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Some New Systems of Exponentially General Equations

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Abstract

Some new systems of exponentially general equations are introduced and investigated, which can be used to study the odd-order, non-positive and non-symmetric exponentially boundary value problems. Some important and interesting results such as Riesz-Frechet representation theorem, Lax-Milgram lemma and system of absolute values equations can be obtained as special cases. It is shown that the system of exponentially general equations is equivalent to nonlinear optimization problem. The auxiliary principle technique is used to prove the existence of a solution to the system of exponentially general equations. This technique is also used to suggest some new iterative methods for solving the system of the exponentially general equations. The convergence analysis of the proposed methods is analyzed. Ideas and techniques of this paper may stimulate further research.

Keywords

General Equations, Lax-Milgram Lemma, Auxiliary Principle, Iterative Method, Convergence

1. Introduction

It is well known [1] [2] [3] that a linear continuous functional on a Hilbert space can be represented by an inner product as well as by an arbitrary bifunction Lax-Milgram [4]. These results are known as representation theorems and can be viewed as the weak formulation of the initial and boundary value problems. One can easily that the minimum of the functional $I[v]$ on the Hilbert space H

$$I[v] = \langle v, v\rangle - 2f(v), \quad v \in H, \tag{1.1}$$

i.e. equivalent to finding $u \in H$ such that

$$\langle u, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{1.2}$$

which is known as the Riesz-Frechet representation theorem [1] [2] [3]. This result had played a significant role in the development of various branches of mathematical and engineering sciences and is continue to inspire new ideas and techniques to tackle complicated and complex problems. See [1]-[13] and the references therein. From the day of discovery of the representation theorems, many important contributions have been made in this direction. In every case, a new approach and method is applied to generalize some of these results and the ideas they used.

For a symmetric, positive, bilinear $a(.,.)$ and f linear continuous functions, the minimum of the functional $J[v]$ defined by

$$J[v] = a(v,v) - 2f(v), \quad v \in H, \tag{1.3}$$

can be characterized by

$$a(u,v) = \langle f, v \rangle, \quad \forall v \in H, \tag{1.4}$$

which is known as the Lax-Milgram Lemma [4].

If the function f is nonlinear Frechet differentiable, then the minimum of the functional $J[v]$ defined by (1.3) can be characterized by

$$a(u,v) = \langle f'(u), v \rangle, \quad \forall v \in H, \tag{1.5}$$

where $f'(u)$ is the differential of f. Problem of the type (1.5) is called the general Lax-Milgram Lemma, introduced and studied by Noor [9]. For motivation, formulation, numerical applications, generalizations and novel aspects of the general Lax-Milgram Lemma, see [6]-[16].

Noor and Noor [10] considered the problem of finding $u \in H$ such that

$$a(u,v) = \langle A(u), v \rangle, \quad \forall v \in H, \tag{1.6}$$

where A is the nonlinear operator. Problem (1.6) is also called the general Lax-Milgram Lemma and have been used in finite element analysis of mildly nonlinear boundary value problems [9] [12].

From the problems (1.2), (1.4), (1.5) and (1.6), it is clear that these representation theorems have variational character, the origin of which can be traced back to Euler, Newton and Bernoulli's brothers.

For given nonlinear operators $T, A : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, v \rangle = \langle A(u), v \rangle, \quad \forall v \in H, \tag{1.7}$$

which is called the general Lax-Milgram Lemma [10]. One can easily show that the problems involving difference of two monotone operators, system of absolute value equations, difference of two convex functions, known as DC-problem and complementarity problems are special cases of the general Lax-Milgram Lemma and Riesz-Frechet representation theorems for different and appropriate choice of the operators. Mangasarian *et al.* [17] considered the systems of absolute value equations. Karamardian [18] had established the equivalence between the complementarity problems and variational inequalities. The equivalence interplay among these different fields enables us to use various techniques, which have been developed for variational inequalities, systems of absolute value problems and complementarity problems for solving the system of general equations and vice versa. For recent numerical methods for solving these different problems, see [4] [5] [7] [9] [10] [12] [14]-[47] and the references therein.

It has been observed that only the even order, positive and symmetric boundary value problems can be studied. For odd-order, non-positive and non-symmetric, these representation theorems cannot be used. To tackle such problems, the operator may be made positive and symmetric with respect to an arbitrary map. Noor [30] [31] [32] [33] [34] introduced the general variational inequalities, which are used to study the odd-order and non-symmetric boundary value problems. From the general variational inequalities, we can obtain system of general equations, see Noor [30] [31] [32] and Noor *et al.* [36]. To be more precise, for given nonlinear operators $T, A, g : H \to H$, consider the problem of fining $u \in H$ such that

$$\langle Tu, g(v) \rangle = \langle A(u), g(v) \rangle, \quad \forall v \in H, \tag{1.8}$$

This system of general equations can be viewed as a weak formulation of the non-positive and non-symmetric odd-order boundary value problems. For more details, see Filippov [27], Noor [30] [31] [32], Noor *et al.* [36], Petryshin [28], Tonti [29], and the references therein.

Motivated and inspired by ongoing research in these fields, we introduce and study some new systems of exponentially general equations. This new system of exponentially general equations can be viewed as a weak formulation of the non-positive and non-symmetric exponentially boundary value problems. It is shown that the system of absolute value equations, complementarity problems and Lax-Milgram Lemma can be obtained as special cases. The auxiliary principle technique, which is mainly due to Lions and Stampacchia [39] and Glowinski *et al.* [40], is used to discuss the existence of a solution for the system of exponentially general equations. This approach is also applied to suggest some hybrid inertial iterative methods for solving the system of general exponentially equations. The convergence analysis of these methods is investigated under weaker conditions.

