Applied Mathematics, 2022, 13, 338-355
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Derivation of the Reduction Formula of Sixth Order and Seven Stages Runge-Kutta Method for the Solution of an Ordinary Differential Equation

Georgios D. Trikkaliotis, Maria Ch. Gousidou-Koutita

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki, Greece
Email: melinatrik@hotmail.com, gousidou@math.auth.gr

How to cite this paper: Trikkaliotis, G.D. and Gousidou-Koutita, M.Ch. (2022) Derivation of the Reduction Formula of Sixth Order and Seven Stages Runge-Kutta Method for the Solution of an Ordinary Differential Equation. *Applied Mathematics*, **13**, 338-355.
https://doi.org/10.4236/am.2022.134024

Received: March 1, 2022
Accepted: April 24, 2022
Published: April 27, 2022

Abstract

This paper is describing in detail the way we define the equations which give the formulas in the methods Runge-Kutta 6^{th} order 7 stages with the incorporated control step size in the numerical solution of Ordinary Differential Equations (ODE). The purpose of the present work is to construct a system of nonlinear equations and then by solving this system to find the values of all set parameters and finally the reduction formula of the Runge-Kutta (6,7) method (6^{th} order and 7 stages) for the solution of an Ordinary Differential Equation (ODE). Since the system of high order conditions required to be solved is complicated, all coefficients are found with respect to 7 free parameters. These free parameters, as well as some others in addition, are adjusted in such a way to furnish more efficient R-K methods. We use the MATLAB software to solve several of the created subsystems for the comparison of our results which have been solved analytically. Some examples for five different choices of the arbitrary values of the systems are presented in this paper.

Keywords

Initial Value Problem, Runge-Kutta Methods, Ordinary Differential Equations

1. Introduction

A system of ordinary differential equations of the form:

$$y' = f(x, y), y(x_0) = y_0 \tag{1}$$

with $x_0 \in \mathbb{R}$, $y, y' \in \mathbb{R}^m$ and $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ is called Initial Value Problem. Some related references are cited in the bibliography part [1] [2].

Carl David Tolmé Runge [3] and Martin Wilhelm Kutta [4] introduced the methods bearing their names almost in the turning of the 19th century. These methods are very practical and popular one-step methods for solving Ordinary Differential Equations.

The reduction formula [5] of the method Runge-Kutta for the ODE $y'(x) = f(x,y)$, $y(x_0) = y_0$ is given by the relation $y_{n+1} = y_n + \sum_{i=1}^{\nu} w_i K_i$ (2) with w_i as weighting factors, ν is the number of stages and $K_i = hf\left(x_n + c_i h, y_n + \sum_{j=1}^{i-1} \alpha_{ij} K_i\right)$ for $i = 1, 2, \cdots, \nu$ and $j = 1, 2, \cdots, i-1$ (3) with h the step size of this method. The values of w_i and expressions K_i will be determined by the values of the unknown variables α_{ij} and c_i. In every R-K method the relations $\sum_{i=1} w_i = 1$ and $c_i = \sum_{j=1}^{i-1} \alpha_{ij}$ for $i = 2, 3, \cdots, \nu$ (4) must be valid.

High order R-K methods have been presented by Butcher, Shanks, Lawson, Fehlberg, Feagin, Hairer, Curtis and others.

The motivation for this work is to create high order R-K (6,7) methods in a simple and practical way. In the present work after the Introduction follow the paragraph with the Presentation of the R-K method 6th order and 7 stages, the choices of the arbitrary parameters (5 choices), the detailed presentation and the solutions of the respective systems that result, as well as the tables with the found values of the coefficients of the R-K method (6,7) and the summary expressions of K_i. Finally, the Conclusions follow.

2. Presentation of the Method

From BUTCHER'S TABLE I (Table 1) [6] the equations of the nonlinear system result, where the first column represent the **order of the method**, the second column the **symbolism** of the function $f(x,y)$ and its derivatives, and the third column and so on the number of **coefficients**, for the study of Runge-Kutta methods [6], of the equations of the nonlinear system. The values of w_i, c_i and α_{ij} will be found by the solving this system as well as the K_i and the reduction formula for the solution of the ordinary differential equation.

The equations of the nonlinear system in their general form are: [7]

$$\sum_{\kappa=1}^{\omega} w_\kappa = 1 \tag{5}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa = \frac{1}{2} \tag{6}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1} = \frac{1}{6} \tag{7}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^2 = \frac{1}{3} \tag{8}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{24} \tag{9}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 2} = \frac{1}{12} \tag{10}$$

Table 1. The resulted Butcher's table.

(I)	F	1	1	1
(II)	$\{f\}$	1	1	2
(III)	$\{_2f\}_2$	1	2	6
	$\{f^2\}$	1	1	3
(IV)	$\{_3f\}_3$	1	6	24
	$\{_2f^2\}_2$	1	3	12
	$\{\{f\}f\}$	3	6	8
	$\{f^3\}$	1	1	4
(V)	$\{_4f\}_4$	1	24	120
	$\{_3f^2\}_3$	1	12	60
	$\{_2\{f\}f\}_2$	3	24	40
	$\{_2f^3\}_2$	1	4	20
	$\{\{_2f\}_2f\}$	4	24	30
	$\{\{f^2\}f\}$	4	12	15
	$\{\{f\}^2\}$	3	12	20
	$\{\{f\}f^2\}$	6	12	10
	$\{f^4\}$	1	1	5
(VI)	$\{_5f\}_5$	1	120	720
	$\{_4f^2\}_4$	1	60	360
	$\{_3\{f\}f\}_3$	3	120	240
	$\{_3f^3\}_3$	1	20	120
	$\{_2\{_2f\}_2f\}_2$	4	120	180
	$\{_3\{f^2\}f\}_3$	4	60	90
	$\{_2\{f\}^2\}_2$	3	60	120
	$\{_2\{f\}f^2\}_2$	6	60	60
	$\{_2f^4\}_2$	1	5	30
	$\{\{_3f\}_3f\}$	5	120	144
	$\{\{_2f^2\}_2f\}$	5	60	72
	$\{\{\{f\}f\}f\}$	15	120	48
	$\{\{f^3\}f\}$	5	20	24
	$\{\{_2f\}_2\{f\}\}$	10	120	72
	$\{\{f^2\}\{f\}\}$	10	60	36
	$\{\{_2f\}_2f^2\}$	10	60	36
	$\{\{f^2\}f^2\}$	10	30	18
	$\{\{f\}^2f\}$	15	60	24
	$\{\{f\}f^3\}$	10	20	12
	$\{f^5\}$	1	1	6

