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Bach-Einstein Gravitational Field Equations as a Perturbation of Einstein Gravitational Field Equations

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Abstract

The Bach equations are a version of higher-order gravitational field equations, exactly they are of fourth-order. In 4-dimensions the Bach-Einstein gravitational field equations are treated here as a perturbation of Einstein's gravity. An approximate inversion formula is derived which admits a comparison of the two field theories. An application to these theories is given where the gravitational Lagrangian is expressed linearly in terms of $R, R^2, |Ric|^2$, where the Ricci tensor $Ric = R_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ is inserted in some formulas which are of geometrical or physical importance, such as; Raychaudhuri equation and Tolman's formula.

Keywords

Gravitational Theory, Higher Order Gravity, Buchdahl's Formula, Bach-Einstein Gravitational Field Equations, Raychaudhuri Equation, Tolman's Formula

1. Introduction

In this paper we study the purely metrical fourth-order theories of gravitation in 4-dimensions which follow from a Lagrangian

$$L := L_{grav} + 2\chi L_{mat}, \tag{1}$$

which is the sum of a gravitational Lagrangian of the form [1] [2] [3]:

$$L_{grav} := -2\Lambda + R + \left(a_0 R^2 + a_1 |Ric|^2\right), \tag{2}$$

and an appropriate matter Lagrangian L_{mat}.

In fact, the most general quadratic gravitational Lagrangian:

$$L_1 := c_0 R^2 + c_1 |Ric|^2 + c_2 |Riem|^2, \tag{3}$$

effectively reduces to:

$$L_2 := a_0 R^2 + a_1 |Ric|^2, \tag{4}$$

with $a_0 = c_0 - c_2$, $a_1 = c_1 + 4c_2$, because of the fact that the Gauss-Bonnet expression

$$B := R^2 - 4|Ric|^2 + |Riem|^2, \tag{5}$$

has vanishing variational derivatives with respect to the metric in 4-dimensions [4]-[18]. Here χ, a_0, a_1 are real coupling constants, Λ is a "cosmological constant" and we abbreviate $|Riem|^2 := R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}$, $|Ric|^2 := R_{\alpha\beta} R^{\alpha\beta}$, where the Ricci tensor Ric has the components $R_{\alpha\beta} := R_{\mu\alpha\beta}{}^{\mu}$, and the scalar curvature reads $R := tr\, Ric \equiv g^{\alpha\beta} R_{\alpha\beta}$, tr denotes the trace with respect to the metric:

$$\mathrm{d}s^2 = g_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta.$$

We adopt the usual conventions of tensor calculus: Greek letters $\alpha, \beta, \gamma, \cdots$ take the values $0,1,2,3$. The signature of the metric g is assumed to be $(+---)$, $Riem = R_{\alpha\beta\mu\nu} \left(\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta\right)\left(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\right)$ denotes the Riemann curvature, their components $R_{\alpha\beta\mu}{}^{\nu}$ are introduced through the Ricci identity for a one-form $u = u_\alpha \mathrm{d}x^\alpha$ in terms of the Levi-Civita covariant derivatives ∇_α to g [19] [20] [21] [22] as:

$$\left(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha\right) u_\mu = R_{\alpha\beta\mu}{}^{\nu} u_\nu.$$

Equivalently, there holds

$$R_{\alpha\beta\mu}{}^{\nu} = \partial_\beta \Gamma_{\alpha\mu}{}^{\nu} - \partial_\alpha \Gamma_{\beta\mu}{}^{\nu} + \Gamma_{\alpha\mu}{}^{\lambda} \Gamma_{\lambda\beta}{}^{\nu} - \Gamma_{\beta\mu}{}^{\lambda} \Gamma_{\lambda\alpha}{}^{\nu},$$

in terms of the Christoffel symbols

$$\Gamma_{\alpha\beta}{}^{\mu} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right),$$

while the Weyl conformal tensor, denoted by Weyl, is defined through its components [21] [22]:

$$W_{\alpha\beta\mu\nu} := R_{\alpha\beta\mu\nu} + g_{\alpha[\mu} R_{\nu]\beta} + g_{\beta[\nu} R_{\mu]\alpha} - \frac{R}{6} g_{\alpha\beta\mu\nu}.$$

Here and in the following ( ) or [ ] indicate the symmetrization or antisymmetrization respectively of indices and we abbreviate:

$$g_{\alpha\beta\mu\nu} = g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu}.$$

