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A New Approach for Solving Fuzzy Linear Multi-Criterion Problems: An Approach Based on Minimization of the Errors Functions

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Abstract

The main purpose of this paper is to build a new approach for solving a fuzzy linear multi-criterion problem by defining a function called "error function". For this end, the concept of level set α is used to construct the error function. In addition, we introduce the concept of deviation variable in the definition of the error function. The algorithm of the new approach is summarized in three main steps: first, we transform the original fuzzy problem into a deterministic one by choosing a specific level α. second, we solve separately each uni-criteria problem and we compute the error function for each criteria. Finally, we minimize the sum of error functions in order to obtain the desired compromise solution. A numerical example is done for a comparative study with some existing approaches to show the effectiveness of the new approach.

Keywords

Deviation Variable, Compromise Solution, Membership Function, Error Function, Decision-Making Function

1. Introduction

The application of theory and methods of multi-criteria linear programming requires that the data determining a linear problem be real numbers known with certainty. However, sometimes we encounter situations that the data describe a real situation that is not known with certainty [1] [2]. This justifies the inefficient information or incomplete information that an investor can receive from the real world.

To reflect this vagueness due to the lack of data, the fuzzy set theory introduced by [3], will be used. The fuzzy set theory provides a framework in which vague concepts can be precisely and rigorously studied in order to take into account the imprecision of data. Some authors proposed approaches and algorithms for solving fuzzy linear multi criteria problems. We can cite [4]-[9]. The vagueness in a multi criteria problem consists of systematic substitution of fuzzy data for deterministic one into the multi criteria problem. In 2018, Y.T. Mangongo *et al.*, [1] used fuzzy concepts to propose a fuzzy multi criteria model to truck the portfolio selection problem. More recently, Y.T. Mangongo and J.D.B. Kampempe [5] proposed two approaches to bring a balance between effectiveness and efficiency while solving a multiobjective programming problem with fuzzy objective functions. In the first approach, they used the Nearest Interval Approximation Operator for fuzzy numbers to propose the deterministic counterpart of the original fuzzy problem. In their second approach, they used the Embedding Theorem for fuzzy numbers.

Following this way, in this paper, we introduce a new approach for solving a fuzzy linear multi-criteria problem based on the new function introduced in this paper, called "error function". The algorithme of our main contribution is summarized in three steps. First of all, we transform the fuzzy initial problem into a deterministic one, secondly, we define the error function for each criteria and finally we minimize the sum of error functions. This new approach improves the approaches of M. Sakawa and H. Yano [8], and B.J. Reardon [6]. A numerical example is done in order to see the effectiveness of the new approach compared with the solutions obtained by the approaches of M. Sakawa and H. Yano [8], and B.J. Reardon [6].

The paper is organized as follows: the first section will be devoted to preliminary concepts and it will be ended by providing the approaches of M. Sakawa and H. Yano [8] and B.J. Reardon [6]. In the second section, we will present the new approach. In the third section, a numerical example will be done in order to compare the three approaches. Finally, the paper will be ended with some concluding remarks.

2. Preliminaries

Definition 1 (3) *Let X be a non empty set, called "universe". A fuzzy set, denoted by $\tilde{A}$ is defined as follow:*

$$\tilde{A} = \left\{ \left(x, f_{\tilde{A}}(x) \right) : x \in X \right\},$$

where

$f_{\tilde{A}}$ is called the membership function of a fuzzy set $\tilde{A}$, defined on X ans takes values on the closed interval $[0,1]$.

A fuzzy measure f is a function defined on the power set $\mathcal{P}(X)$ into [0, 1], satisfying the following conditions [10] [11] [12]:

- Limits cases: $f(\varnothing)=0$ and $f(X)=1$
- Monotonicity: for all $A, B \in \mathcal{P}(X)$ such that $A \subseteq B$ then $f(A) \leq f(B)$
- Continuity: for any nested part $A_1 \subset A_2 \subset \cdots \subset A_n$ or $A_n \subset A_{(n-1)} \subset \cdots \subset A_1$

we have: $\lim_{n\to+\infty} f(A_n) = f(\lim_{n\to+\infty}(A_n))$.

