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Reliability Analysis of Varietal Hypercube

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Abstract

Connectivity is a vital metric to explore fault tolerance and reliability of network structure based on a graph model. Let $G=(V,E)$ be a connected graph. A connected graph G is called supper-κ (resp. supper-λ) if every minimum vertex cut (edge cut) of G is the set of neighbors of some vertex in G. The g-component connectivity of a graph G, denoted by $c\kappa_g(G)$, is the minimum number of vertices whose removal from G results in a disconnected graph with at least g components or a graph with fewer than g vertices. The g-component edge connectivity $c\lambda_g(G)$ can be defined similarly. In this paper, we determine the g-component (edge) connectivity of varietal hypercube VQ_n for small g.

Keywords

Interconnection Networks, Fault Tolerance, g-Component Connectivity

1. Introduction

Graph connectivity is an important topological parameter that reflects the graph structure, and is usually used to evaluate the vulnerability, reliability and fault tolerance of the corresponding network [1]. Given a graph $G=(V,E)$ with vertex set V and edge set E, we use V stand for the set of network nodes and E the set of communication links between nodes. The vertex-cut of a connected graph G is the subset $F\subseteq V$ that make $G-F$ disconnected. The cardinality of the smallest vertex-cut set of a graph G is called the connectivity of the graph G, denoted by $\kappa(G)$. In order to further analyze the detailed situation of disconnected graphs caused by vertex-cut, Harary [2] suggested studying the conditional connectivity with additional restrictions on the vertex-cut F and (or) the component of $G-F$. In 1984, Chartrand *et al.* [3] [4] proposed the concepts of component connectivity, which is essentially extensions of traditional connectivity, so it can also be regarded as a kind of conditional connectivity.

For any positive integer *g*, the *g*-component cut of the graph G is a vertex set $F \subseteq V$ such that $G-F$ has at least $g\left(g \geq 2\right)$ components. The *g*-component connectivity of graph *G*, denoted by $c\kappa_g\left(G\right)$, is the cardinality of a minimum *g*-component cut of graph *G*, that is, $c\kappa_g\left(G\right)=\min\left\{\left|F\right|: F \subseteq V, \omega\left(G-F\right) \geq g\right\}$. Of course, we define that $c\kappa_g\left(G\right)=0$ if G is a complete graph K_n or a disconnected graph. Obviously, $c\kappa_2\left(G\right)=\kappa\left(G\right)$ and $c\kappa_g\left(G\right) \leq c\kappa_{g+1}\left(G\right)$.

In [5] [6] [7], authors determined the *g*-component connectivity of *n*-dimensional bubble-sort star graph BS_n, *n*-dimensional burnt pancake graph BP_n, the hierarchical star networks HS_n, the alternating Group graphs AG_n and split star graph S_2^n. Zhao *et al.* [8] [9] and Xu *et al.* [10] respectively determined the *g*-component connectivity of Cayley graphs generated by *n*-dimensional folded hypercube FQ_n, *n*-dimensional dual cube D_n and transposition tree. In addition, Chang *et al.* [11] determined the *g*-component connectivity of alternating group networks AN_n when $g=3,4$. Ding *et al.* [12] dealt with the *g*-component (edge) connectivity of shuffle-cubes SQ_n for small *g*. Recently, Li *et al.* [13] studied the relationship between extra connectivity and component connectivity of general networks, and Hao *et al.* [14] and Guo *et al.* [15] independently proposed the relationship between extra edge connectivity and component connectivity of regular networks in the literature.