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 1} = \frac{1}{8} \tag{11}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^3 = \frac{1}{4} \tag{12}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{120} \tag{13}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\omega} \alpha_{\kappa\lambda} P_{\lambda 2}\right) = \frac{1}{60} \tag{14}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 1}\right) = \frac{1}{40} \tag{15}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 3} = \frac{1}{20} \tag{16}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{30} \tag{17}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 2} = \frac{1}{15} \tag{18}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1}^2 = \frac{1}{20} \tag{19}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^2 P_{\kappa 1} = \frac{1}{10} \tag{20}$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^4 = \frac{1}{5} \tag{21}$$

$$\sum_{\kappa=6}^{\omega} w_\kappa \left\{\sum_{\lambda=5}^{\kappa-1} \alpha_{\kappa\lambda} \left[\sum_{\mu=4}^{\lambda-1} \alpha_{\lambda\mu} \left(\sum_{\nu=3}^{\mu-1} \alpha_{\mu\nu} P_{\nu 1}\right)\right]\right\} = \frac{1}{720} \tag{22}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 2}\right)\right] = \frac{1}{360} \tag{23}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} c_\mu P_{\mu 1}\right)\right] = \frac{1}{240} \tag{24}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 2}\right) = \frac{1}{120} \tag{25}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{180} \tag{26}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 2}\right) = \frac{1}{90} \tag{27}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}^2\right) = \frac{1}{120} \tag{28}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda^2 P_{\lambda 1}\right) = \frac{1}{60} \tag{29}$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 4} = \frac{1}{30} \tag{30}$$

$$\sum_{\kappa=5}^{\omega} w_\kappa c_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \alpha_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \alpha_{\lambda\mu} P_{\mu 1}\right)\right] = \frac{1}{144} \tag{31}$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 2}\right) = \frac{1}{72} \quad (32)$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} c_\lambda P_{\lambda 1}\right) = \frac{1}{48} \quad (33)$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 3} = \frac{1}{24} \quad (34)$$

$$\sum_{\kappa=4}^{\omega} w_\kappa P_{\kappa 1} \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{72} \quad (35)$$

$$\sum_{\kappa=3}^{\omega} w_\kappa P_{\kappa 1} P_{\kappa 2} = \frac{1}{36} \quad (36)$$

$$\sum_{\kappa=4}^{\omega} w_\kappa c_\kappa^2 \left(\sum_{\lambda=3}^{\kappa-1} \alpha_{\kappa\lambda} P_{\lambda 1}\right) = \frac{1}{36} \quad (37)$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^2 P_{\kappa 2} = \frac{1}{18} \quad (38)$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa P_{\kappa 1}^2 = \frac{1}{24} \quad (39)$$

$$\sum_{\kappa=3}^{\omega} w_\kappa c_\kappa^3 P_{\kappa 1} = \frac{1}{12} \quad (40)$$

$$\sum_{\kappa=2}^{\omega} w_\kappa c_\kappa^5 = \frac{1}{6} \quad (41)$$

with

$$\omega = 7, \quad (42)$$

$$P_{\kappa\lambda} = \alpha_{\kappa 2} c_2^\lambda + \alpha_{\kappa 3} c_3^\lambda + \cdots + \alpha_{\kappa\kappa-1} c_{\kappa-1}^\lambda \text{ with } \kappa = 3,4,\cdots,7 \text{ and } \lambda = 1,2,3,4 \text{ [5]} \quad (43)$$

2.1. 1st Choice: $c_2 = c_4 = 1/3$, $c_3 = 2/3$, $c_5 = c_6 = 1/2$, $c_7 = 1$ [5]

From the system of (5), (6), (8), (12) και (41) results that:

$w_1 = \frac{11}{120}$ (44) $w_2 + w_4 = \frac{27}{40}$ (45) $w_3 = \frac{27}{40}$ (46) $w_5 + w_6 = -\frac{8}{15}$ (47) $w_7 = \frac{11}{120}$ (48) and setting $w_2 = 0$ (49) and $w_5 = w_6$ (50) we have $w_4 = \frac{27}{40}$ (51) and $w_5 = w_6 = -\frac{4}{15}$ (52) while the Equation (21) verified.

Since the above equations become somewhat lengthy, we introduce the following abbreviations: [7]

$$\alpha_{42} c_2 + a_{43} c_3 = \frac{\alpha_{42} + 2\alpha_{43}}{3} = \frac{S_4}{3} \quad (53)$$

$$\alpha_{52} c_2 + a_{53} c_3 + \alpha_{54} c_4 = \frac{\alpha_{52} + 2\alpha_{53} + \alpha_{54}}{3} = \frac{S_5}{3} \quad (54)$$

$$\alpha_{62} c_2 + a_{63} c_3 + \alpha_{64} c_4 + \alpha_{65} c_5 = \frac{2\alpha_{62} + 4\alpha_{63} + 2\alpha_{64} + 3\alpha_{65}}{6} = \frac{S_6}{6} \quad (55)$$

$$\alpha_{72} c_2 + a_{73} c_3 + \alpha_{74} c_4 + \alpha_{75} c_5 + \alpha_{76} c_6 = \frac{2\alpha_{72} + 4\alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}}{6} = \frac{S_7}{6} \quad (56)$$

and

$$\alpha_{42}c_2^2 + a_{43}c_3^2 = \frac{\alpha_{42} + 4\alpha_{43}}{9} = \frac{S_4 + 2\alpha_{43}}{9} \tag{57}$$

$$\alpha_{52}c_2^2 + a_{53}c_3^2 + \alpha_{54}c_4^2 = \frac{S_5 + 2\alpha_{53}}{9} \tag{58}$$

$$\alpha_{62}c_2^2 + a_{63}c_3^2 + \alpha_{64}c_4^2 + \alpha_{65}c_5^2 = \frac{2S_6 + 8\alpha_{63} + 3\alpha_{65}}{36} \tag{59}$$

$$\alpha_{72}c_2^2 + a_{73}c_3^2 + \alpha_{74}c_4^2 + \alpha_{75}c_5^2 + a_{76}c_6^2 = \frac{2S_7 + 8\alpha_{73} + 3\alpha_{75} + 3\alpha_{76}}{36} \tag{60}$$