Again in 4-dimensions, one can easily deduce the special quadratic expression [3]-[13]:

$$|Weyl|^2 := W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu} = \frac{1}{3} R^2 - 2|Ric|^2 + |Riem|^2 \tag{6}$$

is conformably invariant of weight −2, that means, a conformal transformation

$g \to e^{\phi} g$, ϕ variable, implies $|Weyl|^2 \to e^{-2\phi}|Weyl|^2$. Accordingly, that the Gauss-Bonnet expression (5) has vanishing variational derivatives with respect to the metric in 4-dimensions, thus (6) is equivalent to:

$$|Weyl|^2 := W_{\alpha\beta\mu\nu} W^{\alpha\beta\mu\nu} = 2|Ric|^2 - \frac{2}{3} R^2. \tag{7}$$

The most general Einstein's equations [21] are given as:

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \chi T_{\alpha\beta}, \tag{8}$$

where

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta},$$

is the Einstein tensor $G = G_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}$, and $T = T_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}$ is the energy-momentum tensor.

It is obviously, that the most general Einstein's Equations (8) have the alternative formula [20]:

$$R_{\alpha\beta} = \chi \left(T_{\alpha\beta} - \frac{trT}{2} g_{\alpha\beta} \right) + \Lambda g_{\alpha\beta}. \tag{9}$$

A spacetime for which

$$R_{\alpha\beta} = \frac{1}{4} R g_{\alpha\beta}, \quad R = const., \tag{10}$$

is called an Einstein spacetime [22]. Inserting Equation (10) into the identity (7) one obtains

$$|Weyl|^2 = -\frac{1}{6} R^2.$$

In Section 2; we introduce the variation derivatives of the Lagrangian (1) with respect to g which produces the fourth-order gravitational field Equations (14). It well known that the choice $a_1 = -3a_0$ of the gravitational Lagrangian (2), yields the so-called Bach-Einstein gravitational field Equations (21). In Section 3; a general algebraic structure is discussed, where we show that the Ricci tensor components $R_{\alpha\beta}$ to g can be represented by a covariant linear differential operator applied to a linear combination of $T_{\mu\nu}, g_{\mu\nu} trT, \Lambda g_{\mu\nu}$ plus an error term with the factor ε^2, where ε is a real parameter such that $|\varepsilon|$ is so small, that is $|\varepsilon| \ll 1$. In Section 4; the Bach-Einstein gravitational field equations in 4-dimensions are treated as a perturbation of Einstein's gravity, where we derive an approximate inversion Formula (32) which admits a comparison of the two field theories. Exactly, we obtain an approximate inversion formula corresponding of the Bach-Einstein gravitational field equations similar to the alternative Formula (9). Finally, in the last section, an application to both the Einstein gravitational field equations and Bach-Einstein gravitational field equations is given where the gravitational Lagrangian is expressed linearly in terms of $R, R^2, |Ric|^2$ (28), where the Ricci tensor $Ric = R_{\alpha\beta} \mathrm{d}x^{\alpha} \mathrm{d}x^{\beta}$ is inserted in some formulas which are of geometrical or physical importance, such as; Raychaudhuri equation and

Tolman's formula. [1] [23]. D. Barraco and V. H. Hamity [1] mention Tolman's expression as a possible application of approximate inversion formulas, where the gravitational Lagrangian is expressed linearly in terms of R, R^2.

2. The Fourth-Order Gravity

Variation derivatives of the Lagrangian (1) with respect to g produce the field equations

$$E_{\alpha\beta} = \chi T_{\alpha\beta},$$

where the variational derivative tensor $E_{\alpha\beta}$ and the energy-momentum tensor $T_{\alpha\beta}$ are defined by

$$|det\ g|^{\frac{1}{2}} E_{\alpha\beta} := \frac{\delta}{\delta g^{\alpha\beta}}\left(|det\ g|^{\frac{1}{2}} L_{grav}\right),$$

$$|det\ g|^{\frac{1}{2}} T_{\alpha\beta} := -2\frac{\delta}{\delta g^{\alpha\beta}}\left(|det\ g|^{\frac{1}{2}} L_{mat}\right),$$

$$det\ g := det\left(g_{\alpha\beta}\right).$$

Here the symbol δ expresses variational derivatives (cf., e.g. [13] [24]).