All graphs considered in this paper are finite and simple. We refer to the book [16] for graph theoretical notation and terminology not described here. For the graph *G*, let $e\left(G\right)$, $n\left(G\right)$, $\bar{G}$, and $\omega\left(G\right)$ represent respectively the size, the order, the complement and the number of components of *G*. Let $G=\left(V,E\right)$ be a connected graph, $N_G\left(v\right)$ the neighbors of a vertex *v* in G (simply $N\left(v\right)$), $E\left(v\right)$ the edges incident to *v*. For $X, Y \subset V$, denote by $\left[X,Y\right]$ the set of edges of G with one end in *X* and tie other in *Y*. We call *G* *k*-regular if $d_G\left(u\right)=k$ for every vertex $u \in V\left(G\right)$. By $\delta\left(G\right)$ and $\Delta\left(G\right)$ denote the minimum and maximum degree of the graph *G*, respectively. By $\left|S\right|$ denote the number of elements in *S* and $N_G\left(S\right)$ denote the set of vertices of *G* which has neighbour vertex in *S*, that means $N_G\left(S\right)=\bigcup_{v \in S} N_G\left(v\right) \setminus S$.

2. Preliminary

Definition 1 [17] *The n-dimensional hypercube is a connected graph with* 2^n *vertices and denoted by* Q_n. *The vertex set* $V\left(Q_n\right)=\left\{x_1x_2\cdots x_n : x_i=0 \text{ or } 1, 1 \leq i \leq n\right\}$. *Two vertices* $u=u_1u_2\cdots u_n$ *and* $v=v_1v_2\cdots v_n$ *in* Q_n *are adjacent if and only they differ in exact one position.*

Definition 2 [18] *The n-dimensional varietal hypercube, denoted by* VQ_n, *has* 2^n *vertices, each labeled by an n-bit binary string and*
$V\left(VQ_n\right)=\left\{x_nx_{n-1}\cdots x_2x_1 : x_i=0 \text{ or } 1, i=1,2,\cdots,n\right\}$. VQ_1 *is a complete graph* K_2 *of two vertices labeled with* 0 *and* 1, *respectively. For* $n \geq 2$, VQ_n *can be recursively constructed from two copies of* VQ_{n-1}, *denoted by* VQ_{n-1}^0 *and* VQ_{n-1}^1, *and by adding* 2^{n-1} *edges between* VQ_{n-1}^0 *and* VQ_{n-1}^1, *where*
$V\left(VQ_{n-1}^0\right)=\left\{0x_{n-1}\cdots x_2x_1 : x_i=0 \text{ or } 1, i=1,2,\cdots,n\right\}$,

$V\left(VQ_{n-1}^1\right)=\{1x_{n-1}\cdots x_2x_1 : x_i=0$ *or* $1, i=1,2,\cdots,n\}$. *The vertex* $x=0x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1\in V\left(VQ_{n-1}^0\right)$ *is adjacent to the vertex* $y=0y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1\in V\left(VQ_{n-1}^1\right)$ *if and only if*

1) $x_{n-1}x_{n-2}x_{n-3}\cdots x_2x_1=y_{n-1}y_{n-2}y_{n-3}\cdots y_2y_1$ if $3\nmid n$, or;

2) $x_{n-3}\cdots x_2x_1=y_{n-3}\cdots y_2y_1$ and $\left(x_{n-1}x_{n-2},y_{n-1}y_{n-2}\right)\in\{(00,00),(01,01),(10,11),(11,10)\}$ if $3\mid n$.

Obviously, VQ_n is an *n*-regular and its girth is 4. Moreover, it contains circles of length 5 when $n\geq 3$ [19] [20]. The varietal hypercube VQ_1, VQ_2, VQ_3 are illustrated in Figure 1.

From the definition, we can see that each vertex of VQ_n with a leading 0 bit has exactly one neighbor with a leading 1 and vice versa. In fact, some pairs of parallel edge are changed to some pairs of cross edges. Furthermore, VQ_n can be obtained by adding a perfect matching *M* between VQ_{n-1}^0 and VQ_{n-1}^1. Hence VQ_n can be viewed as $G\left(VQ_{n-1}^0,VQ_{n-1}^1,M\right.$ or $VQ_{n-1}^0\odot VQ_{n-1}^1$ briefly. For any vertex $u\in V\left(VQ_n\right)$, $e_M(u)$ is the esge incident to *u* in *M*.