Definition 2 (12) *A possibility measure is an application, Poss* $\equiv \Pi : P(X) \to [0,1]$ *such that*:

- $\Pi(\varnothing)=0$
- $\Pi(X)=1$
- $\forall A_1 \in P(X), \cdots, A_i \in P(X), \cdots$, then $\Pi\left(\bigcup_i A_i\right) = \sup_i \Pi(A_i)$.

One can show that the possibility measure satisfies the following properties:

- $\forall A \in P(X), B \in P(X)$, we have $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$.
- $\forall A \in P(X), B \in P(X)$, we have $\Pi(A \cap B) \leq \min(\Pi(A), \Pi(B))$.
- if $B \subset A$ then $\Pi(A) \geq \Pi(B)$.
- $\forall A \in P(X)$, $\max(\Pi(A), \Pi(A^c)) = 1$

Definition 3 (12)

A distribution of possibility is an application $\pi : X \to [0,1]$ such that $\sup_{x\in X} \pi(x) = 1$.

- A possibility measure verify: $\forall A \in P(X)$, $\Pi(A) = \sup_{x\in A} \pi(x)$
- One can obtain the possibility distribution from a possibility measure as follow:

$$\pi(x) = \Pi(\{x\}), \forall x \in X.$$

Definition 4 (3)

A necessity measure is an application $N : P(X) \to [0,1]$ such that:

- $N(\varnothing)=0$
- $N(X)=1$
- $\forall A_1 \in P(X), \cdots, A_i \in P(X), \cdots$, then $N\left(\bigcap_i A_i\right) = \inf_i N(A_i)$.

It can be shown that the necessity measure satisfies the following properties:

- $\forall A \in P(X), B \in P(X)$, we have $N(A \cap B) = \min(N(A), N(B))$.
- $\forall A \in P(X), B \in P(X)$, we have $N(A \cup B) \geq \max(N(A), N(B))$.
- if $B \subset A$ then $N(A) \geq N(B)$.
- $\forall A \in P(X)$ we have: $\max(N(A), N(A^c)) = 1$.

Also, for all $A \in P(X)$, the following relations hold:

- $N(A) = 1 - \Pi(A^c)$,
- $N(A) = \sup_{x\notin A}(1 - \pi(x))$,
- $N(A) \leq \Pi(A)$.

Definition 5 (12) *A fuzzy linear multi criteria problem is formulated as follows*:

$$\begin{cases} \text{"min"}(\tilde{c}_1 x, \tilde{c}_2 x, \cdots, \tilde{c}_k x) \\ \sum_{j=1}^{n} \tilde{a}_{ij} \odot x_j \leq \tilde{b}_i \\ x_j \geq 0; 1 \leq j \leq n; 1 \leq i \leq m \end{cases}; \quad (1)$$

where: $\tilde{c}_j$ is the $(1\times n)$ matrix; x is the $(n\times 1)$ matrix and $\tilde{c}_{rj}$, $\tilde{a}_{ij}$, $\tilde{b}_i$ the fuzzy intervals with the respective membership functions $\mu_{\tilde{c}_{rj}}$, $\mu_{\tilde{a}_{ij}}$, $\mu_{\tilde{b}_i}$, with $1 \le r \le k$, in addition $\odot$ expresses the multiplication of fuzzy numbers.

Bellow, we recall two methods (approaches), (Sakawa's method [8] and Reardon's method [6]).

The **Sakawa's method** [8] is very interesting because it involves fuzzy logic at all levels of the problem (on the parameters of criteria and on the parameters of the constraints). The set of solutions that will be found will also involve fuzzy logic. The solutions are given with a degree of belonging and have a correlation with the initial criteria; knowing that the correlation with the initial criteria is fixed by the investor.