and

$$\alpha_{42}c_2^3 + a_{43}c_3^3 = \frac{S_4 + 6\alpha_{43}}{27} \tag{61}$$

$$\alpha_{52}c_2^3 + a_{53}c_3^3 + a_{54}c_4^3 = \frac{S_5 + 6\alpha_{53}}{27} \tag{62}$$

$$\alpha_{62}c_2^3 + a_{63}c_3^3 + \alpha_{64}c_4^3 + \alpha_{65}c_5^3 = \frac{4S_6 + 48\alpha_{63} + 15\alpha_{65}}{216} \tag{63}$$

$$\alpha_{72}c_2^3 + a_{73}c_3^3 + \alpha_{74}c_4^3 + \alpha_{75}c_5^3 + a_{76}c_6^3 = \frac{4S_7 + 48\alpha_{73} + 15\alpha_{75} + 15\alpha_{76}}{216} \tag{64}$$

and

$$\alpha_{42}c_2^4 + a_{43}c_3^4 = \frac{S_4 + 14\alpha_{43}}{81} \tag{65}$$

$$\alpha_{52}c_2^4 + a_{53}c_3^4 + \alpha_{54}c_4^4 = \frac{S_5 + 14\alpha_{53}}{81} \tag{66}$$

$$\alpha_{62}c_2^4 + a_{63}c_3^4 + \alpha_{64}c_4^4 + \alpha_{65}c_5^4 = \frac{8S_6 + 224\alpha_{63} + 57\alpha_{65}}{1296} \tag{67}$$

$$\alpha_{72}c_2^4 + a_{73}c_3^4 + \alpha_{74}c_4^4 + \alpha_{75}c_5^4 + a_{76}c_6^4 = \frac{8S_7 + 224\alpha_{73} + 57\alpha_{75} + 57\alpha_{76}}{1296} \tag{68}$$

Then we substitute the defined abbreviations in the original system, as well as the found values of c_2, c_3, c_4, c_5, c_6, c_7, w_2, w_3, w_4, w_5, w_6, w_7, and as a result the system is simplified. Also omitting the equations which are a linear combination of equations of the system the 30 × 15 system is obtained:

$$162\alpha_{32} + 162S_4 - 64S_5 - 32S_6 + 11S_7 = 120 \tag{69}$$

$$\begin{aligned} &162\alpha_{43}\alpha_{32} - 64(\alpha_{53}\alpha_{32} + \alpha_{54}S_4) - 64(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5) \\ &+11(2\alpha_{73}\alpha_{32} + 2\alpha_{74}S_4 + 2\alpha_{75}S_5 + \alpha_{76}S_6) = 30 \end{aligned} \tag{70}$$

$$648\alpha_{43} - 256\alpha_{53} - 32(8\alpha_{63} + 3\alpha_{65}) + 11(8\alpha_{73} + 3\alpha_{75} + 3\alpha_{76}) = 120 \tag{71}$$

$$108\alpha_{32} + 54S_4 - 32S_5 - 16S_6 + 11S_7 = 90 \tag{72}$$

$$\begin{aligned} &-32\alpha_{54}\alpha_{43}\alpha_{32} - 32\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}(\alpha_{53}\alpha_{32} + \alpha_{54}S_4)\right] \\ &+11\left[\alpha_{74}\alpha_{43}\alpha_{32} + \alpha_{75}(\alpha_{53}\alpha_{32} + \alpha_{54}S_4) + \alpha_{76}(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5)\right] = 3 \end{aligned} \tag{73}$$

$$256\left(\alpha_{54}\alpha_{43}+\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)-88\left(\alpha_{74}\alpha_{43}+\alpha_{75}\alpha_{53}+\alpha_{76}\alpha_{63}\right)$$
$$-33\alpha_{76}\alpha_{65}=-12 \quad (74)$$

$$128\alpha_{54}S_4+128\,\alpha_{64}S_4+64\,\alpha_{65}S_5-44\,\alpha_{74}S_4-22\,\alpha_{75}S_5-11\alpha_{76}S_6=-12 \quad (75)$$

$$32\,\alpha_{65}-11\alpha_{75}-11\alpha_{76}=32 \quad (76)$$

$$54\,\alpha_{43}\alpha_{32}-16\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)-16\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)=3 \quad (77)$$

$$108\,\alpha_{43}-32\,\alpha_{53}-32\,\alpha_{63}-12\,\alpha_{65}=3 \quad (78)$$

$$324\,\alpha_{32}^2+324\,S_4^2-128\,S_5^2-32\,S_6^2+11S_7^2=216 \quad (79)$$

$$72\,\alpha_{32}+18\,S_4-16\,S_5-8\,S_6+11S_7=72 \quad (80)$$

$$64\,\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32}-22\left\{\alpha_{75}\alpha_{54}\alpha_{43}\alpha_{32}+\alpha_{76}\left[\alpha_{64}\alpha_{43}\alpha_{32}+\alpha_{65}\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)\right]\right\}=-1$$
(81)

$$32\,\alpha_{65}\alpha_{54}\alpha_{43}-11\left[\alpha_{75}\alpha_{54}\alpha_{43}+\alpha_{76}\left(\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)\right]=0 \quad (82)$$

$$64\,\alpha_{65}\alpha_{54}S_4-22\,\alpha_{75}\alpha_{54}S_4-11\alpha_{76}\left(2\alpha_{64}S_4+\alpha_{65}S_5\right)=-3 \quad (83)$$

$$11\alpha_{76}\alpha_{65}=-8 \quad (84)$$

$$64\,\alpha_{65}\alpha_{53}\alpha_{32}-11\left[2\alpha_{75}\alpha_{53}\alpha_{32}+\alpha_{76}\left(2\alpha_{63}\alpha_{32}+\alpha_{65}S_5\right)\right]=-9 \quad (85)$$

$$648\,\alpha_{43}\alpha_{32}-256\,\alpha_{53}\alpha_{32}-128\left[2\alpha_{63}\alpha_{32}+\alpha_{65}\left(2\alpha_{53}+S_5\right)\right]$$
$$+11\left[8\alpha_{73}\alpha_{32}+4\alpha_{75}\left(2\alpha_{53}+S_5\right)+\alpha_{76}\left(8\alpha_{63}+3\alpha_{65}+2S_6\right)\right]=144 \quad (86)$$

