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Space Discretization of Time-Fractional Telegraph Equation with Mamadu-Njoseh Basis Functions

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Abstract

In this paper, we examine the space discretization of time fractional telegraph equation (TFTE) with Mamadu-Njoseh orthogonal basis functions. For ease and convenience, we deal with the fractional derivative by first converting from Caputo's type to Riemann-Liouville's type. The proposed method was constrained to precise error analysis to establish the accuracy of the method. Numerical experimentation was implemented with the aid of MAPLE 18 to show convergence of the method as compared with the analytic solution.

Keywords

Finite Difference Method, Mamadu-Njoseh Polynomials, Telegraph Equation, Gaussian Elimination Method, Quadrature Formula, Sobolev Space

1. Introduction

The popularity of fractional partial differential equations (FPDEs) gained momentum in science and engineering due to its involvement in many areas of applications ([1]). Many researchers have developed numerical techniques for solving FPDEs. Some of the methods include finite difference method ([2] [3] [4] [5]), spectral method ([6] [7] [8] [9]), spline function method ([10]), finite element method ([11] [12] [13] [14] [15]) variational method ([16]), etc. However, the development of these enormous numerical procedures for FPDEs still poses meaningful challenges such as the use of orthogonal polynomials as basis functions.

A time fractional telegraph equation (TFTE) has the form ([17])

$$\frac{{}_{0}^{C}\partial^{\beta} u(x,t)}{\partial t^{\beta}} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^{2} u(x,t)}{\partial x^{2}} = g(x,t),\ 0 \le x \le T,\ t > 0, \tag{1.1}$$

with the initial conditions

$$u(x,0)=u_0(x), \frac{\partial u(x,0)}{\partial t}=u_1(x), 0\le x\le T, t>0, \tag{1.2}$$

and boundary conditions

$$u(0,t)=u(0,T)=0, 0\le x\le T, t>0, \tag{1.3}$$

where $1<\beta<2$, $g(x,t)$ is the source term and $\frac{{}_0^C\partial^\beta u(x,t)}{\partial t^\beta}$ is Caputo fractional derivative of $u(x,t)$.

The TFTE is a hyperbolic partial differential equation responsible for modeling many physical phenomena, such as wave propagation, signal processing, random walk theory and so on. Consequently, TFTE has been studied by many authors. Riemann-Liouville's method was adopted by Cascaval *et al.* ([18]) for analyzing the solution of TFTE. Orsingher and Beghin ([19]) studied the TFTE governed by a Brownian time. The method of separable variable was used by Chen *et al.* ([20]) for solving TFTE constrained to three nonhomogeneous boundary conditions. Momani ([21]) solved the approximate and analytic solution of space and time fractional telegraph equations via Adomian decomposition method (ADM).

In this paper, we solve (1.1)-(1.3) with Mamadu-Njoseh orthogonal basis functions in a space discretization approach. Here, the process of discretization is quite different from the classical numerical method—*finite difference method*. In FEM, the given differential equation has to be reformulated as a variational problem leading to the solution via the following steps:

1) Finite dimensional space construction, U_h. This is the discretization process;

2) Seeking solution to the resultant discrete problem; and

3) Implementation through a computer programming.

This paper is organized as follows. Section 2 constitutes preliminaries. Finite element method for time fractional telegraph equation is given in Section 3. Error analysis is given in Section 4. Numerical illustrations, tables of results and graphical simulations are given in Section 5 and Section 6. Discussion of results and conclusions are presented in Sections 7 and Section 8, respectively.

2. Preliminaries

Let's use the notation

$$\alpha\le\tau A \text{ and } \alpha\le QA,$$

where τ and Q are constants free of α and A, and are discretization parameters.

Let R and γ be two given Hilbert spaces, $\|.\|_{R\to\gamma}$ is defined as

$$\|G\|_{R\to\gamma} = \sup_{\theta\in R,\theta\ne 0}\frac{\|G(\theta)\|y}{\|\theta\|x}.$$

2.1. Weak Derivative

Suppose $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ represent a multi-index and $|\beta| = \sum_{i=1}^{n} \beta_j$. For a well defined smooth function $U \in \Omega$, D^β, being the differential operator is given by ([22] [23])

$$D^\beta U = \partial^{|\beta|} U .$$

Now, an integrable function V is said to possess a weak derivative U, if U satisfies

$$\int_\Omega U \phi \mathrm{d}x = (-1)^{|\beta|} \int_\Omega D^\beta \phi \mathrm{d}x , \quad \forall \phi \in C_0^\infty(\Omega),$$

where, $C_0^\infty(\Omega)$ denotes the space of infinity differentiable functions supported compactly in Ω. We assume D^β to be weak derivative throughout this research.