Let us now calculate the variational derivative tensor $E = E_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$ in the general. Using Buchdahl's formula: according to [13]-[18] there holds:

$$E_{\alpha\beta} = \nabla^{\mu}\nabla^{\nu} Z_{\alpha\beta\mu\nu} - \frac{2}{3} R_{\alpha\rho\mu\nu} Z_{\beta}{}^{\nu\mu\rho} - \frac{1}{2} g_{\alpha\beta} L_{grav},$$

where

$$Z_{\alpha\beta\mu\nu} = Y_{(\alpha\beta)(\mu\nu)}, \quad Y_{\alpha\beta\mu\nu} = 2X_{[\alpha\nu][\beta\mu]}, \quad X_{\alpha\beta\mu\nu} = \frac{\partial L_{grav}}{\partial R^{\alpha\beta\mu\nu}}.$$

Consequently, the fourth-order gravitational field equations of the Lagrangian (1) read

$$E_{\alpha\beta} \equiv \Lambda g_{\alpha\beta} + G_{\alpha\beta} + a_0 E^{(0)}_{\alpha\beta} + a_1 E^{(1)}_{\alpha\beta} = \chi T_{\alpha\beta}, \tag{11}$$

where

$$\begin{aligned} E^{(0)}_{\alpha\beta} &= 2\nabla^{\mu}\nabla^{\nu}\left(g_{\mu\alpha\beta\nu} R\right) + 2RR_{\alpha\beta} - \frac{1}{2}R^2 g_{\alpha\beta} \\ &\equiv 2\nabla_{\alpha}\nabla_{\beta} R - 2g_{\alpha\beta}\Box R + 2RR_{\alpha\beta} - \frac{R^2}{2} g_{\alpha\beta}, \end{aligned} \tag{12}$$

$$E^{(1)}_{\alpha\beta} = \nabla_{\alpha}\nabla_{\beta} R - \Box R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2}\Box R + 2R_{\mu\alpha\beta\nu} R^{\mu\nu} - \frac{1}{2}|Ric|^2 g_{\alpha\beta}, \tag{13}$$

where $\Box := g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}$ is the covariant d'Alembertian operator.

Thus, the fourth-order field Equation (11) takes the form:

$$\begin{aligned} &\Lambda g_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + a_0\left(2\nabla_{\alpha}\nabla_{\beta} R - 2g_{\alpha\beta}\Box R + 2RR_{\alpha\beta} - \frac{1}{2}R^2 g_{\alpha\beta}\right) \\ &+ a_1\left(\nabla_{\alpha}\nabla_{\beta} R - \Box R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2}\Box R + 2R_{\mu\alpha\beta\nu} R^{\mu\nu} - \frac{1}{2}|Ric|^2 g_{\alpha\beta}\right) = \chi T_{\alpha\beta}. \end{aligned} \tag{14}$$

Inserting Equation (10) into the fourth-order tensors (12), (13), one easily obtains:

$$E_{\alpha\beta}^{(0)} = 0 = E_{\alpha\beta}^{(1)},$$

anyway, in 4-dimensions, the variational derivative tensor $E = E_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$ that corresponding to the most general quadratic Lagrangian (3) on an Einstein spacetime (10) identically vanishes [3]. Consequently, the fourth-order field Equations (14) on an Einstein spacetime and the most general Einstein's Equations (8) are equivalent, where:

$$\left(\Lambda - \frac{1}{4}R\right)g_{\alpha\beta} = \chi T_{\alpha\beta}, \quad R = const.$$

It is obvious that the choice $a_0 = a_1 = 0$ of the gravitational Lagrangian (2), leads to the Einstein -Hilbert gravitational Lagrangian, that is:

$$L_{EH} := -2\Lambda + R,$$

which yields the most general Einstein's Equations (8). On the other hand, the choice $a_1 = -3a_0$ of the gravitational Lagrangian (4), leads to

$$L_2 := a_0\left(R^2 - 3|Ric|^2\right). \tag{15}$$

Comparing (7), (15) we obtain the Bach gravitational Lagrangian, that is:

$$L_2 \equiv L_{Bach} := \frac{-3}{2}a_0|Weyl|^2, \tag{16}$$

which, leads, possibly supplemented by an appropriate choice of L_{mat}, to conformably invariant fourth-order field equations, namely the equations introduced by R. Bach in 1921 [12]:

$$3a_0 B_{\alpha\beta} = \chi T_{\alpha\beta}, \tag{17}$$

that called the Bach field equations, where the Bach tensor $Bach = B_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}$ [7]-[13] is given by:

$$\begin{aligned} B_{\alpha\beta} &= 2\nabla^{\mu}\nabla^{\nu}W_{\mu\alpha\beta\nu} - W_{\mu\alpha\beta\nu}R^{\mu\nu} \\ &= \Box\left(R_{\alpha\beta} - \frac{R}{6}g_{\alpha\beta}\right) - \frac{1}{3}\nabla_{\alpha}\nabla_{\beta}R - \left(W_{\mu\alpha\beta\nu} + R_{\mu\alpha\beta\nu} - R_{\alpha\mu}g_{\beta\nu}\right)R^{\mu\nu}. \end{aligned} \tag{18}$$