$$324\,\alpha_{43}\alpha_{32}^2-128\left(\alpha_{53}\alpha_{32}^2+\alpha_{54}S_4^2\right)-128\left(\alpha_{63}\alpha_{32}^2+\alpha_{64}S_4^2+\alpha_{65}S_5^2\right)$$
$$+11\left(4\alpha_{73}\alpha_{32}^2+4\alpha_{74}S_4^2+4\alpha_{75}S_5^2+\alpha_{76}S_6^2\right)=36 \quad (87)$$

$$1944\,\alpha_{43}\alpha_{32}-768\,\alpha_{53}\alpha_{32}-768\,\alpha_{63}\alpha_{32}-320\,\alpha_{65}S_5+264\,a_{73}a_{32}$$
$$+110\,\alpha_{75}S_5+55\,\alpha_{76}S_6=312 \quad (88)$$

$$11\left[\alpha_{74}\alpha_{43}\alpha_{32}+\alpha_{75}\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)+\alpha_{76}\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)\right]=2 \quad (89)$$

$$-54\,\alpha_{43}\alpha_{32}+128\,\alpha_{54}\alpha_{43}+128\left(\alpha_{64}\alpha_{43}+\alpha_{65}\alpha_{53}\right)$$
$$+11\left(2\alpha_{73}\alpha_{32}+2\alpha_{74}S_4+2\alpha_{75}S_5+2\alpha_{76}S_6\right)=18 \quad (90)$$

$$-216\,\alpha_{43}\alpha_{32}+88\,\alpha_{73}\alpha_{32}+44\,\alpha_{74}S_4+66\,\alpha_{75}S_5+33\,\alpha_{76}S_6=72 \quad (91)$$

$$-54\,\alpha_{43}+22\,\alpha_{65}+22\,\alpha_{73}=47 \quad (92)$$

$$324\,S_4\alpha_{43}\alpha_{32}-128\,S_5\left(\alpha_{53}\alpha_{32}+\alpha_{54}S_4\right)-64\,S_6\left(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5\right)$$
$$+11S_7\left(2\alpha_{73}\alpha_{32}+2\alpha_{74}S_4+2\alpha_{75}S_5+\alpha_{76}S_6\right)=60 \quad (93)$$

$$1296\,\alpha_{43}S_4-512\,\alpha_{53}S_5-\left(256\alpha_{63}+396\alpha_{65}\right)S_6$$
$$+\left(88\alpha_{73}+33\alpha_{75}+33\alpha_{76}\right)S_7=288 \quad (94)$$

$$18\,\alpha_{43}\alpha_{32}=-1 \quad (95)$$

$$72\,\alpha_{43}-16\,\alpha_{53}-16\,\alpha_{63}-4\,\alpha_{65}=1 \quad (96)$$

$$108\,\alpha_{32}^2-108\,S_4^2+11S_7^2=144 \quad (97)$$

$$24\,\alpha_{32}-6\,S_4+11S_7=48 \quad (98)$$

From (69), (72), (80) και (98) result:

$$\alpha_{52} = \frac{2}{3} \quad (99)$$

$$S_4 = \frac{1}{6} \quad (100)$$

$$S_7 = 3 \quad (101)$$

and

$$4S_5 + 2S_6 = 3 \quad (102)$$

From (79) we find:

$$32S_5^2 + 8S_6^2 = 9 \quad (103)$$

so:

$$S_5 = \frac{3}{8} \quad (104)$$

$$S_6 = \frac{3}{4} \quad (105)$$

From (71), (76), (78), (92) and (96) result:

$$a_{43} = -\frac{1}{12} \quad (106)$$

$$a_{65} = \frac{1}{2} \quad (107)$$

$$a_{73} = \frac{63}{44} \quad (108)$$

and

$$a_{53} + a_{63} = -\frac{9}{16} \quad (109)$$

$$a_{75} + a_{76} = -\frac{16}{11} \quad (110)$$

From (84):

$$a_{76} = -\frac{16}{11} \quad (111)$$

and from (110):

$$a_{75} = 0 \quad (112)$$

while the Equations (95) and (97) are verified.

By substitution the found values from (75), (77) and (109) we find

$$a_{74} = \frac{18}{11} \quad (113)$$

and

$$a_{54} + a_{64} = -\frac{9}{8} \quad (114)$$

from (81):

$$4a_{53} + a_{54} = -\frac{9}{8} \tag{115}$$

and from (90):

$$192a_{53} - 32\left(a_{54} + a_{64}\right) = 0 \tag{116}$$

The solution of the system of Equations (109), (114), (115) and (116) is:

$$a_{53} = -\frac{3}{16} \tag{117}$$

$$a_{54} = -\frac{6}{16} \tag{118}$$

$$a_{63} = -\frac{6}{16} \tag{119}$$

and

$$a_{64} = -\frac{12}{16} \tag{120}$$

Using the defined abbreviations $S_4 = \alpha_{42} + 2\alpha_{43}$, $S_5 = \alpha_{52} + 2\alpha_{53} + \alpha_{54}$ $S_6 = 2\alpha_{62} + 4\alpha_{63} + 2\alpha_{64} + 3\alpha_{65}$ and $S_7 = 2\alpha_{72} + 4\alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}$ result:

$$a_{42} = \frac{4}{12} \tag{121}$$

$$a_{52} = \frac{18}{16} \tag{122}$$

$$a_{62} = \frac{9}{8} \tag{123}$$

$$a_{72} = -\frac{9}{11} \tag{124}$$

From the relations $\sum_{j=1}^{i-1}\alpha_{ij} = c_i$, for $i = 2,3,4,5,6,7$ result:

$$a_{21} = \frac{1}{3} \tag{125}$$

$$a_{41} = \frac{1}{12} \tag{126}$$

$$a_{51} = -\frac{1}{16} \tag{127}$$

$$a_{61} = 0 \tag{128}$$

and

$$a_{71} = \frac{9}{44} \tag{129}$$

The remaining equations are verified.

Runge-Kutta methods usually presented in a so-called Butcher table (**Table 2**) [8].

Table 2. The so-called Butcher table.

c	A
	w^{T}

The table contains on the 1st column the coefficients c_i, the matrix A with the coefficients of a_{ij}, which appear in the Formulae of K_i, and w_i the coefficients in Formula of y_{i+1}.