In Section 2, we introduce new system of exponentially general equations and discuss their applications. It is shown that the Reisz-Frechet representation theorem, Lax-Milgram Lemma and system of absolute value exponentially equations can be obtained as special cases. As an interesting case, a new inner product space is derived. This may be starting point to explore the applications of the general inner product space. It is shown that the exponentially third order boundary value problems can be studied in the general framework of exponentially general equations. In Section 3, we use the auxiliary principle technique to discuss the existence of a solution as well as to suggest some iterative methods for solving the general absolute value equations. The convergence criteria of proposed iterative methods are considered under weaker conditions. Our method of proofs is very simple with other techniques. Several new iterative methods for solving the exponentially general equations are obtained as novel applications of the results.

2. Formulations and Basic Facts

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle \cdot,\cdot \rangle$, respectively.

For given operators $L, A, g : H \to H$ and a continuous linear functional *f*, we consider the problem of finding $u \in H$ such that

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle = \left\langle f, g(v) - g(u) \right\rangle, \quad \forall v \in H, \tag{2.1}$$

which is called the system of exponentially general equations. We note that the problem (2.1) is equivalent to finding $u \in H$ such that

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle = \left\langle f, g(v) \right\rangle, \quad \forall v \in H. \tag{2.2}$$

Several important applications are discussed as special cases of the problems (2.1) and (2.2):

1) If $A = 0$, then problem (2.1) collapses to finding $u \in H$, such that

$$\left\langle e^{Lu}, g(v) - g(u) \right\rangle = \left\langle f, g(v) - g(u) \right\rangle, \quad \forall v \in H \tag{2.3}$$

which is called the exponentially general equations.

Systems of the exponentially general equations of type (2.1), (2.2) and (2.3) can be viewed as exponentially general Lax-Milgram lemma. These systems of exponentially general equations are called the weak formulations of the odd-order, non-symmetric and non-positive boundary value problems. We use these systems to discuss the unique existence to a solution of the odd-order and non-symmetric exponentially boundary value problems. This result plays a significant role in the study of function spaces and partial differential equations.

2) If $g = I$, the identity operator and $\left\langle e^{Lu}, v \right\rangle = a(u,v)$, where $a(.,.)$ is bilinear continuous form, then problem (2.2) reduces to finding $u \in H$ such that

$$a(u,v) + \left\langle e^{A(u)}, v \right\rangle = \left\langle f, v \right\rangle, \quad \forall v \in H, \tag{2.4}$$

which is known as the exponentially general Lax-Milgram Lemma.

3) If $g = I$, the identity operator, then problem (2.2) reduced to finding $u \in H$ such that

$$\left\langle e^{Lu} + e^{A(u)}, v \right\rangle = \left\langle f, v \right\rangle, \quad \forall v \in H \tag{2.5}$$

which is known as the weak formulation of the exponentially boundary value problems.

4) From problem (2.5), we have the problem of finding $u \in H$ such that

$$e^{Lu} + e^{A(u)} = f, \tag{2.6}$$

which is called the system of exponentially equations.

5) If $e^{Lu} + e^{A(u)} = e^{Lu} + e^{A|u|}$, then problem (2.1) reduces to finding $u \in H$

such that

$$\left\langle e^{Lu}+e^{A|u|},g(v)-g(u)\right\rangle=\left\langle f,g(v)-g(u)\right\rangle,\quad \forall v\in H, \tag{2.7}$$

which is called the system of exponentially general absolute value equations. From (2.16), we find $u\in H$ such that

$$e^{Lu}+e^{A|u|}=f,\quad \forall v\in H, \tag{2.8}$$

which is called the system of exponentially absolute value equations.

6) If $e^{Lu}=\Phi(u)$, $e^{A(u)}=N(u)$, are nonlinear operators, then problem (2.16) reduces to finding $u\in H$ such that

$$\Phi(u)+N|u|=f,\quad \forall v\in H, \tag{2.9}$$

is called the system of absolute value equations. It is known that the system of absolute value equations is equivalent to the complementarity problems. If the involved convex set in the variational inequalities is a convex cone, then variational inequalities are equivalent to the complementarity problems. Consequently, all these problems are equivalent under some suitable conditions. This fascinating interplay among these problems can be used in developing several numerical methods to solve complicated and complex problems.

7) For $A=0$, $L=I$, the problem (2.2) collapses to finding $u\in H$ such that

$$\left\langle u,g(v)\right\rangle=\left\langle f,g(v)\right\rangle,\quad \forall v\in H, \tag{2.10}$$

which is called the general Riesz-Frechet representation theorem.

8) If $g=I$, then general Riesz-Frechet representation theorem reduces to finding $u\in H$. Such that

$$\left\langle u,v\right\rangle=\left\langle f,v\right\rangle,\quad \forall v\in H, \tag{2.11}$$

which is the celebrated Riesz-Frechet representation theorem, introduced by Riesz [2] [3] and Frechet [1] in 1907, independently. It has been shown by Noor [8] [9] that the Riesz-Frechet representation theorem has a variational character. In fact, $u\in H$ is solution of (2.11), if and only if, $u\in H$ is the minimum of the energy functional

$$I[v]=\left\langle v,v\right\rangle-2\left\langle f,v\right\rangle,\quad \forall v\in H. \tag{2.12}$$

It is obvious that the energy function $I[v]$ is a strongly convex functions. Consequently it has a unique minimum $u\in H$. This equivalent formulation can be used to discuss the unique existence of the Riesz-Frechet representation theorem, which can be viewed as novel way.

9) If $e^{Lu}=\Phi(u)$, $e^{A(u)}=N(u)$, are nonlinear operators, then problem (2.1) reduces to finding $u\in H$, such that

$$\left\langle \Phi(u)+N(u),g(v)-g(u)\right\rangle=\left\langle f,g(v)-g(u)\right\rangle,\quad \forall v\in H, \tag{2.13}$$

which is called the system of general equations, introduced and studied by Noor [30].

10) If $g=I$, the identity operator, then (2.13) is equivalent to fining $u\in H$

such that

$$\langle \Phi(u)+N(u), v-u\rangle = \langle f, v-u\rangle, \quad \forall v \in H, \tag{2.14}$$

is called the generalized Lax-Milgram Lemma, introduced and studied by Noor [30].