The method starts by a fuzzy problem which will be transformed by a deterministic one by assigning an element of a level set α of the form:

$$\begin{cases} \text{"min"}\ f_1(x), \cdots, f_k(x) \\ g_i(x) \le 0 \\ x \ge 0 \end{cases} ; \tag{2}$$

where

$$f_i(x) = c_i x$$

and

$$g_i(x) = \sum_j a_{ij} x_j - b_i .$$

Under the constraints $g_i(x) \le 0$, one minimizes then maximizes, each criteria separately, so that one obtains an interval of variation of the value of criteria.

One defines the membership functions as follows:

$$\mu_i(x) = \frac{f_{i1} - f_i(x)}{f_{i1} - f_{i0}}; \text{ where:} \tag{3}$$

f_{i0} is the least interesting value of the criteria and f_{i1} is the most interesting value of the criteria.

One defines the decision-making functions DM_i as follows:

$$DM_i(x) = \begin{cases} 0, & \text{si } \mu_i(x) \le 0 \\ \mu_i(x), & \text{si } 0 < \mu_i(x) < 1 \\ 1, & \text{si } \mu_i(x) \ge 1 \end{cases} \text{ where:} \tag{4}$$

$DM_i(x) = 1$ when the objective is reached and $DM_i(x) = 0$ when the objective is not reached.

One maximizes the DM_i functions.

If there is no solution, or if the solution does not satisfy the investor, then one must change the membership functions, and return to step 3.

Stop and display the solution.

For further informations the readers can be referenced on the cited paper.

The **Reardon's method** [6] is a less complex method than Sakawa's method which involves also the fuzzy logic at each level.

The deterministic equivalent of the above Problem (1) is this below linear multi-criteria problem:

$$\begin{cases} \min \begin{cases} f_1(x) \\ \cdots \\ f_k(x) \end{cases} \\ g_1(x), \cdots, g_m(x) \le 0 \\ x \in \mathbb{R}^n \\ \left(f_1, \cdots, f_k(x)\right) \in \mathbb{R}^k \\ \left(g_1(x), \cdots, g_m(x)\right) \in \mathbb{R}^m \\ x \ge 0 \end{cases} . \tag{5}$$

For each criterion, one should define a membership function called the *Reardon's membership function*. The representation of Reardon's membership function and more details about it can be found in [6]. Here are some significations of the parameters of Reardon's membership function:

S_{max}: fuzzy scale factor max.

S_{min}: fuzzy scale factor min.

$f_{i\min}$: minimum value of the i^{th} criteria.

$f_{i\max}$: maximum value of the i^{th} criteria.

O_i: experimental value of the i^{th} criteria. We wanted that $f_i = O_i$.

E_i: a margin of acceptable error (the criteria O_i is defined with error E_i).

Reardon define the expressions of his membership functions as follows:

- If $f_i \le (O_i - E_i)$ then:

$$f'(f_i) = \left[\frac{S_{\max}}{f_{i\min} - (O_i - E_i)}\right] \cdot \left(f_i - (O_i - E_i)\right). \tag{6}$$

- If $(O_i - E_i) \le f_i \le (O_i + E_i)$ then:

$$f'(f_i) = 0. \tag{7}$$

- If $(O_i + E_i) \le f_i$ then:

$$f'(f_i) = \left[\frac{S_{\min}}{(O_i + E_i) - f_{i\max}}\right] \cdot \left(f_i - (O_i + E_i)\right). \tag{8}$$

Comparing with the classical methods of fuzzy optimisation, in this algorithm the level is in inverse. Instead of having 1 for the value of membership function when f_i is belongs to the set of desired values (that means in the interval $[O_i - E_i, O_i + E_i]$), the value of membership function is null. This inversion of the degree of belonging allows us to obtain very simple global criteria which aggregate all criterion.

$$F = \frac{1}{k} \sum_{i=1}^{k} f'(f_i) \tag{9}$$

As one can see, F is the mean of the Reardon's membership functions of criterion. All these Reardon's membership's functions has been normalized by an intermediate of $S_{\min}$ and of $S_{\max}$.

3. New Approach Based on Minimization of Errors Functions

In this section, we present the new approach. The approach which involves the systematically definition of *the errors functions*, where their sum will be minimized in order to obtain the compromise solution of the initial fuzzy problem.