In this type of table, we have $\boldsymbol{w}^{T}, \boldsymbol{c} \in \mathbb{R}^{m}$ while $\boldsymbol{A} \in \mathbb{R}^{m\times m}$. Then, the method shares m stages and in case that $c_1 = 0$ and $\boldsymbol{A}$ [5] is strictly lower triangular, it is evaluated explicitly.

Therefore we get **Table 3**:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{130}$$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right) \tag{131}$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right) \tag{132}$$

$$K_4 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1 + 4K_2 - K_3}{12}\right) \tag{133}$$

$$K_5 = hf\left(x_n + \frac{h}{2}, y_n + \frac{-K_1 + 18K_2 - 3K_3 - 6K_4}{16}\right) \tag{134}$$

$$K_6 = hf\left(x_n + \frac{h}{2}, y_n + \frac{9K_2 - 3K_3 - 6K_4 + 4K_5}{8}\right) \tag{135}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{9K_1 - 36K_2 + 63K_3 + 72K_4 - 64K_5}{44}\right) \tag{136}$$

and

$$y_{n+1} = y_n + \frac{11K_1 + 81K_3 + 81K_4 - 32K_5 - 32K_6 + 11K_7}{120} \tag{137}$$

2.2. 2nd Choice: $c_2 = c_3 = 1/4$, $c_4 = 2/4$, $c_5 = c_6 = 3/4$, $c_7 = 4/4$ [5]

From the system of (5), (6), (8), (12) και (41) results that:

$$w_1 = \frac{7}{90} \tag{138}$$

$$w_2 + w_3 = \frac{32}{90} \tag{139}$$

$$w_4 = \frac{12}{90} \tag{140}$$

$$w_5 + w_6 = \frac{32}{90} \tag{141}$$

Table 3. For choices values of arbitrary constants: $c_2 = c_4 = 1/3$, $c_3 = 2/3$, $c_5 = c_6 = 1/2$, $c_7 = 1$.

0							
1/3	1/3						
2/3	0	2/3					
1/3	1/12	4/12	−1/12				
1/2	−1/16	18/16	−3/16	−6/16			
1/2	0	9/8	−3/8	−6/8	4/8		
1	9/44	−36/44	63/44	72/44	0	−64/44	
	11/120	0	81/120	81/120	−32/120	−32/120	11/120

$w_7 = \frac{7}{90}$ (142) and setting $w_2 = 0$ (143) and $w_5 = w_6$ (144) we get

$w_3 = \frac{32}{90}$ (145) and $w_5 = w_6 = \frac{16}{90}$ (146) while the Equation (21) verified.

Since the above equations become somewhat lengthy, we introduce the following abbreviations

$$\alpha_{42}c_2 + a_{43}c_3 = \frac{\alpha_{42} + \alpha_{43}}{4} = \frac{S_4}{4} \tag{147}$$

$$\alpha_{52}c_2 + a_{53}c_3 + \alpha_{54}c_4 = \frac{\alpha_{52} + \alpha_{53} + 2\alpha_{54}}{4} = \frac{S_5}{4} \tag{148}$$

$$\alpha_{62}c_2 + a_{63}c_3 + \alpha_{64}c_4 + \alpha_{65}c_5 = \frac{\alpha_{62} + \alpha_{63} + 2\alpha_{64} + 3\alpha_{65}}{4} = \frac{S_6}{4} \tag{149}$$

$$\alpha_{72}c_2 + a_{73}c_3 + \alpha_{74}c_4 + \alpha_{75}c_5 + \alpha_{76}c_6 = \frac{\alpha_{72} + \alpha_{73} + 2\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}}{4} = \frac{S_7}{4} \tag{150}$$

and

$$\alpha_{42}c_2^2 + a_{43}c_3^2 = \frac{\alpha_{42} + \alpha_{43}}{16} = \frac{S_4}{16} \tag{151}$$

$$\alpha_{52}c_2^2 + a_{53}c_3^2 + \alpha_{54}c_4^2 = \frac{S_5 + 2\alpha_{54}}{16} \tag{152}$$

$$\alpha_{62}c_2^2 + a_{63}c_3^2 + \alpha_{64}c_4^2 + \alpha_{65}c_5^2 = \frac{S_6 + 2\alpha_{64} + 6\alpha_{65}}{16} \tag{153}$$

$$\alpha_{72}c_2^2 + a_{73}c_3^2 + \alpha_{74}c_4^2 + \alpha_{75}c_5^2 + a_{76}c_6^2 = \frac{S_7 + 2\alpha_{74} + 6\alpha_{75} + 6\alpha_{76}}{16} \tag{154}$$

and

$$\alpha_{42}c_2^3 + a_{43}c_3^3 = \frac{S_4}{64} \tag{155}$$

$$\alpha_{52}c_2^3 + a_{53}c_3^3 + \alpha_{54}c_4^3 = \frac{S_5 + 6\alpha_{54}}{64} \tag{156}$$

$$\alpha_{62}c_2^3 + a_{63}c_3^3 + \alpha_{64}c_4^3 + \alpha_{65}c_5^3 = \frac{S_6 + \alpha_{64} + 24\alpha_{65}}{64} \tag{157}$$

$$\alpha_{72}c_2^3 + a_{73}c_3^3 + \alpha_{74}c_4^3 + \alpha_{75}c_5^3 + a_{76}c_6^3 = \frac{S_7 + 6\alpha_{74} + 24\alpha_{75} + 24\alpha_{76}}{64} \tag{158}$$

and

$$\alpha_{42}c_2^4 + a_{43}c_3^4 = \frac{S_4}{256} \tag{159}$$

$$\alpha_{52}c_2^4 + a_{53}c_3^4 + \alpha_{54}c_4^4 = \frac{S_5 + 14\alpha_{54}}{256} \tag{160}$$

$$\alpha_{62}c_2^4 + a_{63}c_3^4 + \alpha_{64}c_4^4 + \alpha_{65}c_5^4 = \frac{S_6 + 4\alpha_{64} + 78\alpha_{65}}{256} \tag{161}$$

$$\alpha_{72}c_2^4 + a_{73}c_3^4 + \alpha_{74}c_4^4 + \alpha_{75}c_5^4 + a_{76}c_6^4 = \frac{S_7 + 14\alpha_{74} + 78\alpha_{75} + 78\alpha_{76}}{256} \tag{162}$$