2.2. Sobolev Spaces

Let $U \in \Omega$ be a lebesque measurable function and $q \ge 1$. The norm $\|.\|_{L^P(\Omega)}$ be defined by ([24])

$$\|U\|_{L^P(\Omega)} = \left(\int_\Omega |u(x)|^q \mathrm{d}x \right)^{1/q},$$

where $L^P(\Omega)$ denotes the set of all U such that $\|U\|_{L^P(\Omega)}$ is finite. Given an integer $k \ge 0$, we have the Sobolev space $W^{k,p}(\Omega)$ given as

$$W^{k,p}(\Omega) = \left\{ U \in L^P(\Omega) : D^\beta U \in L^P(\Omega), \forall |\beta| \le k \right\}.$$

Also,

$$\|U\| W^{k,p}(\Omega) = \left(\sum_{\|\beta\| \le k} \|D^\beta U\|_{L^P(\Omega)} \right)^{1/2},$$

are the corresponding Sobolev and Semi norms of $W^{k,p}(\Omega)$ respectively.

Now for $0 < k < 1$, $|\cdot| W^{k,p}(\Omega)$ is defined by

$$|U| W^{k,p}(\Omega) = \int_\Omega \int_\Omega \frac{|u(x) - y(v)|^p}{|x \forall v|^{d+kp}} (\mathrm{d}x\mathrm{d}p)^{1/p},$$

called the fractional Sobolev Semi norm with

$$W^{k,p}(\Omega) = \left\{ U \in L^P(\Omega) : |U| W^{k,p}(\Omega) < \infty \right\}.$$

For $q \ge 0$, we write $q = n + k$, $n \ge q$, $k \in (0,1)$.

Thus, the Sobolev space becomes

$$W^{q,p}(\Omega) = \left\{ u \in W^{n,p} : D^\beta U \in W^{k,p}(\Omega), \forall |\beta| = n \right\},$$

and

$$\|U\| W^{q,p}(\Omega) = \left(U^p W^{n,p}(\Omega) + \sum_{|\beta|=n} |D^\beta U|^p W^{k,p}(\Omega) \right)^{\frac{1}{2}},$$

is the full norm. For $q \ge 0$, Sobolev space $W^{q,p}(\Omega)$ is a Banach space ([25]).

Similarly,

when $p = 2$, the sobolev space $W^{q,2}(\Omega)$ is a Hilbert space, that is,

$$H^2(\Omega) = W^{q,2}(\Omega).$$

In particular, to solve our model equation we define the Sobolev space as

$$H_0^n(\Omega) = \left\{U \in H^n(\Omega) : \partial\beta|_{\partial\Omega} = 0, \forall|\beta| \le n-1\right\}.$$

To establish the equivalences of certain norms in the subspaces of $H_0^n(\Omega)$, we shall rely in the famous Poincaré inequalities.

Lemma 2.1. ([26]): For $C \ge 0$, then

$$\|V\|_{L^P(\Omega)} \le C|V|H'(\Omega), \forall V \in H_0'(\Omega).$$

Lemma 2.2. ([26]): For $C \ge 0$, then

$$\left\| V - \frac{\int_\Omega v\mathrm{d}x}{meas(\Omega)} \right\|_{L^P(\Omega)} \le C\|V\|H'(\Omega), \forall V \in H'(\Omega).$$

Lemma 2.3 ([26]): For $C \ge 0$, then

$$\|V\|H^S(\Omega) \le C|V|H^n(\Omega), \forall s \le n, \forall V \in H^n(\Omega),$$

which is generalized poincare inequality.

Thus, $|\cdot|_{n,\Omega}$ over the space $H_0^n(\Omega)$ is equivalent to $\|\cdot\|_{n,\Omega}$.

2.3. Caputo Fractional Derivatives

Let $[a,b] \in \mathbb{R}$, $D_{a+}^\beta[U(t)](x) \equiv (D_{a+}^\beta U)(x)$, and $D_{b-}^\beta[U(t)](x) \equiv (D_{-b}^\beta U)(x)$ be the Reimann-Liouville (R-L) fractional derivatives of order β. The fractional derivatives of order $({}^cD_{a+}^\beta U)(x)$ and $({}^cD_{b-}^\beta U)(x)$ of order β on $[a,b] \in \mathbb{R} > 0$, are as ([27])

$$({}^cD_{a+}^\beta U)(x) = \left(D_{a+}^\beta\left[u(t) - \sum_{i=0}^{m-1}\frac{u^{(k)}(a)}{i!}(t-a)^i\right]\right)(x) \tag{2.1}$$

$$({}^cD_{b-}^\beta U)(x) = \left(D_{b-}^\beta\left[u(t) - \sum_{i=0}^{m-1}\frac{u^{(k)}(b)}{i!}(b-a)^i\right]\right)(x) \tag{2.2}$$

respectively, where $m = [\mathbb{R}(\beta)] + 1$ for $\beta \notin \mathbb{N}_0$, $m = \beta$ for $\beta \in \mathbb{N}_0$.

The above Equations (2.1) and (2.2) are called left- and right-sided Caputo fractional derivatives of order β.

Lemma 2.4 ([27]):

Let $r(x) \in C_{-1}^n$, $n \in \mathbb{N} \cup \{0\}$. Then the caputo fractional derivative of $r(x)$ is given as $D^\beta U(x) = I^{\lambda-\gamma}D^n U(x)$, satisfying the following properties:

(a) $D^\beta\left(I^\beta U(x)\right) = U(x)$

(b) $I^\beta\left(D^\beta U(x)\right) = \gamma(x) - \sum_{i=1}^{m-1} U^k(0^+)\left(\frac{x^i}{i!}\right)$

(c) $$D^\beta x^\gamma = \begin{cases} 0, \gamma \in \mathbb{N}_a, \gamma < \beta_a \\ \dfrac{\Gamma(\gamma+1)}{-\Gamma(\gamma-\beta+1)}x^{\gamma-\beta}, \gamma \in \mathbb{N}_a, \gamma \ge \beta_a \end{cases}, \tag{2.3}$$

where $\beta_a \ge a$ and $\mathbb{N}_a = \{0,1,2,3,\cdots\}$.