Remark 2.1. *For suitable and appropriate choice of the operators* T, A, g, *one can obtain various classes of new and known classes of problems as application of the problem* (2.1). *This shows that the system of exponentially general equations is a unified one.*

It is known [27] [28] [29] that, if the operator is not symmetric and non-positive, then it can be made symmetric and positive with respect to an arbitrary operator.

Definition 2.1. *An operator* $T: H \to H$ *with respect to an arbitrary operator* $g: H \to H$ *is said to be*:

1) *Exponentially general symmetric*, if,

$$\langle e^{Tu}, g(v)\rangle = \langle g(u), e^{Tv}\rangle, \quad \forall u, v \in H.$$

2) Exponentially general positive, if,

$$\langle e^{Tu}, g(u)\rangle \geq 0, \quad \forall u \in H.$$

3) *Exponentially general coercive* (*g-elliptic*), if there exists a constant $\alpha > 0$ such that

$$\langle e^{Tu}, g(u)\rangle \geq \alpha \|g(u)\|^2, \quad \forall u \in H.$$

Note that exponentially general coercivity implies exponentially general positivity, but the converse is not true. It is also worth mentioning that there are operators which are not exponentially general symmetric but exponentially general positive. On the other hand, there are *g*-positive, but not *g*-symmetric operators. Furthermore, it is well-known [27] [28] [29] that, if for a linear operator *L*, there exists an inverse operator L^{-1} on $R(L)$, the range of *L*, with $\overline{R(L)} = H$, then one can find an infinite set of auxiliary operators g such that the operator T is both *g*-symmetric and *g*-positive.

Remark 2.2. *If* $e^{Tu} = I(u)$, *the identity operator, then Definition* (2.1) *reduces to*:

Definition 2.2. *An inner produce* $\langle .,. \rangle$ *with respect to an arbitrary operator* $g: H \to H$ *is said to be*

1) *General symmetric*, if, $\langle u, g(v)\rangle = \langle g(u), v\rangle$, $\forall u, v \in H$.

2) General positive, if, $\langle u, g(u)\rangle \geq 0$, $\forall u \in H$.

3) *General positive definite, if there exists a constant* $\alpha > 0$ *such that*

$$\langle u, g(u)\rangle \geq \alpha \|g(u)\|^2, \quad \forall u \in H.$$

Motivated by the Remark (2.2), we can define the general inner product space with respect to an arbitrary function g such that

1) $\langle u, g(u)\rangle \geq 0$, and $\langle u, g(u)\rangle = 0$, $\Leftrightarrow$ $u = g(u)$, $\forall u, v \in H$.

2) $\langle u, g(v)\rangle = \langle g(u), v\rangle, \quad \forall u, v \in H$.

3) $\langle u, g(v)+g(w)\rangle = \langle u, g(v)\rangle + \langle u, g(w)\rangle, \quad \forall u, v, w \in H$.

Also, we can obtain the result for general inner product spaces, that is,

$$\|u+g(v)\|^2 + \|u-g(v)\|^2 = 2\left\{\|u\|^2 + \|g(v)\|^2\right\}, \quad \forall u, v \in H, \tag{2.15}$$

which is known as the parallelogram laws and can be used to characterize the general Hilbert space.

It is interesting problem to consider the completeness of the general inner product spaces and explore their properties in fixed-point theory, differential equations and optimization theory.

If the operators L, A are linear, general positive, general symmetric and the operator g is linear, then the problem (2.1) is equivalent to finding a minimum of the function $I[v]$ on H, where

$$I[v] = \left\langle e^{Lv} + e^{A(v)}, g(v)\right\rangle - 2\langle f, g(v)\rangle, \quad \forall v \in H, \tag{2.16}$$

which is a nonlinear programming problem and can be solved using the known techniques of the optimization theory.

We now consider the problem of finding the minimum of the functional $I[v]$, defined by (2.16). For the sake of completeness and to convey the main ideas, we include all the details.

Theorem 2.1. *Let the operators* $L, A : H \to H$ *be linear, exponentially general symmetric and exponentially general positive. If the operator* $g : H \to H$ *is linear, then the function* $u \in H$ *minimizes the functional* $I[v]$, *defined by* (2.16), *if and only if,*

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\right\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2.17}$$

Proof. Let $u \in H$ satisfy (2.17). Then, using the exponentially general positivity of the operators L, A, we have

$$\left\langle e^{Lv} + e^{A(v)}, g(v) - g(u)\right\rangle \ge \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{2.18}$$

$\forall u, v \in H$, $\varepsilon \ge 0$, let $v_\varepsilon = u + \varepsilon(v-u) \in H$. Taking $v = v_\varepsilon$ in (2.18) and using the fact that g is linear, we have

$$\left\langle e^{Lv_\varepsilon} + e^{A(v_\varepsilon)}, g(v_\varepsilon) - g(u)\right\rangle \ge \langle f, g(v_\varepsilon) - g(u)\rangle. \tag{2.19}$$

We now define the function

$$\begin{aligned} h(\varepsilon) = {} & \varepsilon\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\right\rangle + \frac{\varepsilon^2}{2}\left\langle e^{L(v-u)} + e^{A(v-u)}, g(v) - g(u)\right\rangle \\ & - \varepsilon\langle f, g(v) - g(u)\rangle, \end{aligned} \tag{2.20}$$

such that

$$\left.\begin{aligned} h'(\varepsilon) = {} & \left\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\right\rangle \\ & + \varepsilon\left\langle e^{L(v-u)} + e^{A(v-u)}, g(v) - g(u)\right\rangle - \langle f, g(v) - g(u)\rangle \end{aligned}\right. \ge 0, \text{ by (2.19).}$$

Using the g symmetry of L, A we see that $h(\varepsilon)$ is an increasing function on $[0,1]$ and so $h(0) \le h(1)$ gives us

$$\left\langle e^{Lu} + e^{A(u)}, g(u) \right\rangle - 2\left\langle f, g(u) \right\rangle \le \left\langle e^{Lv} + e^{A(v)}, g(v) \right\rangle - 2\left\langle f, g(v) \right\rangle,$$

that is,

$$I[u] \le I[v], \quad \forall v \in H,$$

which shows that $u \in H$ minimizes the functional $I[v]$, defined by (2.16).