Then we substitute the defined abbreviations in the original system, as well as the found values of c_2, c_3, c_4, c_5, c_6, c_7, w_2, w_3, w_4, w_5, w_6, w_7, and as a result the system is simplified. Also omitting the equations which are a linear combination of equations of the system and the 30 × 15 system is obtained:

$$32\alpha_{32} + 12S_4 + 16S_5 + 16S_6 + 7S_7 = 60 \tag{163}$$

$$\begin{aligned} &12\alpha_{43}\alpha_{32} + 16(\alpha_{53}\alpha_{32} + \alpha_{54}S_4) + 16(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5) \\ &+ 7(\alpha_{73}\alpha_{32} + \alpha_{74}S_4 + \alpha_{75}S_5 + \alpha_{76}S_6) = 15 \end{aligned} \tag{164}$$

$$16(2\alpha_{54}) + 16(2\alpha_{64} + 6\alpha_{65}) + 7(2\alpha_{74} + 6\alpha_{75} + 6\alpha_{76}) = 60 \tag{165}$$

$$32\alpha_{32} + 24S_4 + 48S_5 + 48S_6 + 28S_7 = 180 \tag{166}$$

$$\begin{aligned} &16\alpha_{54}\alpha_{43}\alpha_{32} + 16\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}(\alpha_{53}\alpha_{32} + \alpha_{54}S_4)\right] \\ &+ 7\left[\alpha_{74}\alpha_{43}\alpha_{32} + \alpha_{75}(\alpha_{53}\alpha_{32} + \alpha_{54}S_4) + \alpha_{76}(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5)\right] = 3 \end{aligned} \tag{167}$$

$$16\alpha_{65}(2\alpha_{54}) + 7\left[\alpha_{75}(2\alpha_{54}) + \alpha_{76}(2\alpha_{64} + 6\alpha_{65})\right] = 9 \tag{168}$$

$$16\alpha_{54}S_4 + 16(\alpha_{64}S_4 + 2\alpha_{65}S_5) + 7(\alpha_{74}S_4 + 2\alpha_{75}S_5 + 2\alpha_{76}S_6) = 21 \tag{169}$$

$$16\alpha_{65} + 7(\alpha_{75} + \alpha_{76}) = 8 \tag{170}$$

$$6\alpha_{43}\alpha_{32} + 4(\alpha_{53}\alpha_{32} + \alpha_{54}S_4) + 4(\alpha_{63}\alpha_{32} + \alpha_{64}S_4 + \alpha_{65}S_5) = 3 \tag{171}$$

$$7(\alpha_{74} + 3\alpha_{75} + 3\alpha_{76}) = 12 \tag{172}$$

$$32\alpha_{32}^2 + 12S_4^2 + 16S_5^2 + 16S_6^2 + 7S_7^2 = 72 \tag{173}$$

$$2\alpha_{32} + 3S_4 + 9S_5 + 9S_6 + 7S_7 = 36 \tag{174}$$

$$32\alpha_{65}\alpha_{54}\alpha_{43}\alpha_{32} + 14\left\{\alpha_{75}\alpha_{54}\alpha_{43}\alpha_{32} + \alpha_{76}\left[\alpha_{64}\alpha_{43}\alpha_{32} + \alpha_{65}(\alpha_{53}\alpha_{32} + \alpha_{54}S_4)\right]\right\} = 1 \tag{175}$$

$$7\alpha_{76}\alpha_{65}2\alpha_{54} = 1 \tag{176}$$

$$16\alpha_{65}\alpha_{54}S_4 + 7\left[\alpha_{75}\alpha_{54}S_4 + \alpha_{76}(\alpha_{64}S_4 + 2\alpha_{65}S_5)\right] = 3 \tag{177}$$

$$7\alpha_{76}\alpha_{65} = 1 \tag{178}$$

$$64\alpha_{65}\alpha_{53}\alpha_{32} + 7\alpha_{75}\alpha_{53}\alpha_{32} + 7\left[\alpha_{76}(\alpha_{63}\alpha_{32} - \alpha_{65}S_5)\right] = -1 \tag{179}$$

$$16(6\alpha_{65})\alpha_{54}+7(6\alpha_{75})\alpha_{54}+7(6\alpha_{76})(\alpha_{64}+3\alpha_{65})=27 \tag{180}$$

$$12\alpha_{43}\alpha_{32}^2+16(\alpha_{53}\alpha_{32}^2+\alpha_{54}S_4^2)+16(\alpha_{63}\alpha_{32}^2+\alpha_{64}S_4^2+\alpha_{65}S_5^2)$$
$$+7(\alpha_{73}\alpha_{32}^2+\alpha_{74}S_4^2+\alpha_{75}S_5^2+\alpha_{76}S_6^2)=12 \tag{181}$$

$$16\alpha_{54}S_4+16\alpha_{64}S_4+7\alpha_{74}S_4=3 \tag{182}$$

$$7\left[\alpha_{74}\alpha_{43}\alpha_{32}+\alpha_{75}(\alpha_{53}\alpha_{32}+\alpha_{54}S_4)+\alpha_{76}(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5)\right]=1 \tag{183}$$

$$8\alpha_{65}\alpha_{54}=1 \tag{184}$$

$$7(\alpha_{74}S_4+2\alpha_{75}S_5+2\alpha_{76}S_6)=9 \tag{185}$$

$$64\alpha_{65}+7\alpha_{73}=20 \tag{186}$$

$$12S_4\alpha_{43}\alpha_{32}+16S_5(\alpha_{53}\alpha_{32}+\alpha_{54}S_4)+16S_6(\alpha_{63}\alpha_{32}+\alpha_{64}S_4+\alpha_{65}S_5)$$
$$+7S_7(\alpha_{73}\alpha_{32}+\alpha_{74}S_4+\alpha_{75}S_5+\alpha_{76}S_6)=20 \tag{187}$$

$$16S_5\alpha_{54}+16S_6(\alpha_{64}+3\alpha_{65})+7S_7(\alpha_{74}+3\alpha_{75}+3\alpha_{76})=42 \tag{188}$$

$$6\alpha_{43}\alpha_{32}=1 \tag{189}$$

$$8\alpha_{54}+8(\alpha_{64}+3\alpha_{65})=9 \tag{190}$$

$$8\alpha_{32}^2+6S_4^2+12S_5^2+12S_6^2+7S_7^2=60 \tag{191}$$

$$2\alpha_{32}+6S_4+27S_5+27S_6+28S_7=120 \tag{192}$$

From (163), (166), (174) and (192) result:

$$\alpha_{32}=\frac{1}{8} \tag{193}$$

$$S_4=\frac{1}{2} \tag{194}$$

$$S_7=2 \tag{195}$$

and

$$S_5+S_6=\frac{9}{4} \tag{196}$$

From both Equations (173) and (191) we find

$$S_5^2+S_6^2=\frac{81}{32} \tag{197}$$

From the system $S_5^2+S_6^2=\frac{81}{32}$

and $S_5+S_6=\frac{9}{4}$ result:

$$S_5=S_6=\frac{9}{8} \tag{198}$$

From (176) and (178):

$$\alpha_{54}=\frac{1}{2} \tag{199}$$

From (184): $\alpha_{65} = \frac{1}{4}$ (200) from (178): $\alpha_{76} = \frac{4}{7}$ (201)