2.4. Mamadu-Njoseh Polynomials

These are orthoponal polynomials generated with reference to the properties ([28] [29] [30])

$$\varphi_m(x)=\sum_{j=0}^{m}a_jx^j\,,\quad x\in[-1,1]\,, \tag{2.4}$$

$$B[j_+]=\int_a^b\left(1+x^2\right)\varphi_{j-1}(x)\left(\sum_{j=0}^{m}a_jx^j\right)\mathrm{d}x=0\,,\quad j=1(2)n\,, \tag{2.5}$$

subject to the initial conditions

$$\varphi_0(x)=1\ \text{ and }\ \varphi_n(1)=1\,, \tag{2.6}$$

where j_+ denotes a unit step increment, $w(x)$ is weight function.

Lemma **2.5:** For any $Z_+\cup\{0\}$ value of *j*, $\exists$ a partition $j_0<j_1<j_2<\cdots<j_{n-1}<j_n$ with a unit step size.

Theorem **2.1.** For *m* = *j*, there exists *n* system of linear algebraic Equations generated from using (2.4)-(2.6) at the $(j_0,m),(j_1,m),\cdots,(j_{n-1},m),(j_n,m)$, respectively.

Proof. Let $B[j_+]$ be given by lemma (2.5), we have $j_0<j_1<\cdots<j_{r-1}<j_r$. Thus, for $m=j=r$, the grid points of the partition by refinement would $(j_0,m),(j_1,m),\cdots,(j_{r-1},m),(j_r,m)$. Hence, we have

$$B[j_{n+1}]=\int_a^b w(x)\varphi_n(x)\varphi_m(x)\mathrm{d}x=0\ \text{ at }\ (j_n,m).$$

The first Mamadu-Njoseh polynomials are general via MAPLE 18 via theorem 2.1, and are presented in Figure 1 and Table 1, respectively.

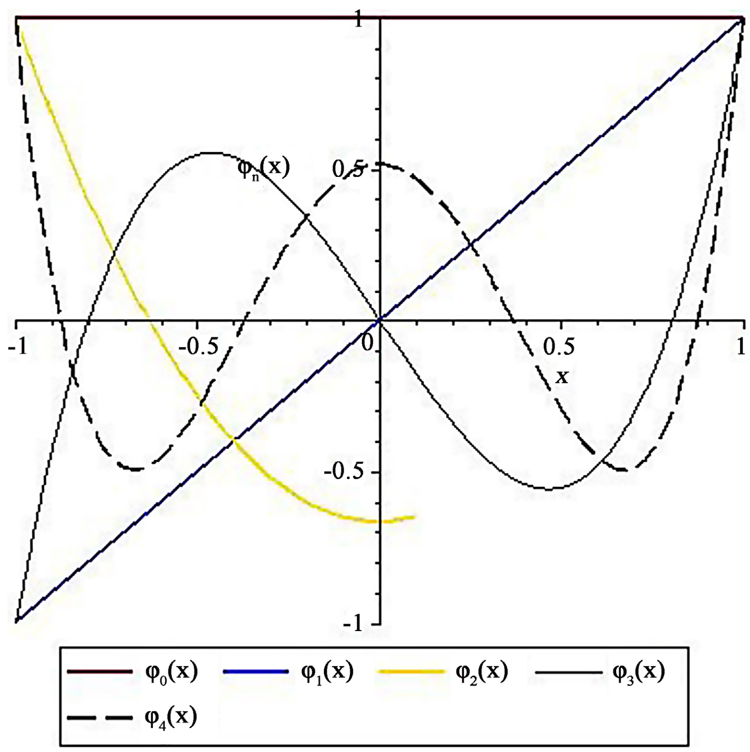

Figure 1. Graphical view Mamadu-Njoseh Polynomials $\varphi_n(x)$.

Table 1. First seven Mamadu-Njoseh Polynomials.

n	Mamadu-Njoseh polynomials, $\varphi_n(x)$
0	1
1	x
2	$\frac{1}{3}(5x^2-2)$
3	$\frac{1}{5}(14x^3-9x)$
4	$\frac{1}{648}(333-2898x^2+3213x^4)$
5	$\frac{1}{136}(325x-1410x^3+1221x^5)$
6	$\frac{1}{1064}(-460+8685x^2-24750x^4+17589x^6)$

3. Finite Element Method for Time Fractional Telegraph Equation

We consider the space discretization time functional telegraph Equations (1.1)-(1.3) with Mamadu-Njoseh basis function using the finite element method.

Let a piecewise finite element space that is linear and continuous be given as V_h. Let [0, 1] be partitioned as

$$0 = x_0 < x_1 < x_2 < \cdots < x_n = 1,$$

called the space partitioning of $[a,b]$.