Conversely, assume that $u \in H$ is the minimum of $I[v]$, then

$$I[u] \le I[v], \quad \forall v \in H. \tag{2.21}$$

Taking $v = v_\varepsilon \equiv u + \varepsilon(v-u) \in H, \forall u, v \in H$ in (2.21), we have $I[u] \le I[v_\varepsilon]$.

Using (2.16), *g*-positivity and the linearity of L, A, we obtain

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle + \frac{\varepsilon}{2}\left\langle e^{L(g(v)-g(u))} + e^{A(g(v)-g(u))}, g(v) - g(u) \right\rangle$$
$$\ge \left\langle f, g(v) - g(u) \right\rangle,$$

from which, as $\varepsilon \to 0$, we have

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle \ge \left\langle f, g(v) - g(u) \right\rangle, \quad \forall v \in H. \tag{2.22}$$

Replacing $g(v) - g(u)$ by $g(u) - g(v)$ in inequality (2.22), we have

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle \le \left\langle f, g(v) - g(u) \right\rangle, \quad \forall v \in H. \tag{2.23}$$

From (2.22) and (2.23), it follows that $u \in H$ satisfies

$$\left\langle e^{Lu} + e^{A(u)}, g(v) - g(u) \right\rangle = \left\langle f, g(v) - g(u) \right\rangle, \quad \forall v \in H, \tag{2.24}$$

the required result (2.17). □

We now show that the third order exponentially boundary value problems can be studied via problem (2.1).

Example 2.1. *Consider the third order exponentially boundary value problem of finding* u *such that*

$$-e^{\frac{d^3u}{dx^3}} + \frac{dv}{dx} = f(x), \quad \forall x \in [a,b], \tag{2.25}$$

with boundary conditions

$$u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0, \tag{2.26}$$

where $f(x)$ *is a continuous function. This problem can be studied in the general framework of the problem* (2.1) *To do so, let*

$$H = \left\{ u \in H_0^2[a,b] : u(a) = 0, u'(a) = 0, u'(b) = 0 \right\}$$

be a Hilbert space, see [5]. *One can easily show that*

$$-\int_a^b e^{\frac{d^3v}{dx^3}} \frac{dv}{dx} dx + \int_a^b \frac{dv}{dx}\frac{dv}{dx} dx - \int_a^b f \frac{dv}{dx} dx, \quad \forall \frac{dv}{dx} \in H_0^2[a,b]$$
$$= \int_a^b \left(\frac{d^2v}{dx^2} \right)^2 + \int_a^b \frac{dv}{dx}\frac{dv}{dx} - \int_a^b f \frac{dv}{dx} dx$$
$$= \left\langle e^{Lv}, g(v) \right\rangle + \left\langle e^{Au}, g(v) \right\rangle - \left\langle f, g(v) \right\rangle = 0,$$

from which, it follows that

$$\langle e^{Lv}, g(v)\rangle + \langle e^{Au}, g(v)\rangle = \langle f, g(v)\rangle.$$

This is the weak formulation of the third order exponentially boundary (2.25).

Definition 2.3. *An operator* $L: H \to H$ *is said to be*;

1) *Strongly exponentially monotone, if there exists a constant* $\alpha > 0$, *such that*

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

2) *Exponentially Lipschitz continuous, if there exists a constant* $\beta > 0$, *such that*

$$\|e^{Lu} - e^{Lv}\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

3) *Exponentially monotone, if*

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

4) *Firmly strongly exponentially monotone, if*

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq \|u - v\|^2, \quad \forall u, v \in H.$$

We remark that, if the operator L is both strongly exponentially monotone with constant $\alpha > 0$ and exponentially Lipschitz continuous with constant $\beta > 0$, respectively, then from (1) and (2), it follows that $\alpha \leq \beta$.

3. Main Results

In this section, we use the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [39] and Glowinski *et al.* [40], as developed by Noor [30] [31] [32], Noor *et al.* [13] [14] [15] [16] [22] [32] and Zhu *et al.* [41]. The main idea of this technique is to consider an auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. In recent several inertial type algorithms have been analyzed for solving variational inequalities and optimization problems, which are mainly due to Polyak [46]. These methods help us to improve convergence rate of the iterative methods. In this section, we use the auxiliary principle technique to suggest some new inertial iterative methods for solving the system of exponentially general equations. These inertial methods do not involve the evaluations of the projection methods, resolvent methods and their variant forms. This is advantage of this technique.

We now consider the problem of the uniqueness and existence of the solution of (2.1) using the technique of the auxiliary principle approach, which is subject of our nest result.

Theorem 3.1. *Let the operator L be a strongly exponentially monotone with constant* $\alpha > 0$ *and exponentially Lipschitz continuous with constant* $\beta > 0$, *respectively. Let the operator g be firmly strongly monotone and Lipschitz con-*

tinuous with constant β_1. *If the operator A is Lipschitz continuous with constant* $\lambda > 0$ *and there exists a constant* $\rho > 0$ *such that*

$$\left|\rho - \frac{\alpha+\nu-1}{\beta^2-\lambda^2}\right| < \frac{\sqrt{(\alpha+\nu-1)^2 - (\beta^2-\lambda^2)\nu(2-\nu)}}{\beta^2-\lambda^2}, \tag{3.1}$$

$$\alpha > 1-\nu+\sqrt{(\beta^2-\lambda^2)\nu(2-\nu)}, \quad \rho\lambda < 1-\nu, \quad \nu < 1, \tag{3.2}$$

where

$$\nu = \sqrt{\beta_1^2 - 1} \tag{3.3}$$

then the problem (2.1) *has a solution.*

Proof. We use the auxiliary principle technique to prove the existence of a solution of (2.1).