From (189): $\alpha_{43} = \frac{4}{3}$ (202) from (170): $\alpha_{75} = 0$ (203)

From (172): $\alpha_{74} = 0$ (204) from (182): $\alpha_{64} = -\frac{1}{8}$ (205) from (186): $\alpha_{73} = \frac{4}{7}$ (206)

From (171) and (189): $\alpha_{53} = 0$ (207) and $\alpha_{63} = \frac{1}{4}$ (208)

The remaining equations are verified.

From $S_4 = \frac{1}{2}$: $\alpha_{42} = -\frac{5}{6}$ (209) from $S_5 = \frac{9}{8}$: $\alpha_{52} = \frac{1}{8}$ (210)

From $S_6 = \frac{9}{8}$: $\alpha_{62} = \frac{3}{8}$ (211) from $S_7 = 2$: $\alpha_{72} = -\frac{2}{7}$ (212)

From the relations $\sum_{j=1}^{i-1} \alpha_{ij} = c_i$, $i = 2,3,4,5,6,7$ result:

$$\alpha_{21} = \frac{1}{4} \tag{213}$$

$$\alpha_{31} = \frac{1}{8} \tag{214}$$

$$\alpha_{41} = 0 \tag{215}$$

$$\alpha_{51} = \frac{1}{8} \tag{216}$$

$$\alpha_{61} = 0 \tag{217}$$

and

$$\alpha_{71} = \frac{1}{7} \tag{218}$$

Therefore we get **Table 4**:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{219}$$

Table 4. For choices values of arbitrary constants: $c_2 = c_3 = 1/4$, $c_4 = 2/4$, $c_5 = c_6 = 3/4$, $c_7 = 4/4$.

0							
1/4	1/4						
1/4	1/8	1/8					
2/4	0	−5/6	8/6				
3/4	1/8	1/8	0	4/8			
3/4	0	3/8	2/8	−1/8	2/8		
4/4	1/7	−2/7	4/7	0	0	4/7	
	7/90	0	32/90	12/90	16/90	16/90	7/90

$$K_2 = hf\left(x_n + \frac{h}{4}, y_n + \frac{K_1}{4}\right) \tag{220}$$

$$K_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{K_1 + K_2}{8}\right) \tag{221}$$

$$K_4 = hf\left(x_n + \frac{2h}{4}, y_n + \frac{-5K_2 + 8K_3}{6}\right) \tag{222}$$

$$K_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{K_1 + K_2 + 4K_4}{8}\right) \tag{223}$$

$$K_6 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{3K_2 + 2K_3 - K_4 + 2K_5}{8}\right) \tag{224}$$

$$K_7 = hf\left(x_n + h, y_n + \frac{K_1 - 2K_2 + 4K_3 + 4K_6}{7}\right) \tag{225}$$

and

$$y_{n+1} = y_n + \frac{7K_1 + 32K_3 + 12K_4 + 16K_5 + 16K_6 + 7K_7}{90} \tag{226}$$

Working in a similar way we are presented three other choices:

2.3. 3rd Choice: $c_2 = c_4 = 4/12$, $c_3 = 8/12$, $c_5 = 3/12$, $c_6 = 9/12$, $c_7 = 12/12$ [5]

For 3rd choice we get **Table 5**:

with

$$K_1 = hf\left(x_n, y_n\right) \tag{227}$$

$$K_2 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1}{3}\right) \tag{228}$$

$$K_3 = hf\left(x_n + \frac{2h}{3}, y_n + \frac{2K_2}{3}\right) \tag{229}$$

$$K_4 = hf\left(x_n + \frac{h}{3}, y_n + \frac{K_1 + 4K_2 - K_3}{12}\right) \tag{230}$$

Table 5. For choices values of arbitrary constants. $c_2 = c_4 = 4/12$, $c_3 = 8/12$, $c_5 = 3/12$, $c_6 = 9/12$, $c_7 = 12/12$.

0							
4/12	1/3						
8/12	0	2/3					
4/12	1/12	4/12	−1/12				
3/12	10/64	−9/64	0	15/64			
9/12	−3/64	−27/64	27/64	−45/64	96/64		
12/12	55/116	−81/145	−459/580	528/145	−384/145	128/145	
	29/360	0	27/200	27/200	64/225	64/225	29/360

$$K_5 = hf\left(x_n + \frac{h}{4}, y_n + \frac{10K_1 - 9K_2 + 15K_4}{64}\right) \quad (231)$$

$$K_6 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{-3K_1 - 27K_2 + 27K_3 - 45K_4 + 96K_5}{64}\right) \quad (232)$$

$$K_7 = hf\left(x_n + h, y_n + \frac{\frac{275K_1}{4} - 81K_2 - \frac{459K_3}{4} + 528K_4 - 384K_5 + 128K_6}{145}\right) \quad (233)$$

and

$$y_{n+1} = y_n + \frac{145K_1 + 243K_3 + 243K_4 + 512K_5 + 512K_6 + 145K_7}{1800} \quad (234)$$

2.4. 4th Choice: $c_2 = c_3 = 1/5$, $c_4 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ *και* $c_7 = 5/5 = 1$ [5]

For 4th choice we get Table 6:
with

$$K_1 = hf\left(x_n, y_n\right) \quad (235)$$

$$K_2 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1}{5}\right) \quad (236)$$

$$K_3 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1 + K_2}{10}\right) \quad (237)$$

$$K_4 = hf\left(x_n + \frac{2h}{5}, y_n + \frac{-9K_2 + 25K_3}{40}\right) \quad (238)$$

$$K_5 = hf\left(x_n + \frac{3h}{5}, y_n + \frac{4K_1 - 17K_2 + 21K_3 + 16K_4}{40}\right) \quad (239)$$

$$K_6 = hf\left(x_n + \frac{4h}{5}, y_n + \frac{-8K_1 + 21K_2 + 35K_3 - 40K_4 + 40K_5}{60}\right) \quad (240)$$

$$K_7 = hf\left(x_n + h, y_n + \frac{66K_1 - 5K_2 - 165K_3 + 240K_4 - 120K_5 + 60K_6}{76}\right) \quad (241)$$

Table 6. For choices values of arbitrary constants: $c_2 = c_3 = 1/5$, $c_4 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$.