Let $V_h = \{S_h(x) : S_h(x) \text{ is continuous and linear in } [0,1]\}$.

The variational formulation for the time – fractional telegraph Equation (1.1) is to compute $u(t) \in H_0^1(a,b)$ such that

$$\begin{aligned}&\left({}_0^R D_t^\beta \left[u(x,t)-U_0\right], S(x)\right) + \left(U_t, S(x)\right) - \left(U_x, S(x)\right)\\&= \left(g(x,t), S(x)\right),\ S(x) \in H_0^1.\end{aligned} \tag{3.1}$$

The essence of FEM is to compute $U_h(t) \in V_h$, such that

$$\left({}_0^R D_t^\beta \left[u(x,t)-U_0\right], \gamma\right) + \left(\frac{\partial u}{\partial t}, \gamma\right) - \left(\frac{\partial u}{\partial x}, \frac{\partial \gamma}{\partial x}\right) = \left(\gamma, \frac{\partial \gamma}{\partial x}\right),\ \gamma \in V_h \tag{3.2}$$

Let $B_h = -\Delta_h : V_h \to V_h$ satisfies

$$\left(B_h U_h, \gamma\right) = \left(\frac{\partial u}{\partial t}, \gamma\right) - \left(\frac{\partial u}{\partial x}, \frac{\partial \gamma}{\partial x}\right),\quad \gamma \in V_h. \tag{3.3}$$

Suppose $G_h : G \to V_h$ defined a L_2 operator given by

$$\left(G_h s, \gamma\right) = (s, \gamma),\quad \forall \gamma \in V_h,\quad s \in L_2.$$

Thus, Equation (3.2) can be written in the abstract sense as

$$\left({}_0^R D_t^\beta \left[u(x,t)-U_0\right], S(x)\right) + B_h U_h = G_h g,\quad t > 0, \tag{3.4}$$

where

$$ {}_0^R D_t^\beta \left(u(x,t)\right) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \int_0^t \frac{u(s-x)}{(t-s)^{-\beta}} \mathrm{d}s, \quad \beta t(0,1), \tag{3.5} $$

called the Riemann-Liouville fractional derivative, and Γ is the Gamma function.

Using quadrative formula (([31])] on (3.4), we obtain

$$ {}_0^R D_t^\beta u(t_j) = \sum_{r=0}^{j} w_{rj} \left[u(t_j - t_r) - U_0 \right] + \frac{t_j^{-\beta}}{\Delta t^\beta \Gamma(-\beta)} G_j(g), \tag{3.6} $$

where w_{rj}

$$ w_{rj} = \frac{1}{\Gamma(2-\beta)} \begin{cases} 1, & r = 0 \\ -2r^{1-\beta} + (r-1)^{1-\beta} + (r+1)^{1-\beta}, & r = 1, 2, \cdots, j \end{cases}, \tag{3.6a} $$

and $G_j(g)$ satisfies

$$ \left\| G_j(g) \right\| \le K_j^{\beta-2} \sup_{0 \le t \le T} \left\| U''(t_J - t_{jw}) \right\|, \quad w \in [0,1]. $$

Now, let $u(x,t) = U_j \approx U_h(t_j) = \sum_{J=1}^{N-1} \alpha_j \varphi_j(x, t_j)$, be an approximation of $U_h(t_j)$, where $\varphi_j(x)$, $j = 0(1)(N-1)$, are Mamadu-Njoseh Basis function of V_h.

Also, let $g_j = g(t_j)$ defines the time discretization such that

$$ \begin{aligned} &\Delta t^{-\beta} \sum_{r=0}^{j} w_{rj} \left(U_{j-r} - U_0, \gamma\right) + \left(\frac{\partial U_j}{\partial t}, \gamma\right) - \left(\frac{\partial U_j}{\partial x}, \frac{\partial \gamma}{\partial x}\right) \\ &= \left(g_j, \frac{\partial \gamma}{\partial x}\right), \ j = 0(1)n, \forall \gamma \in V_h. \end{aligned} \tag{3.7} $$

Now, we consider the following steps for $j = 0(1)n$.

Step 1: Suppose $j = 0$, then $U_j = 0$.

Step 2: Set $j = 1$, we get,

$$ \begin{aligned} &\Delta t^{-\beta} W_{0,1}(U_1, \gamma) + \left(\frac{\partial U_1}{\partial t}, \gamma\right) - \left(\frac{\partial U_1}{\partial x}, \frac{\partial \gamma}{\partial x}\right) \\ &+ \Delta t^{-\beta} \left(\sum_{r=1}^{g} W_{r1}\left((U_j - U_0), \gamma\right) - W_{01}(U_0, \gamma) \right), \gamma \in V_h. \end{aligned} \tag{3.8} $$

