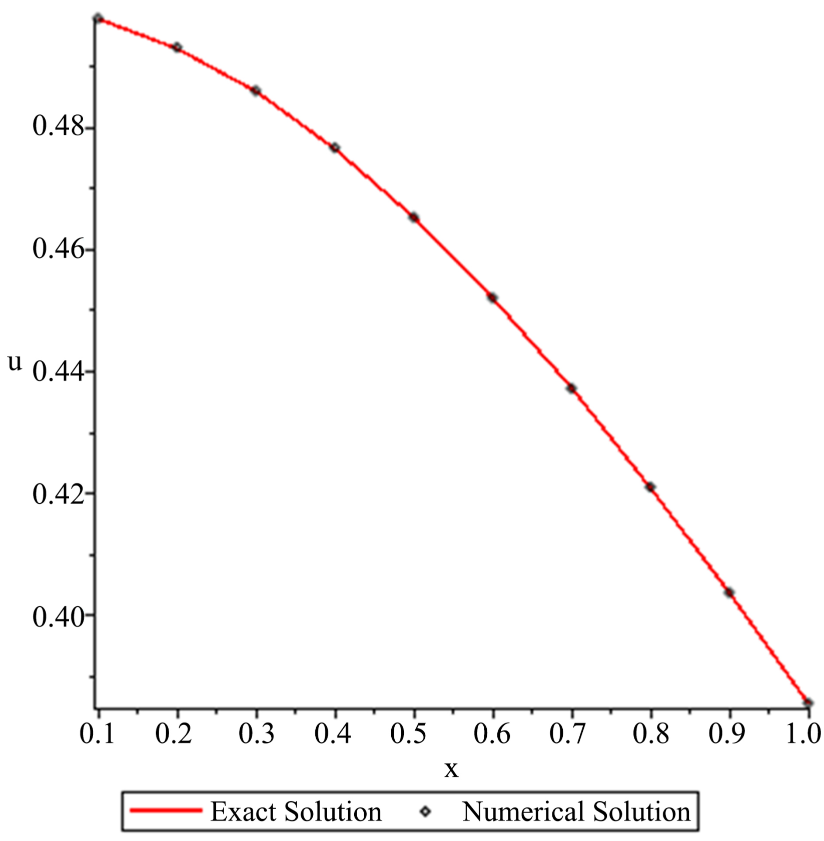

Figure 1. Graph showing the correspondence between exact and approximate solutions result of Boussinesq equations in Example 1.

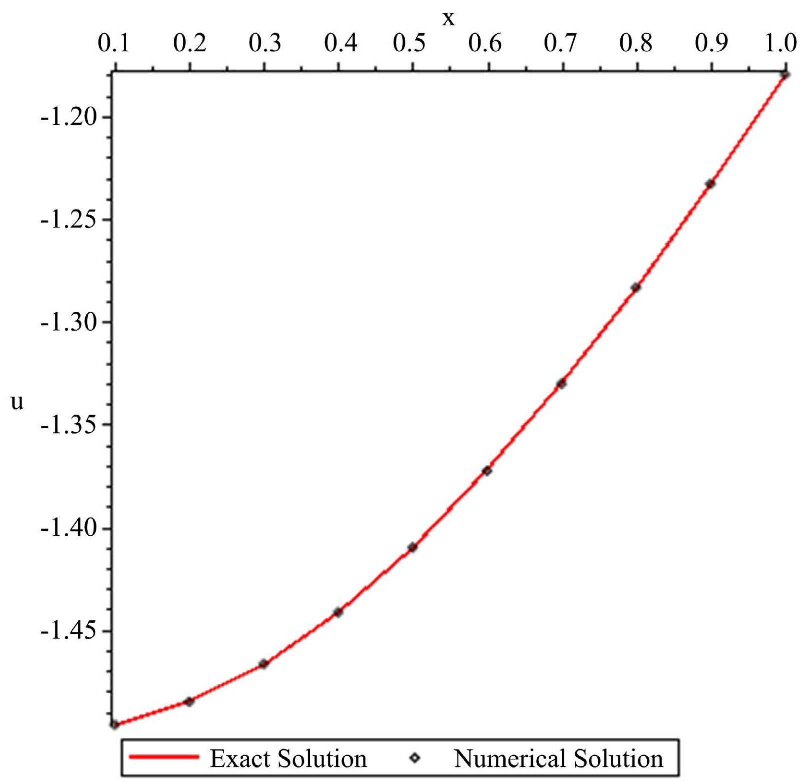

Figure 2. Graph showing the correspondence between exact and approximate solutions result of Boussinesq equations in Example 2.

Table 1. Numerical results and exact solution of Boussinesq equation for Example 1.

x	$\text{Exact} = 2\dfrac{e^{kx+\sqrt{1+k^2}t}}{\left(1+e^{kx+k\sqrt{1+k^2}t}\right)^2}$	$u(x)$	Error
0.10000	0.4974729	0.4976949	0.0002221
0.20000	0.4927250	0.4929399	0.0002148
0.30000	0.4856248	0.4858283	0.0002035
0.40000	0.4763090	0.4764976	0.0001886
0.50000	0.4649528	0.4651236	0.0001707
0.60000	0.4517627	0.4519132	0.0001505
0.70000	0.4369689	0.4370975	0.0001286
0.80000	0.4208171	0.4209230	0.0001059
0.90000	0.4035609	0.4036438	0.0000830
1.00000	0.3854539	0.3855143	0.0000605

Table 2. Numerical results and exact solution of Boussinesq equation for Example 2.

x	$\text{Exact} = -\dfrac{3}{2}k^2\operatorname{sech}\left(\dfrac{1}{2}k\left(x+\sqrt{-k^2+1t}\right)\right)^2$	$u(x)$	Error
0.10000	−1.4962562	−1.4962562	0.0000000

Continued

0.20000	−1.4850994	−1.4850994	0.0000000
0.30000	−1.4667499	−1.4667499	0.0000000
0.40000	−1.4415645	−1.4415645	0.0000000
0.50000	−1.4100223	−1.4100223	0.0000000
0.60000	−1.3727054	−1.3727054	0.0000000
0.70000	−1.3302772	−1.3302772	0.0000000
0.80000	−1.2834582	−1.2834582	0.0000000
0.90000	−1.2330018	−1.2330018	0.0000000
1.00000	−1.1796716	−1.1796716	0.0000000

4. Conclusion

The Adomian decomposition strategy is used to solve the Boussinesq equations using Maple18 software. The results were created using tables and figures. Table 1 and Table 2 show the numerical solution as well as the right solution. We can see that the numerical solution is generally relevant to the precise answer by comparing the numerical results, proving the method's efficacy and the ability to obtain the numerical solution relating swiftly and efficiently to the exact solution using Maple 18 software. Furthermore, the results obtained are pretty precise. The primary goal of the present paper is to mechanize the computing process of the decomposition method by the Maple program, so that we can obtain approximate solutions, which makes it easy in the future to expand the use of the Mabel program to solve more complex equations as quickly as possible.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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