One can easily show that the Bach tensor (18) is symmetric, trace-free; that is, $g^{\alpha\beta}B_{\alpha\beta} = 0$, divergence-free; that is, $\nabla^{\alpha}B_{\alpha\beta} = 0$, and is conformably invariant of weight -1 [3] [8].

We can rewrite the gravitational Lagrangian (2) in terms of (16) as

$$L_{grav} := -2\Lambda + R + \frac{-3}{2}a_0|Weyl|^2, \tag{19}$$

which leads to the Bach-Einstein field equations

$$\Lambda g_{\alpha\beta} + G_{\alpha\beta} + 3a_0 B_{\alpha\beta} = \chi T_{\alpha\beta}. \tag{20}$$

Using Equations (14)-(17) the Bach-Einstein field Equations (20) can be rewritten as:

$$\Lambda g_{\alpha\beta} + R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + a_0 \Bigg(3\Box R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \Box R - \nabla_\alpha \nabla_\beta R$$
$$+ 2RR_{\alpha\beta} - 6R_{\mu\alpha\beta\nu} R^{\mu\nu} - \frac{L_2}{2} g_{\alpha\beta} \Bigg) = \chi T_{\alpha\beta}. \tag{21}$$

3. Algebraic Structure

Generally, let us consider fourth-order gravitational field equations take the form:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} + \varepsilon D_{\alpha\beta}^{\mu\nu} R_{\mu\nu} = \chi T_{\alpha\beta}, \tag{22}$$

where ε is a real parameter such that $|\varepsilon|$ is so small, that is $|\varepsilon| \ll 1$. The tensor field T is assumed to be divergence-free:

$$\nabla^\beta T_{\alpha\beta} = 0.$$

According to that we require the identity

$$\nabla^\beta \left(D_{\alpha\beta}^{\mu\nu} R_{\mu\nu} \right) = 0.$$

We assume, without restriction of generality that, $D_{\alpha\beta}^{\mu\nu}$ is symmetric in α and β as well as in μ and ν

$$D_{\alpha\beta}^{\mu\nu} = D_{(\alpha\beta)}^{\mu\nu} = D_{\alpha\beta}^{(\mu\nu)},$$

It is easy to see that (22) is a singular perturbation of (8) since the small parameter ε appears as a factor of the higher-order term $D_{\alpha\beta}^{\mu\nu} R_{\mu\nu}$. Now, we show that the Ricci tensor components $R_{\alpha\beta}$ to g can be represented by a covariant linear differential operator applied to a linear combination of $T_{\mu\nu}, g_{\mu\nu} trT, \Lambda g_{\mu\nu}$ plus an error term with the factor ε^2.

Contraction of (22) with $g^{\alpha\beta}$ yields

$$R = 4\Lambda + \varepsilon D^{\mu\nu} R_{\mu\nu} - \chi trT.$$

Inserting this value for R in (22), we get

$$\left(\delta_\alpha^\mu \delta_\beta^\nu + \varepsilon \tilde{D}_{\alpha\beta}^{\mu\nu} \right) R_{\mu\nu} = \chi \tilde{T}_{\alpha\beta} + \Lambda g_{\alpha\beta}, \tag{23}$$

where

$$\tilde{T}_{\alpha\beta} := T_{\alpha\beta} - \frac{trT}{2} g_{\alpha\beta}, \tag{24}$$

$$\tilde{D}_{\alpha\beta}^{\mu\nu} := D_{\alpha\beta}^{\mu\nu} - \frac{1}{2} D^{\mu\nu} g_{\alpha\beta}, \quad D^{\mu\nu} := g^{\alpha\beta} D_{\alpha\beta}^{\mu\nu}. \tag{25}$$