Since $U_j \approx u(t_j) = \sum_{r=j}^{N-1} a_r \varphi_r(x, t_j)$, we have that,

$$ \Delta t^{-\beta} \left(\sum_{r=j}^{N-1} a_r \left(\varphi_r(x,t_j), \gamma\right) \right) + \sum_{r=j}^{N-1} a_r \left(\frac{\partial \varphi_r(x,t_j)}{\partial t}, \gamma \right) - \sum_{r=j}^{N-1} a_r \left(\frac{\partial \varphi_r(x,t_j)}{\partial x}, \frac{\partial \gamma}{\partial x} \right) $$

$$ = \left(g_1, \frac{\partial \gamma}{\partial x}\right) - \Delta t^{-\beta} \left(W_{11}(U_0 - u_0, \gamma) + W_{01}(U_0, \gamma) \right), \forall \gamma \in V_h $$

$$ \begin{aligned} &\Rightarrow \Delta t^{-\beta} \left(\sum_{r=j}^{N-1} a_r \left(\varphi_r(x,t_j), \gamma\right) \right) + \sum_{r=j}^{N-1} a_r \left(\left(\frac{\partial \varphi_r(x,t_j)}{\partial t}, \gamma \right) - \left(\frac{\partial \varphi_r(x,t_j)}{\partial x}, \frac{\partial \gamma}{\partial x} \right) \right) \\ &= \left(g_1, \frac{\partial \gamma}{\partial x}\right) - \Delta t^{-\beta} \left(W_{11}(U_0 - u_0, \gamma) + W_{01}(U_0, \gamma) \right), \forall \gamma \in V_h \end{aligned} \tag{3.9} $$

Let $\gamma = \varphi_k(x, t_j)$, $k = 1(2)(N-1)$, we have,

$$
\begin{aligned}
&\Delta t^{-\beta}\left(\sum_{r=j}^{N-1} a_r\left(\varphi_r(x),\varphi_k\left(x,t_j\right)\right)\right)\\
&+\sum_{r=j}^{N-1} a_r\left(\left(\frac{\partial\varphi_r\left(x,t_j\right)}{\partial t},\varphi_k\left(x,t_j\right)\right)-\left(\frac{\partial\varphi_r\left(x,t_j\right)}{\partial x},\frac{\varphi_k\left(x,t_j\right)}{\partial x}\right)\right)\\
&=\left(g_1,\frac{\partial\gamma}{\partial x}\right)-\Delta t^{-\beta}\left(W_{11}\left(U_0-u_0,\gamma\right)+W_{01}\left(U_0,\varphi_k\left(x,t_j\right)\right)\right),\ \forall\gamma\in V_h
\end{aligned}
\tag{3.10}
$$

Thus,

$$
\Delta t^{-\beta}W_{0,1}\left(M*R^1\right)+Q*R^1=G^1-\Delta t^{-\beta}W_{11}R^0+\Delta t^{-\beta}\sum_{r=0}^{1}W_{r1}U_0 \tag{3.11}
$$

where,

$$
\boldsymbol{M}=\begin{pmatrix}
\left(\varphi_1\left(x,t_j\right),\varphi_1\left(x,t_j\right)\right) & \left(\left(\varphi_2\left(x,t_j\right),\varphi_1\left(x,t_j\right)\right) & \cdots & \left(\varphi_{N-1}\left(x,t_j\right),\varphi_1\left(x,t_j\right)\right)\\
\left(\varphi_1\left(x,t_j\right),\varphi_2\left(x,t_j\right)\right) & \left(\varphi_2\left(x,t_j\right),\varphi_2\left(x,t_j\right)\right) & \cdots & \left(\varphi_{N-1}\left(x,t_j\right),\varphi_2\left(x,t_j\right)\right)\\
\vdots & \vdots & & \vdots\\
\left(\varphi_1\left(x,t_j\right),\varphi_{(N-1)}\left(x,t_j\right)\right) & \left(\varphi_2\left(x,t_j\right),\varphi_{(N-1)}\left(x,t_j\right)\right) & \cdots & \left(\varphi_1\left(x,t_j\right),\varphi_{(N-1)}\left(x,t_j\right)\right)
\end{pmatrix},
$$

$$
\boldsymbol{R}^1=\begin{pmatrix}
\left(\frac{\partial\varphi_1\left(x,t_j\right)}{\partial x},\varphi_1\left(x,t_j\right)\right)-\left(\frac{\partial\varphi_1\left(x,t_j\right)}{\partial x},\frac{\partial\varphi_1\left(x,t_j\right)}{\partial x}\right) & \cdots & \left(\frac{\partial\varphi_{N-1}\left(x,t_j\right)}{\partial t},\varphi_1\left(x,t_j\right)\right)-\left(\frac{\partial\varphi_{N-1}\left(x,t_j\right)}{\partial x},\frac{\partial\varphi_{N-1}\left(x,t_j\right)}{\partial x}\right)\\
\vdots & & \vdots\\
\left(\frac{\partial\varphi_1\left(x,t_j\right)}{\partial x},\varphi_{N-1}\left(x,t_j\right)\right)-\left(\frac{\partial\varphi_1\left(x,t_j\right)}{\partial x},\frac{\partial\varphi_{N-1}\left(x,t_j\right)}{\partial x}\right) & \cdots & \left(\frac{\partial\varphi_{N-1}\left(x,t_j\right)}{\partial t},\varphi_{N-1}\left(x,t_j\right)\right)-\left(\frac{\varphi_{N-1}\left(x,t_j\right)}{\partial x},\frac{\varphi_{N-1}\left(x,t_j\right)}{\partial x}\right)
\end{pmatrix}
$$