For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\begin{aligned}&\left\langle \rho\left(e^{Lu}+e^{A(u)}\right), g(v)-g(w)\right\rangle + \langle g(w)-g(u), g(v)-g(w)\rangle \\ &= \langle \rho f, g(v)-g(w)\rangle, \quad \forall v \in H,\end{aligned} \tag{3.4}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (3.4) defines a mapping w connecting the both problems (2.1) and (3.4). To prove the existence of a solution of (2.1), it is enough to show that the mapping w defined by (3.4) is a contraction mapping.

Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) satisfy the auxiliary problem (3.4). Then

$$\begin{aligned}&\left\langle \rho\left(e^{Lu_1}+e^{A(u_1)}\right), g(v)-g(w_1)\right\rangle + \langle g(w_1)-g(u_1), g(v)-g(w_1)\rangle \\ &= \langle \rho f, g(v)-g(w_1)\rangle, \quad \forall v \in H,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\left\langle \rho\left(e^{Lu_2}+e^{A(u_2)}\right), g(v)-g(w_2)\right\rangle + \langle g(w_2)-g(u_2), g(v)-g(w_2)\rangle \\ &= \langle \rho f, g(v)-g(w_2)\rangle, \quad \forall v \in H.\end{aligned} \tag{3.6}$$

Taking $v = w_2$ in (3.5) and $v = w_1$ in (3.6) and adding the resultant, we have

$$\begin{aligned}&\|g(w_1)-g(w_2)\|^2 = \langle g(w_1)-g(w_2), g(w_1)-g(w_2)\rangle \\ &= \left\langle g(u_1)-g(u_2)-\rho\left(e^{Lu_1}-e^{Lu_2}\right)+\rho\left(e^{A(u_1)}-e^{A(u_2)}\right), g(w_1)-g(w_2)\right\rangle.\end{aligned} \tag{3.7}$$

From (3.7), we have

$$\begin{aligned}&\|g(w_1)-g(w_2)\|^2 \\ &\leq \left\|g(u_1)-g(u_2)-\rho\left(e^{Lu_1}-e^{Lu_2}\right)+\rho\left(e^{A(u_1)}-e^{A(u_2)}\right)\right\| \|g(w_1)-g(w_2)\|\end{aligned}$$

from which, using the exponentially Lipschitz continuity of the operator A with constant $\lambda > 0$, it follows that

$$\begin{aligned}&\|w_1-w_2\| \leq \|g(w_1)-g(w_2)\| \\ &\leq \left\|g(u_1)-g(u_2)-\rho\left(e^{Lu_1}-e^{Lu_2}\right)\right\| + \rho\left\|e^{A(u_1)}-e^{A(u_2)}\right\| \\ &\leq \|u_1-u_2-g(u_1)-g(u_2)\| + \left\|u_1-u_2-\rho\left(e^{Lu_1}-e^{Lu_2}\right)\right\| + \rho\lambda\|u_1-u_2\|.\end{aligned} \tag{3.8}$$

Using the strongly exponentially monotonicity and exponentially Lipschitz continuity of the operator L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}
&\left\| u_1 - u_2 - \rho\left(e^{Lu_1} - e^{Lu_2}\right)\right\|^2 \\
&= \left\langle u_1 - u_2 - \rho\left(e^{Lu_1} - e^{Lu_2}\right), u_1 - u_2 - \rho\left(e^{Lu_1} - e^{Lu_2}\right)\right\rangle \\
&= \left\langle u_1 - u_2, u_1 - u_2\right\rangle - 2\rho\left\langle e^{Lu_1} - e^{Lu_2}, u_1 - u_2\right\rangle + \rho^2\left\langle e^{Lu_1} - e^{Lu_2}, e^{Lu_1} - e^{Lu_2}\right\rangle \\
&\le \left(1 - 2\rho\alpha + \rho^2\beta^2\right)\left\|u_1 - u_2\right\|^2 .
\end{aligned} \tag{3.9}$$

Similarly, using the strongly firmly monotonicity and Lipschitz continuity of the operator g with constant β_1, we have

$$\left\| u_1 - u_2 - \left(g(u_1) - g(u_2)\right)\right\|^2 \le \left\{\sqrt{\beta_1^2 - 1}\right\}\left\|u_1 - u_2\right\|^2 . \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we have

$$\left\|w_1 - w_2\right\| \le \left(\sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right)\left\|u_1 - u_2\right\| = \theta\left\|u_1 - u_2\right\|, \tag{3.11}$$

where

$$\begin{aligned}
\theta &= \sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \\
&= \nu + \rho\lambda + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2},
\end{aligned}$$

and

$$\nu = \sqrt{\beta_1^2 - 1}.$$

From (3.13) and (3.2), it follows that $\theta < 1$, so the mapping w is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (2.1). ☐

Remark 3.1. *We point out that the solution of the auxiliary problem* (3.4) *is equivalent to finding the minimum of the functional* $I[w]$, *where*

$$I[w] = \frac{1}{2}\left\langle g(w) - g(u), g(w) - g(u)\right\rangle - \rho\left\langle e^{Lu} + e^{A(u)} - f, g(w) - g(u)\right\rangle,$$

which is a differentiable convex functional associated with the inequality (3.4), *if the operator g is differentiable. This alternative formulation can be used to suggest iterative methods for solving the general absolute value equations. This auxiliary functional can be used to find a kind of gap function, whose stationary points solves the problem* (2.2).

Iterative Methods

We now use the auxiliary principle to suggest some iterative methods for solving the system of exponentially general Equations (2.1). It is clear that, if $w = u$, then w is a solution of (2.1). This observation shows that the auxiliary principle technique can be used to propose the following iterative method for solving the system of general Equations (2.1).

Algorithm 3.1. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\left\langle e^{Lu_n}+e^{A(u_n)},g(v)-g(u_{n+1})\right\rangle+\left\langle g(u_{n+1})-g(u_n),g(v)-g(u_{n+1})\right\rangle$$
$$=\left\langle f,g(v)-g(u_{n+1})\right\rangle,\quad \forall v\in H.$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (2.1).