The linear tensor-operator with the components $\delta_\alpha^\mu \delta_\beta^\nu + \varepsilon \tilde{D}_{\alpha\beta}^{\mu\nu}$ on the left-hand side in (23) has an approximate inverse with the components $\delta_\alpha^\mu \delta_\beta^\nu - \varepsilon \tilde{D}_{\alpha\beta}^{\mu\nu}$, in analogy to the formula

$$(1 + \varepsilon q)^{-1} = 1 - \varepsilon q + \varepsilon^2 r,$$

where the remainder term

$$r = (1 + \varepsilon q)^{-1} q^2,$$

is continuous in ε if q continuously depends on ε and $|\varepsilon|$ is so small such that $|\varepsilon q| < 1$. Thus, in general, the Ricci tensor components $R_{\alpha\beta}$ to g can be represented approximately by a covariant linear differential operator applied to a linear combination of $T_{\mu\nu}, g_{\mu\nu} trT, \Lambda g_{\mu\nu}$ as:

$$R_{\alpha\beta} \cong \left(\delta_\alpha^\mu \delta_\beta^\nu - \varepsilon \tilde{D}_{\alpha\beta}^{\mu\nu}\right)\left(\chi \tilde{T}_{\mu\nu} + \Lambda g_{\mu\nu}\right), \tag{26}$$

where $\cong$ means equality up to terms with the factor ε^2. It is obvious that for $\varepsilon = 0$ both (22) and (26) reduce to the most general Einstein's Equations (8).

4. Perturbation on the Bach-Einstein Field Equations

Let us apply the approximation procedure of section 3 to a class of fourth-order gravitational field equations in 4-dimensions, whence, the Bach-Einstein field Equations (21). Namely, let us consider a Lagrangian

$$L := L_{grav} + 2\chi L_{mat}, \tag{27}$$

here the gravitational Lagrangian has the form

$$L_{grav} := -2\Lambda + R + \varepsilon\left(a_0 R^2 + a_1 |Ric|^2\right), \tag{28}$$

ε is a small parameter.

Thus, the fourth-order field equations take, simply, the symbol form:

$$E_{\alpha\beta} \equiv \Lambda g_{\alpha\beta} + G_{\alpha\beta} + \varepsilon\left(a_0 E_{\alpha\beta}^{(0)} + a_1 E_{\alpha\beta}^{(1)}\right) = \chi T_{\alpha\beta}, \tag{29}$$

where $E_{\alpha\beta}^{(0)}$ and $E_{\alpha\beta}^{(1)}$ are given respectively in (12) and (13).

The field equations derived from the Lagrangian (27), with the gravitational Lagrangian (28) have the form (22) with $D_{\alpha\beta}^{\mu\nu}$ in the form

$$\begin{aligned} D_{\alpha\beta}^{\mu\nu} &= a_0\left(2\nabla_\alpha \nabla_\beta - 2g_{\alpha\beta}\Box + 2R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\right) g^{\mu\nu} \\ &\quad - a_1\left(\delta_\alpha^\mu \delta_\beta^\nu \Box + \frac{1}{2} g_{\alpha\beta} g^{\mu\nu}\Box - 2\delta_{(\alpha}^\nu \nabla^\mu \nabla_{\beta)} - 2\delta_{(\alpha}^\nu R_{\beta)}^\mu + \frac{1}{2} R^{\mu\nu} g_{\alpha\beta}\right). \end{aligned}$$

It is noticeable that, the Riemann curvature tensor has been eliminated by means of the Ricci identity

$$2R_{\mu\alpha\beta\nu} R^{\mu\nu} = 2\nabla_\mu \nabla_\beta R_\alpha^\mu - \nabla_\alpha \nabla_\beta R + 2R_{\alpha\mu} R_\beta^\mu.$$

Applying the results of (24)-(26) to the present situation yields

$$R_{\alpha\beta} \cong \left(\delta_\alpha^\mu \delta_\beta^\nu - \varepsilon \tilde{D}_{\alpha\beta}^{\mu\nu}\right)\left(\chi \tilde{T}_{\mu\nu} + \Lambda g_{\mu\nu}\right),$$

where, in this case

$$\tilde{T}_{\alpha\beta} := T_{\alpha\beta} - \frac{trT}{2} g_{\alpha\beta},$$

$$\begin{aligned} \tilde{D}_{\alpha\beta}^{\mu\nu} &= a_0\left(2\nabla_\alpha \nabla_\beta + g_{\alpha\beta}\Box + 2R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\right) g^{\mu\nu} \\ &\quad - a_1\left(\delta_\alpha^\mu \delta_\beta^\nu \Box - g_{\alpha\beta} g^{\mu\nu}\Box + g_{\alpha\beta}\nabla^\mu \nabla^\nu - 2\delta_{(\alpha}^\nu \nabla^\mu \nabla_{\beta)} - 2\delta_{(\alpha}^\nu R_{\beta)}^\mu + \frac{1}{2} R^{\mu\nu} g_{\alpha\beta}\right). \end{aligned}$$