$$
\boldsymbol{G}^1=\left[\left(g_1,\varphi_1\left(x,t_j\right)\right)\ \left(g_1,\varphi_2\left(x,t_j\right)\right)\ \left(g_1,\varphi_3\left(x,t_j\right)\right)\ \cdots\ \left(g_1,\varphi_{N-1}\left(x,t_j\right)\right)\right]^{\mathrm{T}},
$$

$$
\boldsymbol{R}^0=\left[\left(U_0,\varphi_1\left(x,t_j\right)\right)\ \left(U_0,\varphi_2\left(x,t_j\right)\right)\ \left(U_0,\varphi_3\left(x,t_j\right)\right)\ \cdots\ \left(U_0,\varphi_{N-1}\left(x,t_j\right)\right)\right]^{\mathrm{T}},
$$

$$
\boldsymbol{U}_0=\left[\left(u_0,\varphi_1\left(x,t_j\right)\right)\ \left(u_0,\varphi_2\left(x,t_j\right)\right)\ \left(u_0,\varphi_3\left(x,t_j\right)\right)\ \cdots\ \left(u_0,\varphi_{N-1}\left(x,t_j\right)\right)\right]^{\mathrm{T}},
$$

$$
\boldsymbol{R}^1=\left[a_0\ \ a_1\ \ a_2\ \ \cdots\ \ a_{N-1}\right]^{\mathrm{T}}.
$$

Step 3: To compute $U_n \approx U_h\left(t_j\right)$ we repeat the above steps as 1 and 2. Thus, with the above idea, the finite element method can be formulated and solve the resulting system via MAPLE 18 Software.

4. Error Analysis

We consider the lemma below

***Lemma* 4.1:** Let u_j be the approximate solution of

$$
{}_0^R D_t^{\beta} u\left(t_j\right)=\sum_{r=0}^{j} w_{rj}\left[u\left(t_j-t_r\right)-u_0\right]+\frac{t_j^{-\beta}}{\Delta t^{\beta}\Gamma(-\beta)}G_j(g) \tag{4.1}
$$

Then we have

$$
\left\|u_j\right\|\le 2u_j+\frac{\sin\pi\beta}{\pi}\left|\Gamma(-\beta)\right|t_j^{\beta}\left\|g\right\|_{L_\infty}.
$$

***Theorem* 4.1:** Let $u\left(t_j\right)$ and U_j be the solutions (3.4) and (4.1), then we have

$$\left\|U_j - u\left(t_j\right)\right\| \le 2\left\|U_0 - Q_h u_0\right\| + O\left(\Delta t^{2-\beta} + h^2\right),$$

where h is the space step size. Let $Q_h : H_0^1 \to V_h$ defines an elliptic or Ritz propectim given by

$$\left(\nabla Q_h s, \nabla \gamma\right) = \left(\nabla s, \nabla \gamma\right), \quad \forall \gamma \in V_h .$$

Let $e_j = U_j - u\left(t_j\right) = U_j - Q_h u\left(t_j\right) + Q_h u\left(t_j\right) - u\left(t_j\right) = \alpha_j + q_j$, $j = 1, 2, 3, \cdots$

where, $\alpha_j = U_j - Q_h u\left(t_j\right)$, $q_j = Q_h u\left(t_j\right) - u\left(t_j\right)$.

Now, the error equation obtained from (4.1),

$$\begin{aligned}
&\frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{i} w_{rj} \left(\alpha_{j-r} - \alpha_0\right) + B_h \alpha_j \\
&= \frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} w_{rj} \left[\left(U_{j-r} - u_0\right) + B_h U_j - Q_h \left(u\left(t_{j-r}\right) - u_0\right) + B_h Q_h u\left(t_j\right)\right] \\
&= G_h g_j + G_h B_h u\left(t_j\right) - Q_h \left[\frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} w_{rj} \left(u\left(t_{j-r}\right) - u_0\right)\right] \\
&= -G_h y_j
\end{aligned}$$

where,

$$Y_j = -{}_0^R D_t^{\beta} \left[u\left(t_j\right) - u_0\right] + Q_h \frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} w_{rj} \left(u\left(t_{j-r} - u_0\right) - u_0\right) = P_j + K_j,$$

where,

$$P_j = \left(Q_h - t\right) \frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} w_{rj} \left(u\left(t_{j-r} - u_0\right)\right),$$

$$K_j = \frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{j} w_{rj} \left(u\left(t_{j-r} - u_0\right)\right) - {}_0^R D_t^{\beta} \left[u\left(t_j\right) - u_0\right].$$

Thus, we have,

$$\frac{t_j^{-\beta}}{\Gamma(-\beta)} \sum_{r=0}^{i} w_{rj} u\left(\alpha_{j-r} - \alpha_0\right) + B_h \alpha_j = G_h \left(P_j + K_j\right).$$

By *Lemma* 4.1, we have

$$\left\|\alpha_j\right\| + 2\left\|\alpha_0\right\| + \frac{\sin \pi\beta}{\pi} \left|\Gamma(-\beta)\right| t_j^{-\beta} \left\|G_h \left(P_j + K_j\right)\right\|.$$

Here

$$\left\|\alpha_j\right\| \le K_t^{-\beta} \left\|U''\left(t_j - t_j t\right)\right\| = K \Delta t^{2-\beta} \left\|U''\left(t_j - t_j t\right)\right\|,$$

and

$$\left\|K_j\right\| \le K h^2 \left\|\sum_{r=0}^{j} w_{rj} \left(t_{j-r}\right)\right\|_{G^2} + \left\|\sum_{r=0}^{j} w_{rj} u_0\right\|_{G^2},$$

where $\|.\|_{G^2}$ is Sobolev norm.