As a variant of the hypercube, the *n*-dimensional varietal hypercube VQ_n, which has the same number of vertices and edges as Q_n, not only has the most ideal characteristics of Q_n, including some characteristics such as recursive structure, strong connectivity, and symmetry but also has a smaller diameter than Q_n, and its average distance is smaller than the hypercube [18].

Proposition 1 [18] $diam\left(VQ_n\right)=\lceil 2n/3\rceil$ *for* $n\geq 3$.

Proposition 2 [21] [22] [23] VQ_n *is a vertex-transitive and edge-transitive.*

Proposition 3 [19] $\kappa\left(VQ_n\right)=\lambda\left(VQ_n\right)=n$ *for* $n\geq 1$.

Proposition 4 [24] *Any two vertices of* Q_n *have exactly two common neighbors for* $n\geq 3$ *if they have any.*

Proposition 5 [25] *Let x and y be any two vertices of* $V\left(Q_n\right)(n\geq 3)$ *such that have two common neighbors.*

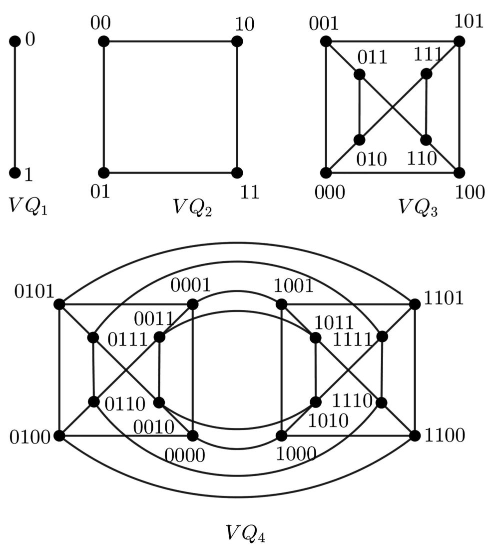

Figure 1. The varietal hypercube VQ_1, VQ_2, VQ_3, and VQ_4.

1) If $x \in V\left(Q_{n-1}^{0}\right)$, $y \in V\left(Q_{n-1}^{1}\right)$, then the one common neighbor is in Q_{n-1}^{0}, and the other one is in Q_{n-1}^{1}.

2) If $x, y \in V\left(Q_{n-1}^{0}\right)$ or $V\left(Q_{n-1}^{1}\right)$, then the two common neighbors are in Q_{n-1}^{0} or Q_{n-1}^{1}.

3. Main Result

The varietal hypercube VQ_n has an important property as follows.

The varietal hypercube is obtained by interchanging a pair of edges of the hypercube. Then it appears two vertices which have only one common neighbor. So we have the following result similar to proposition 4.

Theorem 1. *Any two vertices of* VQ_n *have at most two common neighbors for* $(n \geq 3)$ *if they have.*

Corollary 2. *For any two vertices* $x, y \in V\left(vVQ_n\right)(n \geq 3)$,

1) if $d(x, y) = 2$, then they have at most two common neighbors;

2) if $d(x, y) \neq 2$, then they do not have common neighbors.

According to the definition of VQ_n, if any two vertices of $V\left(VQ_n\right)$ have only one common neighbor, then it is obtained by interchanging a pair of edges of the hypercube. Hence similar to proposition 5, we have

Theorem 3. *Let x and y be any two vertices of* $V\left(VQ_n\right)(n \geq 3)$ *such that have only two common neighbors.*

1) If $x \in V\left(VQ_{n-1}^{0}\right)$, $y \in V\left(VQ_{n-1}^{1}\right)$, then the one common neighbor is in VQ_{n-1}^{0}, and the other one is in VQ_{n-1}^{1}.

2) If $x, y \in V\left(VQ_{n-1}^{0}\right)$ or $V\left(VQ_{n-1}^{1}\right)$, then the two common neighbors are in VQ_{n-1}^{0} or VQ_{n-1}^{1}.