For a given $u\in H$, consider the problem of finding $w\in H$ such that,

$$\begin{aligned}&\left\langle \rho\left(e^{L(w+\zeta(u-w))}+e^{A(w+\zeta(u-w))}\right),g(v)-g(w)\right\rangle\\&+\left\langle g(w)-g(u)+\eta\left(g(u)-g(u)\right),g(v)-g(w)\right\rangle\\&=\rho\left\langle f,g(v)-g(w)\right\rangle,\quad \forall v\in H,\end{aligned}\tag{3.12}$$

which is called the auxiliary problem, where $\eta\geq 0$, $\zeta\geq 0$, are parameter. We note that the auxiliary problems (3.4) and (3.12) are quite different.

Clearly $w=u\in H$ is a solution of (2.1). This observation allows us to suggest the following iterative method for solving the problem (2.1).

Algorithm 3.2. *For given initial values* u_0,u_1, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\begin{aligned}&\left\langle \rho\left(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))}+e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))}\right)+g(u_{n+1})\right.\\&\left.\quad -g(u_n)+\eta\left(g(u_n)-g(u_{n-1})\right),g(v)-g(u_{n+1})\right\rangle\\&=\left\langle \rho f,g(v)-g(u_{n+1})\right\rangle,\ \forall v\in H,\end{aligned}$$

which is an inertial implicit method.

If $\zeta=\frac{1}{2}$, then Algorithm 3.2 reduces to.

Algorithm 3.3. *For given initial values* u_0,u_1, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\begin{aligned}&\left\langle \rho\left(e^{L\left(\frac{u_{n+1}+u_n}{2}\right)}+e^{A\left(\frac{u_{n+1}+u_n}{2}\right)}\right)+g(u_{n+1})-g(u_n)\right.\\&\left.\quad +\eta\left(g(u_n)-g(u_{n-1})\right),g(v)-g(u_{n+1})\right\rangle\\&=\left\langle \rho f,g(v)-g(u_{n+1})\right\rangle,\ \forall v\in H,\end{aligned}$$

which is an inertial midpoint implicit method.

From Algorithm 3.2, we can obtain the following iterative method for solving (2.1).

Algorithm 3.4. *For a given initial value* u_0,u_1 *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\begin{aligned}g(u_{n+1})&=g(u_n)-\eta\left(g(u_n)-g(u_{n-1})\right)\\&\quad-\rho\left(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))}+e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))}-f\right).\end{aligned}$$

This is a new implicit method for solving the system of exponentially general Equations (2.1).

To implement the implicit method (3.2) with $\eta\left(g\left(u_n\right)-g\left(u_{n-1}\right)\right)=0$, one uses the explicit method as a predictor and implicit method as a predictor. Consequently, we obtain the two-step method for solving the problem (2.1).

Algorithm 3.5. *For a given initial value* u_0, *compute the approximate solution* u_{n+1} *by the iterative schemes*

$$\begin{aligned}&\left\langle \rho e^{Lu_n}+\rho e^{A(u_n)}+g\left(y_n\right)-g\left(u_n\right),g\left(v\right)-g\left(u_n\right)\right\rangle\\&=\left\langle \rho f,g\left(v\right)-g\left(y_n\right)\right\rangle,\ \forall v\in H,\end{aligned}$$

$$\begin{aligned}&\left\langle \rho e^{L\left(y_n+\zeta\left(u_n-y_n\right)\right)}+\rho e^{A\left(y_n+\zeta\left(u_n-y_n\right)\right)}+g\left(u_{n+1}\right)-g\left(u_n\right),g\left(v\right)-g\left(y_n\right)\right\rangle\\&=\left\langle \rho f,g\left(v\right)-g\left(y_n\right)\right\rangle,\ \forall v\in H,\end{aligned}$$

which is known as two-step iterative method for solving problem (2.1).

Based on the above arguments, we can suggest a new two-step (predictor-corrector) method for solving the system of exponentially general Equations (2.1).

Algorithm 3.6. *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative schemes*

$$g\left(y_n\right)=g\left(u_n\right)-\rho\left(e^{Lu_n}+e^{A(u_n)}-f\right)$$

$$g\left(u_{n+1}\right)=g\left(u_n\right)-\rho\left(e^{L\left(y_n+\zeta\left(u_n-y_n\right)\right)}+e^{A\left(y_n+\zeta\left(u_n-y_n\right)\right)}-f\right),\quad n=0,1,2,\cdots$$

We again apply the auxiliary principle technique to suggest another iterative method for solving (2.1).

For a given $u\in H$, consider the problem of finding $w\in H$ such that,

$$\begin{aligned}&\left\langle \rho\left(e^{Lw}+e^{A(w)}\right),g\left(v\right)-g\left(w\right)\right\rangle+\left\langle w-u+\xi\left(u-u\right),v-w\right\rangle\\&=\left\langle \rho f,g\left(v\right)-g\left(w\right)\right\rangle,\quad \forall v\in H,\end{aligned}\tag{3.13}$$

which is called the auxiliary problem, where $\rho>0,\xi>0$ are constants. It is clear that (3.4) defines a mapping w connecting the both problems (2.1) and (3.13).

If $w=u$, then w is a solution of (2.1). This observation is used to propose the iterative method.

Algorithm 3.7. *For given initial values* u_0,u_1 *compute the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}&\left\langle \rho e^{Lu_{n+1}}+\rho e^{A(u_{n+1})},g\left(v\right)-g\left(u_{n+1}\right)\right\rangle+\left\langle u_{n+1}-u_n+\xi\left(u_n-u_{n-1}\right),v-u_{n+1}\right\rangle\\&=\left\langle \rho f,g\left(v\right)-g\left(u_{n+1}\right)\right\rangle,\ \forall v\in H,\end{aligned}\tag{3.14}$$

which is an inertial implicit method.

If $\xi=0$, then Algorithm 3.7 reduces to

Algorithm 3.8. *For a given initial value* u_0, *compute the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}&\left\langle \rho e^{Lu_{n+1}}+\rho e^{A(u_{n+1})},g\left(v\right)-g\left(u_{n+1}\right)\right\rangle+\left\langle u_{n+1}-u_n,v-u_{n+1}\right\rangle\\&=\left\langle \rho f,g\left(v\right)-g\left(u_{n+1}\right)\right\rangle,\ \forall v\in H,\end{aligned}\tag{3.15}$$

which is an inertial implicit method.