We need to transform this non deterministic problem (1) into a deterministic one by defining the level set α; see [8]. For an exigent investor, we fix the level α very closer to 1 and obtain the following deterministic problem:

$$\begin{cases} \text{"max"}\left(c_1 x, c_2 x, \cdots, c_k x\right) \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \\ x_j \ge 0; 1 \le j \le n; 1 \le i \le m \end{cases}. \tag{10}$$

Then, we solve each problem separately with one criterion and we compute f_r^α; which is the value of a criterion to a level α. If one problem does not have a solution; we should revise the value of α. After that, we compute for each criterion the error function defined below.

Definition 6 (error function) *We define an "error function" by*:

$$u_r(x) = \frac{f_r^\alpha - f_r(x) + E_r}{f_r^\alpha + \varepsilon}; \tag{11}$$

where ε is an arbitrary positive constant, $f_r(x) = c_r x$ (criterion), E_r the deviation variable of a given criteria with the denominator $f_r^\alpha + \varepsilon \ne 0$ for $1 \le r \le k$. The presence of ε ensure that, the denominator can not be zero.

Finally, we minimize the sum of all these errors functions on the criterion under the constraints below:

$$\begin{cases} 0 \le u_r(x) \le 1 \\ f_r^\alpha - E_r \le f_r(x) \le f_r^\alpha + E_r \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \\ x_j \ge 0; 1 \le j \le n; 1 \le i \le m \end{cases}.$$

Thus, the founded solution of the above problem is the compromise solution of the Problem (1). The algorithm of this new approach can be summarized as follows:

- The fuzzy Problem (1) is transformed into the deterministic one, Problem (10):
- For $r = 1, \cdots, k$,

1) Solve the problem:

$$\begin{cases} \max c_r x \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \\ x_j \ge 0; 1 \le j \le n; 1 \le i \le m \end{cases} \tag{12}$$

and obtain f_r^α the value of criteria $f_r(x)$ at a given level α. If no solution is obtained, then go back to 1.

2) Compute for each criteria, the error function:

$$u_r(x) = \frac{f_r^\alpha - f_r(x) + E_r}{f_r^\alpha + \varepsilon}.$$

• Minimize the sum of all errors functions:

$$\begin{cases} \min \sum_{r=1}^{k} u_r(x) \\ \begin{cases} 0 \le u_r(x) \le 1 \\ f_r^\alpha - E_r \le f_r(x) \le f_r^\alpha + E_r \\ \sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \\ x_j \ge 0; E_i \ge 0; 1 \le j \le n; 1 \le i \le m \end{cases} \end{cases} \tag{13}$$

Proposition 1 *The set of solutions of the above problem, Problem* (13) *is*:

$$D = \left\{ x \in \mathbb{R}^n : 0 \le u_r(x) \le 1;\ f_r^\alpha - E_r \le f_r(x) \le f_r^\alpha + E_r;\ \sum_{j=1}^{n} a_{ij} \cdot x_j \le b_i \text{ and } x \ge 0 \right\} \ne \varnothing.$$

Proof

The set D is non empty by the fact that there exists always an $\alpha \in [0,1]$ such that $f_r^\alpha \in \mathbb{R}$, $\forall r \in \{1, \cdots, k\}$. Which ensure the existence of solution. Thus $D \ne \varnothing$ ■

The complexity of this new approach depends on the number of criterion that the problem contains. By the definition of error functions, the values of criterion are adjusted by taking into account of criterion with their associated constraints simultaneously.

Below, we do one numerical example to show the effectiveness of our new approach, comparing with some existing approaches.

4. Numerical Example

In order to see the effectiveness of this new approach, a numerical example will be done and the solution will be compared with the solutions obtained by the Sakawa and Reardon approaches.