By the definition of VQ_n and above results, we have:

Theorem 4. *If any two vertices of* $V\left(VQ_n\right)$ *have only one common neighbor, then the two vertices and their common neighbor are in some* VQ_3.

In [19], Xu *et al.*, proved that VQ_n is super-λ and super-κ. Here, we present another proof of this result.

Theorem 5. VQ_n *is super-*λ *for* $n \geq 3$.

By induction. It is true $n \leq 4$. Let $n \geq 5$. Assume that it holds for $n < k$.We will show that it is true for $n = k$.

Let $F \subseteq E\left(VQ_n\right)$, $|F| = n$ and $VQ_n - F$ be not connected. Furthermore, $VQ_n - F$ has only two connected components. Without loss of generality, suppose $\left|F \cap E\left(VQ_{n-1}^{0}\right)\right| \leq \lfloor n/2 \rfloor$. Then $VQ_{n-1}^{0} - F$ is connected.

Note that $\left|\left[VQ_{n-1}^{0}, VQ_{n-1}^{1}\right]\right| = 2^{n-1} > n$ ($n \geq 5$). If $VQ_{n-1}^{1} - F$ is connected, then $VQ_n - F$ is connected, a contradiction.

Assume that $VQ_{n-1}^{1} - F$ is not connected. We have $\left|F \cap E\left(VQ_{n-1}^{1}\right)\right| \geq n-1$. If $\left|F \cap E\left(VQ_{n-1}^{1}\right)\right| = n$, then $F \cap E\left(VQ_{n-1}^{0}\right) = \varnothing$ and $\left[VQ_{n-1}^{0}, VQ_{n-1}^{1}\right] \cap F = \varnothing$. And each vertex of $VQ_{n-1}^{1} - F$ has one neighbor in $VQ_{n-1}^{0} - F$, that is, $VQ_n - F$ is connected, a contradiction.

Hence $\left|F \cap E\left(VQ_{n-1}^{1}\right)\right| = n-1$. According to the inductive hypothesis, $VQ_{n-1}^{1} - F$ is super-λ. Suppose the isolated vertex x and G_1 are the only two components of $VQ_{n-1}^{1}-F$. And G_1 is connected to $VQ_{n-1}^{0}-F$. If $e_M(x)\notin F$, then VQ_n-F is connected, a contradiction. So $e_M(x)\in F$. We have $F=e(x)$ and VQ_n-F has only two components, one component ia *x*. Hence VQ_n is super-λ.

Theorem 6. VQ_n *is super-κ for* $n\geq 3$.

The proof is similar to Theorem 5.

Theorem 7. $c\kappa_2(VQ_n)=\kappa(VQ_n)=n$ *for* $n\geq 2$.

By definition of $c\kappa_g(G)$, we have $c\kappa_2(VQ_n)=\kappa(VQ_n)$, and we have $\kappa(VQ_n)=n$ for $n\geq 1$ by proposition 3, thus $c\kappa_2(VQ_n)=\kappa(VQ_n)=n$ for $n\geq 2$.

Theorem 8. $c\kappa_3(VQ_n)=2n-2$ *for* $n\geq 3$.

We choose two nonadjacent vertices *x*, *y* in a cycle C_4 which has two common neighbors. Then $VQ_n-N(\{x,y\})$ has at least three connected components and $|N(\{x,y\})|=2n-2$. That is $c\kappa_3(VQ_n)\leq 2n-2$.

We will show $c\kappa_3(VQ_n)\geq 2n-2$ by induction. It is easy to check that it is true for $n=3,4$. So we suppose $n\geq 5$. Suppose it is true for $n<k$. Let $n=k$.