Let denote $f(t) = u\left(t_j - t_j t\right)$, then,

$$\sum_{r=0}^{j} w_{rj}\left(t_{j-r}\right)=\int_0^1 u\left(t_j-t_j t\right)t^{-1-\beta}\mathrm{d}t+G_j=\int_0^1 f\left(t\right)t^{-1-\beta}\mathrm{d}t+G_j,$$

obtained via Hamamard d integral formulation ([32]), and

$$\left|G_j\right|\le j^{\beta-2}\left\|f_t\right\|_{G^2}\le \Delta t^2\left\|u\right\|_{G^2}\le \Delta t_j^{2-\beta}\left\|u_t\right\|_{G^2}.$$

Let $q=t_j-t_j t$ into $\int_0^1 f\left(t\right)t^{-1-\beta}\mathrm{d}t$, to obtain,

$$\int_0^1 f\left(t\right)t^{-1-\beta}\mathrm{d}t=t_j^{\beta}\,{}_0^R D_t^{\beta}u\left(t_j\right)\Gamma\left(-\beta\right).$$

Thus,

$$\left\|\sum_{r=0}^{j} w_{rj}u\left(t_{j-r}\right)\right\|\le t_j^{\beta}\left|\Gamma\left(-\alpha\right)\right|\left\|{}_0^R D_t^{\beta}u\left(t_j\right)\right\|_{G^2}+\Delta t^{2-\beta}\left\|U_{tt}\right\|_{G^2},$$

$$t_j^{\beta}\left\|\alpha_j\right\|\le Kh^2 t_j^{\beta}\left(\left|\Gamma\left(-\beta\right)\right|\left\|{}_0^R D_t^{\beta}u\left(t_j\right)\right\|_{G^2}+\Delta t^{2-\beta}\left\|U_{tt}\right\|_{G^2}\right).$$

Thus, we have that,

$$\begin{aligned}\left\|\alpha_j\right\| &\le 2\left\|\alpha_0\right\|+\frac{\sin\pi\beta}{\pi}\left|\Gamma\left(-\beta\right)\right|t_j^{-\beta}\left\|G_h\left(\alpha_j+q_j\right)\right\| \\ &\le 2\left\|\alpha_0\right\|+Kt_j^{\beta}\Delta t^{2-\beta}\left\|U_{tt}\right\|_G^2+Kh^2 t_j^{\beta}\left(\left\|{}_0^R D_t^{\beta}u\left(t_j\right)\right\|_{G^2}+\Delta t^{2-\beta}\left\|U_{tt}\right\|_{G^2}\right) \\ &\le 2\left\|\alpha_0\right\|+O\left(\Delta t^{2-\beta}+h^2\right)\end{aligned}$$

Hence,

$$\left\|e_j\right\|\le\left\|\alpha_j\right\|+\left\|G_j\right\|\le 2\left\|\alpha_0\right\|+O\left(\Delta t^{2-\beta}+h^2\right)+\left\|G_j\right\|.$$

Therefore,

$$\left\|G_j\right\|=\left\|Q_h u\left(t_j\right)-u\left(t_j\right)\right\|=Kh^2\left\|u\left(t_j\right)\right\|_{G^2}.$$

Obtained via elliptic projection of error estimation. Thus, we finally obtain

$$\left\|e_j\right\|\le 2\left\|\alpha_j\right\|+B\left(\Delta t^{2-\beta}+h^2\right).$$

5. Numerical Illustration

In this section, we carry out numerical simulations to verify the accuracy of the proposed method.

Let in (1.1) be given

$$g\left(x,t\right)=2\left(x^2-x\right)t\left(\frac{\Gamma\left(3-\beta\right)+t^{1-\beta}}{\Gamma\left(3-\beta\right)}\right)-2t^2,\ 0\le x\le 1,\ t\in\left(0,1\right] \tag{5.1}$$

with initial conditions

$$u\left(x,0\right)=\frac{\partial u\left(x,0\right)}{\partial t}=0,\ 0\le x\le 1 \tag{5.2}$$

and boundary conditions

$$u\left(0,t\right)=u\left(1,t\right)=0,\ 0\le x\le 1. \tag{5.3}$$

The exact solution is given as $u\left(x,t\right)=\left(x^2-x\right)t^2$.

Using (3.11) on (5.1) at $N=3$ with w_{r3}, $r=0\left(1\right)3$, estimated using (3.6a), and $\Delta t = 1/1000$ at $t = 1$, results are presented below with the aid of MAPLE 18.