For the convergence analysis of the iterative methods, we need the following concept.

Definition 3.1. *The operator L is said to be pseudo exponentially general monotone with respect to A, if*

$$\left\langle e^{Lu}+e^{A(u)},g(v)-g(u)\right\rangle=\left\langle f,g(v)-g(u)\right\rangle,\quad\forall v\in H,$$

$$\Rightarrow$$

$$\left\langle e^{Lv}+e^{A(v)},g(v)-g(u)\right\rangle\ge\left\langle f,g(v)-g(u)\right\rangle,\quad\forall v\in H.$$

We now consider the convergence analysis of Algorithm 3.8, which is the main motivation of our next result.

Theorem 3.2. *Let* $u\in H$ *be a solution of problem* (2.1) *and let* u_{n+1} *be the approximate solution obtained from Algorithm* 3.8. *If L is an exponentially monotone operator with respect to* $A(.)$, *then*

$$\left\|u_{n+1}-u\right\|^2\le\left\|u_n-u\right\|^2-\left\|u_{n+1}-u_n\right\|^2. \tag{3.16}$$

Proof. Let $u\in H:g(u)\in H$ be a solution of (2.1). Then

$$\left\langle e^{Lu}+e^{A(u)},g(v)-g(u)\right\rangle=\left\langle f,g(v)-g(u)\right\rangle,\quad\forall v\in H,$$

which implies that

$$\left\langle e^{Lv}+e^{A(v)},g(v)-g(u)\right\rangle\ge\left\langle f,g(v)-g(u)\right\rangle,\quad\forall v\in H, \tag{3.17}$$

since the operator L is an exponentially monotone operator with respect to $\lambda|\cdot|$.

Taking $v=u_{n+1}$ in (3.17) and $v=u$ in (3.15), we have

$$\left\langle e^{Lu_{n+1}}+e^{A(u_{n+1})},g(u_{n+1})-g(u)\right\rangle\ge\left\langle f,g(u_{n+1})-g(u)\right\rangle,\quad\forall v\in H, \tag{3.18}$$

and

$$\begin{aligned}&\left\langle \rho e^{Lu_{n+1}}+\rho e^{A(u_{n+1})},g(u)-g(u_{n+1})\right\rangle+\left\langle u_{n+1}-u_n,u-u_{n+1}\right\rangle\\&=\left\langle \rho f,g(u)-g(u_{n+1})\right\rangle,\ \forall v\in H.\end{aligned} \tag{3.19}$$

From (3.19), we have

$$\begin{aligned}&\left\langle u_{n+1}-u_n,u-u_{n+1}\right\rangle\\&\ge\rho\left\langle\left(e^{Lu_{n+1}}+e^{A(u_{n+1})}\right),g(u_{n+1})-g(u)\right\rangle-\rho\left\langle f,g(u_{n+1})-g(u)\right\rangle\ge0,\end{aligned} \tag{3.20}$$

where we have used (3.18).

Using the relation $2\langle a,b\rangle=\|a+b\|^2-\|a\|^2-\|b\|^2$, $\forall a,b\in H$, the Cauchy inequality and from (3.20), we have

$$\left\|u-u_{n+1}\right\|^2\le\left\|u-u_n\right\|^2-\left\|u_n-u_{n+1}\right\|^2,$$

which is the required (3.16). □

Theorem 3.3. *Let* $\bar{u}\in H$ *be a solution of* (2.1) *and let* u_{n+1} *be the approximate solution obtained from Algorithm* 3.2. *Let L be an exponentially monotone operator with respect to A and the operator g is continuous. Then*

$$\lim_{n\to\infty} u_{n+1} = \overline{u}. \tag{3.21}$$

Proof. Let $\overline{u} \in H$ be a solution of (2.1). From (3.16), it follows that the sequence $\{\|\overline{u} - u_n\|\}$ is noncreasing and consequently the sequence $\{u_n\}$ is bounded. Also, from (3.16), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \le \|u_0 - \overline{u}\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0 \Rightarrow \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.22}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\overline{u} \in H$. Replacing u_n by u_{n_j} in (3.15), taking the limit as $n_j \to \infty$ and using (3.22), we have

$$\left\langle e^{L\hat{u}} + e^{A(\hat{u})}, g(v) - g(\hat{u}) \right\rangle = \langle f, g(v) - g(\hat{u}) \rangle, \quad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (2.1) and

$$\|u_{n+1} - u_n\|^2 \le \|u_n - \hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$. □

The auxiliary principle technique is used to suggest another iterative method for solving (2.1).

For a given $u \in H$, enwinding $w \in H$ such that,

$$\begin{aligned}
&\left\langle \rho e^{L((1-\xi)w+\xi u)} + e^{A((1-\xi)w+\xi u)}, g(v) - g((1-\xi)w + \xi u) \right\rangle \\
&+ \left\langle g((1-\xi)w + \xi u) - g(u), g(v) - g((1-\xi)w + \xi u) \right\rangle \\
&= \left\langle \rho f, g(v) - g((1-\xi)w + \xi u) \right\rangle, \quad \forall v \in H, \xi \in [0,1],
\end{aligned} \tag{3.23}$$

which is called the auxiliary problem, where $\rho > 0$, $\xi \ge 0$, are constants. It is clear that (3.4) defines a mapping w connecting the both problems (2.1) and (3.13). Clearly, $w = u$ is a solution of the problem (2.1). This allows us to suggest the inertial iterative method.

Algorithm 3.9. *For a given* $u_1, u_2 \in H$, *calculate the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}
&\left\langle \rho\left(e^{L((1-\xi)u_n+\xi u_{n-1})} + e^{A((1-\xi)u_n+\xi u_{n-1})}\right), g(v) - g((1-\xi)u_n + \xi u_{n_1}) \right\rangle \\
&+ \left\langle g((1-\xi)u_n + \xi u_{n-1}) - g(u_n), g(v) - g((1-\xi)u_n + \xi u_{n-1}) \right\rangle \\
&= \left\langle \rho f, g(v) - g((1-\xi)u_n + \xi u_{n-1}) \right\rangle, \quad \forall v \in H, \xi \in [0,1]
\end{aligned}$$

This is an inertial type implicit method for solving the problem (2.1), which is equivalent to the following two-step method.