Let $F\subseteq V(VQ_n)$ with $|F|\leq 2n-3$. And VQ_n-F has at least three connected components, say, G_1, G_2, G_3. We have $|F\cap V(VQ_{n-1}^{0})|\leq n-2$ or $|F\cap V(VQ_{n-1}^{0})|\leq n-2$. Without loss of generality, we set $|F\cap V(VQ_{n-1}^{0})|\leq n-2$. Hence $VQ_{n-1}^{0}-F$ is connected.

If $VQ_{n-1}^{1}-F$ has at least three components, from the inductive hypothesis, then $|F\cap V(VQ_{n-1}^{1})|\geq 2n-4$ and $|F\cap V(VQ_{n-1}^{0})|\leq 1$. Because each vertex of VQ_{n-1}^{1} has one neighbor in VQ_{n-1}^{0}, at most one vertex of $VQ_{n-1}^{1}-F$ has no neighbors in VQ_{n-1}^{0}. So VQ_n-F has at most two connected components, a contradiction.

Hence $VQ_{n-1}^{1}-F$ has at most two components. At most one component of $VQ_{n-1}^{1}-F$ is not connected to $VQ_{n-1}^{0}-F$. And VQ_n-F has at most two connected components, a contradiction.

Theorem 9. $c\lambda_2(VQ_n)=\lambda(VQ_n)=n$ *for* $n\geq 2$.

By definition of $c\lambda_g(G)$, we have $c\lambda_2(VQ_n)=\lambda(VQ_n)$, and we have $\lambda(VQ_n)=n$ for $n\geq 1$ by proposition 3, thus $c\lambda_2(VQ_n)=\lambda(VQ_n)=n$ for $n\geq 2$.

Theorem 10. $c\lambda_3(VQ_n)=2n-1$ *for* $n\geq 2$.

Take an edge $e=uv$, then $|E(u)\cup E(v)|=2n-1$. And $VQ_n-E(u)-E(v)$ has at least three connected components. That is $c\lambda_3(VQ_n)\leq 2n-1$.

Next we will show that $c\lambda_3(VQ_n)\geq 2n-1$ by induction. It is easy to check it is true for $n=2,3,4$. So we suppose $n\geq 5$. Suppose it is true for all $n<k$. We will prove that is true for $n=k$.

Let $F\subseteq E(VQ_n)$ with $|F|\leq 2n-2$, and VQ_n-F has at least three components. Now since $VQ_{n-1}^{0}\odot VQ_{n-1}^{1}$, we have $|F\cap E(VQ_{n-1}^{0})|\leq n-1$ or $|F\cap E(VQ_{n-1}^{0})|\leq n-1$, say, $|F\cap E(VQ_{n-1}^{0})|\leq n-1$. Since $\lambda(VQ_{n-1})=n-1$, we have two cases.

Case 1 $VQ_{n-1}^{0}-F$ is not connected.

Then $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|=n-1$ and $VQ_{n-1}^{0}-F$ has only two components.

If $VQ_{n-1}^{1}-F$ is not connected, then $\left|F\cap E\left(VQ_{n-1}^{1}\right)\right|=n-1$. That is $\left[VQ_{n-1}^{0},VQ_{n-1}^{1}\right]\cap F=\varnothing$. But each vertex of $VQ_{n-1}^{1}-F$ is connected to one component of $VQ_{n-1}^{0}-F$. Hence $VQ_{n}-F$ has only two components, a contradiction.

Note that $\left|\left[VQ_{n-1}^{0},VQ_{n-1}^{1}\right]\right|=2^{n-1}>n-1$ ($n\geq 5$). If $VQ_{n-1}^{1}-F$ is connected, then $VQ_{n-1}^{1}-F$ is connected to one component of $VQ_{n-1}^{0}-F$. Hence $VQ_{n}-F$ has only two components, a contradiction.

Case 2 $VQ_{n-1}^{0}-F$ is connected.

If $VQ_{n-1}^{1}-F$ is connected, then we are done. We assume that $VQ_{n-1}^{1}-F$ is not connected. And $VQ_{n-1}^{1}-F$ has at most one isolated vertex since $|F|\leq 2n-2$.