0							
1/5	1/5						
1/5	1/10	1/10					
2/5	0	−9/40	25/40				
3/5	4/40	−17/40	21/40	16/40			
4/5	−8/60	21/60	35/60	−40/60	40/60		
1	66/76	−5/76	−165/76	60/19	−30/19	15/19	
	19/288	0	75/288	50/288	50/288	75/288	19/288

Table 7. For choices values of arbitrary constants: $c_2 = c_4 = 1/5$, $c_3 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ και $c_7 = 5/5 = 1$.

0							
1/5	1/5						
2/5	0	2/5					
1/5	11/60	−4/60	5/60				
3/5	−3/20	4/20	3/20	8/20			
4/5	0	−12/15	−8/15	22/15	10/15		
1	51/76	140/76	225/76	−280/76	−120/76	60/76	
	19/288	0	50/288	75/288	50/288	75/288	19/288

and

$$y_{n+1} = y_n + \frac{19K_1 + 75K_3 + 50K_4 + 50K_5 + 75K_6 + 19K_7}{288} \quad (242)$$

2.5. 5th Choice: $c_2 = c_4 = 1/5$, $c_3 = 2/5$, $c_5 = 3/5$, $c_6 = 4/5$ *και* $c_7 = 5/5 = 1$ [5]

For 5th choice we get Table 7:
with

$$K_1 = hf\left(x_n, y_n\right) \quad (243)$$

$$K_2 = hf\left(x_n + \frac{h}{5}, y_n + \frac{K_1}{5}\right) \quad (244)$$

$$K_3 = hf\left(x_n + \frac{2h}{5}, y_n + \frac{2K_2}{5}\right) \quad (245)$$

$$K_4 = hf\left(x_n + \frac{h}{5}, y_n + \frac{11K_1 - 4K_2 + 5K_3}{60}\right) \quad (246)$$

$$K_5 = hf\left(x_n + \frac{3h}{5}, y_n + \frac{-3K_1 + 4K_2 + 3K_3 + 8K_4}{20}\right) \quad (247)$$

$$K_6 = hf\left(x_n + \frac{4h}{5}, y_n + \frac{-12K_2 - 8K_3 + 22K_4 + 10K_5}{15}\right) \quad (248)$$

$$K_7 = hf\left(x_n + h, y_n + \frac{51K_1 + 140K_2 + 225K_3 - 280K_4 - 120K_5 + 60K_6}{76}\right) \quad (249)$$

and

$$y_{n+1} = y_n + \frac{19K_1 + 50K_3 + 75K_4 + 50K_5 + 75K_6 + 19K_7}{288} \quad (250)$$

3. Conclusions

In the R-K methods it is $c_1 = 0$ and we choose $c_\nu = 1$, that is, in method (6,7) presented here, it is $c_7 = 1$. For methods up to 4th order the number of steps is equal to the order of the method. For the higher order method, the number of steps exceeds its order. In the 6th order R-K method the steps are 7 and 8 or more. We look for the method with the fewest steps because increasing the steps also increases the number of parameters to be calculated. Because the system from which the parameters will be calculated is not linear, some "arbitrary" values [5] must be given to parameters to solve the system. We have given 5 choices as examples, to the values of the arbitrary constants and the solutions of the respective systems that result are made analytically.

The process of solving the system starts by giving values to c_i to calculate the w_i. The resulting system is a linear 6 × 7 system. To solve it we give the same value in two c_i eg $c_2 = c_3$ *or* $c_2 = c_4$ etc. while the values of c_i we choose to be in ascending order. As the variable c_2 does not occur often we prefer one of the two equals c_i to be c_2. In the choice *eg* $c_2 = c_3$ we consider $w_2 + w_3$ as one parameter and after it is found that $w_2 + w_3 = k$ we set $w_2 = 0$ and $w_3 = k$. In choices 2.1 and 2.2 the same values were given to two pairs of parameters c_i and we omitted one equation to make the system 5 × 5 and whose solution must verify the omitted equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Fried, I. (2019) Consistency and Stability Issues in the Numerical Integration of the First and Second Order Initial Value Problem. *Applied Mathematics*, **10**, 676-690. https://doi.org/10.4236/am.2019.108048

[2] Abdullah Alzaid, N. and Omar Bakodah, H. (2018) Numerical Treatment of Initial-Boundary Value Problems with Mixed Boundary Conditions. *American Journal of Computational Mathematics*, **8**, 153-174. https://doi.org/10.4236/ajcm.2018.82012

[3] Runge, C. (1895) Ueber die numerische Auflöung von Differentialgleichungen. *Mathematische Annalen*, **46**, 167-178. https://doi.org/10.1007/BF01446807

[4] Kutta, M.W. (1901) Beitrag zur Naherungsweisen Integration von Differentialgleichungen. *Mathematical Physics*, **46**, 435-453.

[5] Gousidou-Koutita, M. (2008/9) Numerical Methods with Applications of Ordinary and Partial Differential Equations. University Lectures 2008/9, Aristotle University of Thessaloniki, Thessaloniki, Greece.

[6] Butcher, J.C. (1963) Coefficients for the study of Runge-Kutta Integration Processes. *Journal of the Australian Mathematical Society*, **3**, 185-201. https://doi.org/10.1017/S1446788700027932

[7] Fehlberg, E. (1968) Classical Fifth-, Sixth-, Seventh-, and Eight-Order Runge-Kutta Formulas with Stepsize Control. NASA Technical Report, R-287, 4-13.

[8] Butcher, J.C. (1964) On Runge-Kutta Processes of High Order. *Journal of the Australian Mathematical Society*, **4**, 179-194. https://doi.org/10.1017/S1446788700023387