6. Numerical Illustrations

The proposed method has been successively implemented for the time fractional telegraph equation. Maximum errors in L_2 and L_∞ were obtained as shown in Table 2. The L_2 and L_∞ errors and the numerical order are in agreement in space for $\beta = 1.5$ and 1.8. It can be seen that the order of convergence of the proposed method is in total agreement with the theoretical analysis as shown in Figure 2 and Figure 3, respectively.

Table 2. Maximum error.

N	L_2 Error (Proposed method)	L_∞ Error	β
20	3.8141E−006	3.6141E−006	
40	1.0594E−005	1.2242E−005	1.5
80	3.0311E−005	4.0311E−005	
160	4.0142E−005	3.0142E−004	
20	3.7337E−003	3.3327E−003	
40	5.2802E−003	4.2982E−003	1.8
80	1.6125E−003	2.0125E−003	
160	2.2804E−003	3.2907E−003	

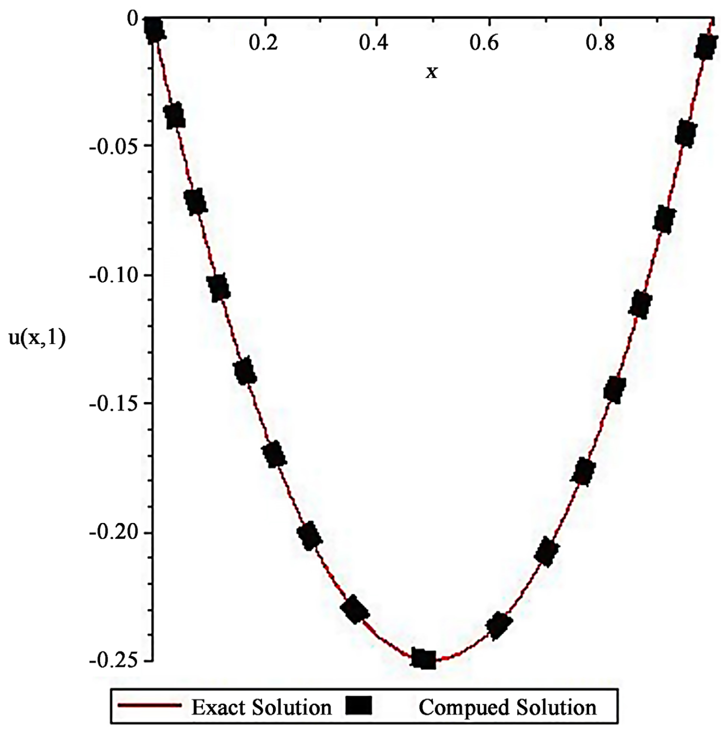

Figure 2. Comparison of computed solutions and Exact solutions at $\Delta t = 1/1000$ at $t = 1, \beta = 1.5$.

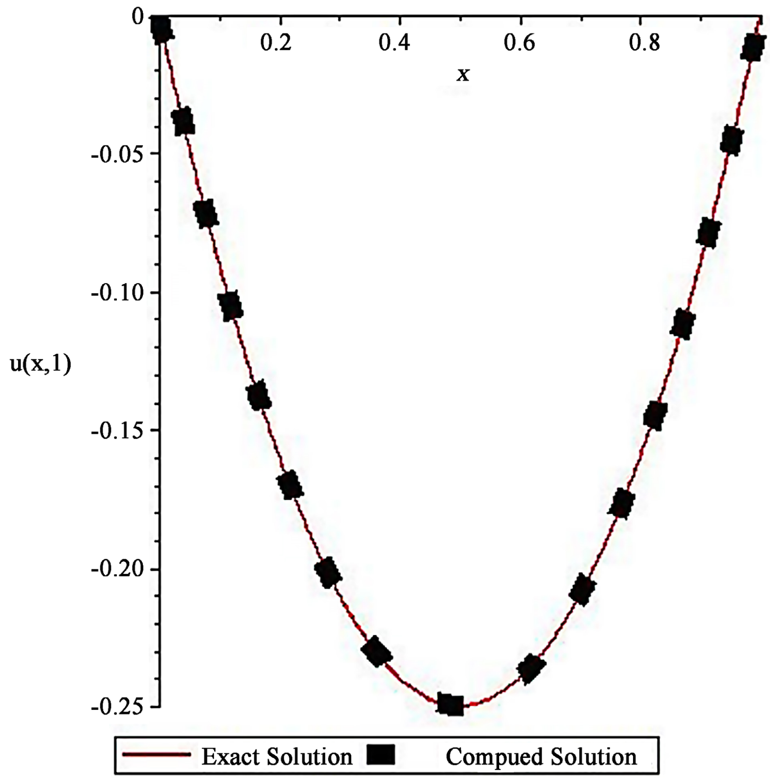

Figure 3. Comparison of computed solutions and exact solutions at $\Delta t = 1/1000$ at $t = 1, \beta = 1.8$.

7. Conclusion

The space discretization scheme was developed and implemented with the aid of Mamadu-Njoseh orthogonal basis functions. Satisfactory numerical evidence was obtained as the order of convergence of the proposed method is in total agreement with the theoretical analysis. Also, The L_2 and L_∞ errors and the numerical order are in agreement in space for $\beta = 1.5$ and 1.8.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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