If $VQ_{n-1}^{1}-F$ has at least three components, from the inductive hypothesis, then $\left|F\cap E\left(VQ_{n-1}^{1}\right)\right|\geq 2n-3$. Hence at most one of components of $VQ_{n-1}^{1}-F$ is not connected to $VQ_{n-1}^{0}-F$, $VQ_{n}-F$ has at most two components, a contradiction.

Therefore we assume that $VQ_{n-1}^{1}-F$ has only two components. But $2^{n-1}-(2n-2)>0$ ($n\geq 5$), $VQ_{n}-F$ has at most two components, a contradiction.

Theorem 11. $c\lambda_{4}\left(VQ_{n}\right)=3n-2$ *for* $n\geq 2$.

Take a path $P_{3}=uvw$. Then $\left|E(u)\cup E(v)\cup E(w)\right|=2n-1$. And $VQ_{n}-E(u)-E(v)-E(w)$ has at least four connected components. That is $c\lambda_{4}\left(VQ_{n}\right)\leq 3n-2$.

Next we will show that $c\lambda_{3}\left(VQ_{n}\right)\geq 2n-1$ by induction. It is easy to check it is true for $n=2,3,4$. So we suppose $n\geq 5$. Suppose it is true for all $n<k$. We will prove that is true for $n=k$.

Let $F\subseteq E\left(VQ_{n}\right)$ with $|F|\leq 3n-3$, and $VQ_{n}-F$ has at least four components. Now since $VQ_{n-1}^{0}\odot VQ_{n-1}^{1}$, we have $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|\leq[3n/2]-2$ or $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|\leq[3n/2]-2$, say, $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|\leq[3n/2]-2$. Since $c\lambda_{3}\left(VQ_{n-1}\right)=2n-3>[3n/2]-2$ ($n\geq 5$), $VQ_{n-1}^{0}-F$ has at most two components.

Case 1 $VQ_{n-1}^{0}-F$ is connected.

If $VQ_{n-1}^{1}-F$ has at least four components, then $c\lambda_{4}\left(VQ_{n-1}\right)\geq 3n-5$ by the inductive hypothesis. We need delete at most two edges again. Since each vertex of VQ_{n-1}^{1} has a neighbor in VQ_{n-1}^{0} and $\left|\left[VQ_{n-1}^{0},VQ_{n-1}^{1}\right]\right|=2^{n-1}>2$ ($n\geq 5$), $VQ_{n}-F$ has at most three components, a contradiction.

Suppose $VQ_{n-1}^{1}-F$ has at most three components. Because of $\left|\left[VQ_{n-1}^{0},VQ_{n-1}^{1}\right]\right|=2^{n-1}>3n-3$ ($n\geq 5$), $VQ_{n}-F$ has at most three components, a contradiction.

Case 2 $VQ_{n-1}^{0}-F$ has only two connected components.

Then $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|\geq\lambda\left(VQ_{n-1}\right)=n-1$ and $\left|F\cap E\left(VQ_{n-1}^{1}\right)\right|\leq 2n-2$. Note that $c\lambda_{3}\left(VQ_{n-1}\right)=2n-3$.

If $VQ_{n-1}^{1}-F$ has at least three components, then $\left|F\cap E\left(VQ_{n-1}^{1}\right)\right|\geq 2n-3$ and $\left|F\cap E\left(VQ_{n-1}^{0}\right)\right|\leq n$. But $\left|\left[VQ_{n-1}^{0},VQ_{n-1}^{1}\right]\cap F\right|\leq 1$ and $2^{n-1}>1$ ($n\geq 5$),

$VQ_n - F$ has at most three components, a contradiction.

Hence $VQ_{n-1}^1 - F$ has at most two components. We have $\left|\left[VQ_{n-1}^0, VQ_{n-1}^1\right]\right| > 3n-3\left(n \ge 5\right)$, and $VQ_n - F$ has at most three components, a contradiction.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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