We arrive at

$$R_{\alpha\beta} \cong \chi\tilde{T}_{\alpha\beta} + \Lambda g_{\alpha\beta} + \varepsilon a_0\left(2\nabla_\alpha\nabla_\beta + g_{\alpha\beta}\Box + 2R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right)\left(\chi trT - 4\Lambda\right)$$
$$+\varepsilon a_1\left[\delta^\mu_\alpha\delta^\nu_\beta\Box - 2\delta^\nu_{(\alpha}\nabla^\mu\nabla_{\beta)} - 2\delta^\nu_{(\alpha}R^\mu_{\beta)}\right.$$
$$\left.-g_{\alpha\beta}\left(g^{\mu\nu}\Box - \nabla^\mu\nabla^\nu - \frac{1}{2}R^{\mu\nu}\right)\right]\left(\chi\tilde{T}_{\mu\nu} + \Lambda g_{\mu\nu}\right). \tag{30}$$

Since we neglect the terms of order ε^2, then we substitute by the following expressions for $R_{\alpha\beta}$ and R:

$$R_{\alpha\beta} \cong \chi\tilde{T}_{\alpha\beta} + \Lambda g_{\alpha\beta}, \quad R \cong -\chi trT + 4\Lambda,$$

in (30), so we get the perturbation of (29) as:

$$R_{\alpha\beta} \cong \chi\left(T_{\alpha\beta} - \frac{trT}{2}g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta} + \varepsilon a_0\chi\left[2\nabla_\alpha\nabla_\beta trT + 2\left(\chi trT - 4\Lambda\right)T_{\alpha\beta}\right.$$
$$\left.+g_{\alpha\beta}\left(\Box trT - \frac{1}{2}\chi\left(trT\right)^2 + 2\Lambda trT\right)\right] + \varepsilon a_1\chi\left[\Box T_{\alpha\beta} + \nabla_\alpha\nabla_\beta trT\right.$$
$$-2\nabla^\mu\nabla_{(\alpha}T_{\beta)\mu} - 2\chi T_{\mu\alpha}T^\mu_\beta + 2\left(\chi trT - 2\Lambda\right)T_{\alpha\beta}$$
$$\left.+g_{\alpha\beta}\left(\frac{1}{2}\chi|T|^2 - \frac{1}{2}\chi\left(trT\right)^2 + \Lambda trT\right)\right], \tag{31}$$

up to terms with the factor ε^2. The trace part of (31) reads

$$R \cong \chi\left(2\varepsilon\left(3a_0 + a_1\right)\Box - 1\right)trT + 4\Lambda.$$

Accordingly, (15)-(20), (27)-(29) and (31), we can easily deduce:

$$R_{\alpha\beta} \cong \chi\left(T_{\alpha\beta} - \frac{trT}{2}g_{\alpha\beta}\right) + \Lambda g_{\alpha\beta} - \varepsilon a_0\chi\left[\nabla_\alpha\nabla_\beta trT - 6\nabla^\mu\nabla_{(\alpha}T_{\beta)\mu}\right.$$
$$+3\Box T_{\alpha\beta} - g_{\alpha\beta}\left(\Box trT - \frac{3}{2}\chi|T|^2 + \chi\left(trT\right)^2 - \Lambda trT\right) - 6\chi T_{\mu\alpha}T^\mu_\beta$$
$$\left.+4\left(\chi trT - \Lambda\right)T_{\alpha\beta}\right], \tag{32}$$

which are a perturbation on the Bach-Einstein field equations.

Simply, the choice $a_1 = -3a_0$ of the perturbation Equation (31), leads to a perturbation on the Bach-Einstein field Equations (32). On the other hand, the choice $\varepsilon = 0$ or $a_0 = a_1 = 0$ of the perturbation Equation (31), leads to the most general Einstein's Equations (9). Of course, the choice $\varepsilon = 0$ or $a_0 = 0$ of a perturbation on the Bach-Einstein field Equations (32) leads, also, to the most general Einstein's Equations (9).

5. Conclusions and Discussions

There is a well-established theory and a broad literature on singular perturbations of differential equations [3]. We circumvent here this theory by assuming the existence of solutions regular in the perturbation parameter ε, and we deduce the result (32) on the latter.