Algorithm 3.10. *For a given* $u_1, u_1 \in H$, *calculate the approximate solution* u_{n+1} *by the iterative schemes*

$$g(y_n) = (1-\xi)u_n + \xi u_{n-1}$$

$$g\left(u_{n+1}\right)=g\left(u_n\right)-\rho\left(e^{Ly_n}+e^{A(y_n)}\right)-\rho f,\ n=1,2,\cdots,\ \xi\in[0,1]$$

We now use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal method for solving exponentially general Equations (2.1). For the sake of completeness and to convey the main ideas of the Bregman distance functions, we recall the basic concepts and applications.

The Bregman distance function is defined as

$$\begin{aligned}B(u,w)&=E\left(g(u)\right)-E\left(g(w)\right)-\left\langle E'\left(g(w)\right),g(u)-g(w)\right\rangle\\&\geq\nu\left\|g(u)-g(w)\right\|^2,\end{aligned}\tag{3.24}$$

using the strongly general convexity with modulus ν.

The function $B(u,w)$ is called the general Bregman distance function associated with general convex functions.

For $g=I$, we obtain the original Bregman distance function

$$B(u,w)=E(u)-E(w)-\left\langle E'(w),u-w\right\rangle\geq\nu\|u-w\|^2.$$

It is important to emphasize that various types of convex function E gives different Bregman distance.

For a given $u\in H$, find $w\in H$ satisfying the auxiliary system of exponentially general Equations (2.1).

$$\begin{aligned}&\left\langle\rho\left(e^{L(w+\zeta(u-w))}+e^{A(w+\zeta(u-w))}\right)+E'\left(g(w)\right)-E'\left(g(u)\right),g(v)-g(w)\right\rangle\\&=\left\langle\rho f,g(v)-g(w)\right\rangle,\ \forall v\in H,\end{aligned}$$

where $E'(u)$ is the differential of a strongly general convex function E.

Note that, if $w=u$, then w is a solution of (2.1). Thus, we can suggest the following iterative method for solving (2.1).

Algorithm 3.11. *For a given* $u_0\in H$, *calculate the approximate solution by the iterative scheme*

$$\begin{aligned}&\left\langle\rho\left(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))}+e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))}\right)+E'\left(g(u_{n+1})\right)-E'\left(g(u_n)\right),g(v)-g(u_{n+1})\right\rangle\\&=\left\langle\rho f,g(v)-g(u_{n+1})\right\rangle,\ \forall v\in H,\end{aligned}\tag{3.25}$$

which is known as the proximal point method.

For $\zeta=0$ and $\zeta=1$, Algorithm 3.11 reduces to:

Algorithm 3.12. *For a given* $u_0\in H$, *calculate the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}&\left\langle\rho\left(e^{Lu_{n+1}}+e^{A(u_{n+1})}\right)+E'\left(g(u_{n+1})\right)-E'\left(g(u_n)\right),g(v)-g(u_{n+1})\right\rangle\\&=\left\langle\rho f,g(v)-g(u_{n+1})\right\rangle,\ \forall v\in H,\end{aligned}$$

which is known as the proximal implicit proximal method.

Algorithm 3.13. *For a given* $u_0\in H$, *calculate the approximate solution* u_{n+1} *by the iterative scheme*

$$\left\langle \rho\left(e^{Lu_n}+e^{A(u_n)}\right)+E'\left(g\left(u_{n+1}\right)\right)-E'\left(g\left(u_n\right)\right), g(v)-g\left(u_{n+1}\right)\right\rangle$$
$$=\left\langle \rho f, g(v)-g\left(u_{n+1}\right)\right\rangle, \quad \forall v \in H,$$

which is an explicit method.

For $\zeta=\frac{1}{2}$, Algorithm 3.11 collapses to:

Algorithm 3.14. *For a given* $u_0 \in H$, *calculate the approximate solution by the iterative scheme*

$$\left\langle \rho\left(e^{L\left(\frac{u_{n+1}+u_n}{2}\right)}+e^{A\left(\frac{u_{n+1}+u_n}{2}\right)}\right)+E'\left(g\left(u_{n+1}\right)\right)-E'\left(g\left(u_n\right)\right), g(v)-g\left(u_{n+1}\right)\right\rangle$$
$$=\left\langle \rho f, g(v)-g\left(u_{n+1}\right)\right\rangle, \quad \forall v \in H, \tag{3.26}$$

which is known as the mid-point proximal method.

Remark 3.2. *We would like to emphasize that for appropriate choice of the operators* L, A, g *one can suggest and analyze several new iterative methods for solving system of exponentially general equations and related problems. The implementation and comparison with other techniques need further efforts.*

4. Conclusion

In this paper, we have considered a new class of system of exponentially general equations involving three operators. Several important problems such as system of absolute value equations, complementarity problems, Lax-Milgram Law and Reisz-Frechet representation theorem can be obtained as special cases. It is shown that the third order exponentially boundary value problems can be studied in the framework of general equations. We have used the auxiliary principle technique to study the existence of the unique solution of the system of general equations. Some new hybrid inertial iterative methods are suggested for solving the system of exponentially general equations using the auxiliary principle technique. The convergence analysis of these iterative methods is investigated under suitable conditions. This is a new approach for solving the system of exponentially general equations, see [48] [49] [50] [51]. We have only discussed the theoretical aspects of the proposed methods. The implementation and comparison with other numerical methods is the subject of the future research efforts. We would like to emphasize that the results obtained and discussed in this paper may motivate novel applications and extensions in these areas.

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Authors Contributions

All authors contributed equally such as conception of the topic, discussion, writing, typing etc.

Conflicts of Interest

Authors have no conflict of interest.

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