The approximate inversion Formulas (31) and (32) derived here stress the role of the Ricci tensor in the class of alternative gravitational theories under consideration. Let us recall that the Ricci tensor $Ric = R_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$ appears in several formulas of geometrical or physical importance:

- The volume of geodesic balls in Riemannian geometry can be expanded with respect to the radius [25]; analogously the volume of truncated light cones in Lorentzian geometry can be expanded with respect to the truncation time parameter [7]. The leading terms of the deviations from the flat space or flat spacetime values are linear in the Ricci tensor. Moreover, some estimate for Ric leads to estimates for the volume of geodesic balls [26] [27].
- The Raychaudhuri equation for the so-called geometrical expansion θ of a family of timelike geodesics with tangent vector field $u = u^\alpha \partial_\alpha$ reads

$$R_{\alpha\beta}u^\alpha u^\beta = \nabla_\alpha \dot{u}^\alpha + \omega^2 - \sigma^2 - \dot{\theta} - \frac{1}{3}\theta^2,$$

where the dot abbreviates the derivative $u^\alpha \nabla_\alpha$ and where the rotation ω, the shear σ, and the expansion θ of u arise from the decomposition of $\nabla_\alpha u_\beta + \dot{u}_\alpha u_\beta$ into irreducible parts (cf. e.g., [28] [29] [30]).

- Singularity theorems of Hawking-Penrose type are based on assumptions on the Ricci tensor [31] [32].
- Tolman's formula expresses the total active mass of a static, asymptotically flat spacetime as

$$M = \frac{2}{\chi}\int_S R_{\alpha\beta}n^\alpha u^\beta \mathrm{d}\sigma,$$

where $n = n^\alpha \partial_\alpha$ denotes the unit normal to the spacelike hypersurface, $u = u^\alpha \partial_\alpha$ is the timelike Killing vector field, and $\mathrm{d}\sigma$ is the natural volume element of the hypersurfaces [1] [23]. D. Barraco and V. H. Hamity [1] mention Tolman's expression as a possible application of approximate inversion formulas.

The Formulas (31) and (32) express the Ricci tensor in terms of the energy-momentum tensor $T = T_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$. Such a result can be inserted into each of the above-mentioned geometrical or physical formulas where the Ricci tensor plays a dominant role. By this, the influence of the energy-momentum tensor becomes transparent.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Barraco, D. and Hamity, V.H. (1999) A Theorem Relating Solutions of a Fourth-Order Theory of Gravity to General Relativity. *General Relativity and Gravitation*, **31**, 213-218. https://doi.org/10.1023/A:1018892110584

[2] Campanelli, M., Lousto, C.O. and Audretsch, J. (1994) A Perturbative Method to Solve Fourth-Order Gravity Field Equations. *Physical Review D*, **49**, 5188-5193. https://doi.org/10.1103/PhysRevD.49.5188

[3] Abdel-Bassier, F.I. (2002) Structure of Higher-Order Gravitational Field Equations on n Dimensional Spacetimes. Ph.D. Thesis, Mathematics Departement, Minia University, El-Minia, Egypt.

[4] Schmidt, H.-J. (1986) The Newtonian Limit of Fourth-Order Gravity. *Astronomische Nachrichten*, **307**, 339-340. https://doi.org/10.1002/asna.2113070526

[5] Larin, S.A. (2018) Fourth-Derivative Relativistic Quantum Gravity. EPJ Web Conferences.

[6] Bergman, J. (2004) Conformal Einstein Spaces and Bach Tensor Generalizations in n Dimensions. Linkoeping Studies in Science and Technology. Theses No. 1113, Matematiska Institutionen, Likoepings Universitet, Likoeping, Sweden.

[7] Schimming, R. (1988) Lorentzian Geometry as Determined by the Volume of Small Truncated Light Cones. *Archivum Mathematicum*, **24**, 5-15. https://eudml.org/doc/18228

[8] Schimming, R. (1998) On the Bach and the Bach-Einstein Gravitational Field Equations. World Scientific, Singapore, 39-46.

[9] Fiedler, B. and Schimming, R. (1980) Exact Solutions of the Bach Field Equations of General Relativity. *Reports on Mathematical Physics*, **17**, 15-36. https://doi.org/10.1016/0034-4877(80)90073-7

[10] Anselli, A. (2021) On the Bach and Einstein Equations in Presence of a Field. *International Journal of Geometric Methods in Modern Physics*, **18**, Article ID: 2150077. https://doi.org/10.1142/S0219887821500778

[11] Kan, N., Kobayashi, K. and Shiraishi, K. (2013) "Critical" Cosmology in Higher Order Gravity. *International Scholarly Reasearch Notices*, **2013**, Article ID: 651684. https://doi.org/10.1155/2013/651684

[12] Bach, R. (1921) Zur Weylschen Relativitaetstheorie und der Weylschen Erweiterung des Kruemmungsbegriffs. *Mathematische Zeitschrift*, **9**, 110-135. https://doi.org/10.1007/BF01378338

[13] Schimming, R., Abdel-Megied, M. and Ibrahim, F. (2003) Cauchy Constraints and Particle Content of Fourth-Order Gravity in n Dimensions. *Chaos, Solitons & Fractals*, **15**, 57-74. https://doi.org/10.1016/S0960-0779(02)00090-5

[14] Buchdahl, H. (1948) The Hamiltonian Derivatives of a Class of Fundamental Invariants. *The Quarterly Journal of Mathematica*, **os-19**, 150-159. https://doi.org/10.1093/qmath/os-19.1.150

[15] Buchdahl, H. (1951) On the Hamilton Derivatives Arising from a Class of Gauge-Invariant Action Principles in a W_n. *Journal of the London Mathematical Society*, **26**, 139-149. https://doi.org/10.1112/jlms/s1-26.2.139

[16] Buchdahl, H. (1973) Functional Derivatives of Invariants of the Curvature Tensor of Unitary Spaces. *Tensor New Series*, **27**, 247-256.

[17] Guendelman, E.I. and Katz, O. (2003) Inflation and Transition to a Slowly Accelerating Phase from SSB of Scale Invariance. *Classical Quantum Gravitation*, **20**, 1715-1728. https://doi.org/10.1088/0264-9381/20/9/309

[18] Lee C.H. and Lee, H.K. (1988) Kasner-Type Solution in a Higher-Derivative Gravity Theory. *Modern Physics Letters A*, **3**, 1035-1039. https://doi.org/10.1142/S0217732388001215

[19] Raschewski, P.K. (1959) Riemannsche Geometrie und Tensoranalysis. Veb Deutscher Verlag der Wissenschaften, Berlin.

[20] Foster, J. and Nightingale, J.D. (2006) A Short Course in General Relativity. 3rd Edition, Springer Science + Business Media, Inc., Berlin. https://doi.org/10.1007/978-0-387-27583-3

[21] Lord, E.A. (1976) Tensors, Relativity and Cosmology. Tata McGraw-Hill Company Ltd., New Delhi.

[22] Eisenhart, L.P. (1964) Riemannian Geometry. Princeton University Press, New Jersey.

[23] Tolman, R.C. (1930) On the Use of the Energy-Momentum Principle in General Relativity. *Physical Review Journals Archive*, **35**, 875-895. https://doi.org/10.1103/PhysRev.35.875

[24] Schmutzer, E. (1968) Relativistische Physik. Klassiche Theorie, Leipzig, German.

[25] Gray, A. and Vanhecke, L. (1979) Riemannian Geometry as Determined by the Volumes of Small Geodesic Balls. *Acta Mathematica*, **142**, 157-198. https://doi.org/10.1007/BF02395060

[26] Günther, P. (1960) Einige Sätze ueber das Volumenelement eines Riemannschen Raumes. *Debrecen*, **7**, 78-93. https://doi.org/10.5486/PMD.1960.7.1-4.08

[27] Bishop, R. and Crittenden, R. (1964) Geometry of manifolds. Academic Press, New York and London.

[28] Raychaudhuri, A. (1979) Theoretical Cosmology. Oxford University Press, New York.

[29] Komar, A. (1959) Covariant Consevation Laws in General Relativity. *Physical Review Journals Archive*, **113**, 934-936. https://doi.org/10.1103/PhysRev.113.934

[30] Kramer, D., Stephani, H., MacCallum, M. and Herlt, E. (1980) Exact Solutions of Einstein's Field Equations. Cambridge University Press, Cambridge, MA.

[31] Hawking, S.W. and Ellis, G.F.R. (1973) The Large-Scale Structure of Space-Time. Cambridge University Press, Cambridge, MA. https://doi.org/10.1017/CBO9780511524646

[32] Hawking, S.W. and Penrose, R. (1970) The Singularities of Gravitational Collapse and Cosmology. *Proceedings of the Royal Society A*, **314**, 529-548. https://royalsocietypublishing.org https://doi.org/10.1098/rspa.1970.0021