4.1. Problem's Presentation

Let us consider the following problem problem which contains two decisions variables $x = (x_1, x_2)$; three criterion $\tilde{f}_1(x) = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2$, $\tilde{f}_2(x) = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2$ and $\tilde{f}_3(x) = 6x_1 + \tilde{c}_{23}x_2$; two constraints $\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \le \tilde{b}_1$ and $\tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \le \tilde{b}_2$. The membership functions of each fuzzy set are defined as below:

$$\mu_{\tilde{c}_{11}} = \begin{cases} 0, & x \le 5 \\ \frac{x^2 - 25}{24}, & 5 < x \le 7 \\ 1, & 7 < x \le 9 \\ \frac{100 - x^2}{19}, & 9 < x < 10 \\ 0, & 10 \le x \end{cases}; \quad \mu_{\tilde{c}_{12}} = \begin{cases} 0, & x \le 2 \\ \frac{x - 2}{5}, & 2 < x \le 7 \\ 1, & 7 < x \le 12 \\ \frac{196 - x^2}{52}, & 12 < x < 14 \\ 0, & 14 \le x \end{cases};$$

$$\mu_{\tilde{c}_{21}} = \begin{cases} 0, & x \le 14 \\ \dfrac{x^2-196}{188}, & 14 < x \le 18 \\ 1, & 18 < x \le 22 \\ \dfrac{576-x^2}{92}, & 22 < x < 24 \\ 0, & 24 \le x \end{cases}; \quad \mu_{\tilde{c}_{22}} = \begin{cases} 0, & x \le 30 \\ \dfrac{x^2-900}{325}, & 30 < x \le 35 \\ 1, & 35 < x \le 40 \\ \dfrac{2025-x^2}{425}, & 40 < x < 45 \\ 0, & 45 \le x \end{cases};$$

$$\mu_{\tilde{a}_{11}} = \begin{cases} 0, & x \le 0 \\ 2x, & 0 < x \le 0.5 \\ 1, & 0.5 < x \le 2 \\ \dfrac{25-5x}{15}, & 2 < x < 5 \\ 0, & 5 \le x \end{cases}; \quad \mu_{\tilde{a}_{12}} = \begin{cases} 0, & x \le 0 \\ 5x, & 0 < x \le 0.2 \\ 1, & 0.2 < x \le 1 \\ \dfrac{9-3x}{6}, & 1 < x < 3 \\ 0, & 3 \le x \end{cases};$$

$$\mu_{\tilde{a}_{21}} = \begin{cases} 0, & x \le 0 \\ 4x, & 0 < x \le 0.25 \\ 1, & 0.25 < x \le 5 \\ \dfrac{64-8x}{24}, & 5 < x < 8 \\ 0, & 8 \le x \end{cases}; \quad \mu_{\tilde{a}_{22}} = \begin{cases} 0, & x \le 5 \\ \dfrac{2x-10}{2}, & 5 < x \le 6 \\ 1, & 6 < x \le 7 \\ \dfrac{64-8x}{8}, & 7 < x < 8 \\ 0, & 8 \le x \end{cases};$$

$$\mu_{\tilde{b}_1} = \begin{cases} 0, & x \le 12 \\ \dfrac{x-12}{3}, & 12 < x \le 15 \\ 1, & 15 < x \le 18 \\ \dfrac{42-2x}{6}, & 18 < x < 21 \\ 0, & 21 \le x \end{cases}; \quad \mu_{\tilde{b}_2} = \begin{cases} 0, & x \le 56 \\ \dfrac{2x-112}{24}, & 56 < x \le 68 \\ 1, & 68 < x \le 74 \\ \dfrac{258-3x}{36}, & 74 < x < 86 \\ 0, & 86 \le x \end{cases};$$

$$\mu_{\tilde{c}_{23}} = \begin{cases} 0, & x \le 5 \\ x-5, & 5 < x \le 6 \\ -x+7, & 6 < x < 7 \\ 0, & 7 \le x \end{cases}.$$

The problem described above can be formulated as follow:

$$\begin{cases} \text{Max } \tilde{f}(x) \equiv \text{"max"} \begin{cases} \tilde{f}_1(x) = \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 \\ \tilde{f}_2(x) = \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 \\ \tilde{f}_3(x) = 6x_1 + \tilde{c}_{23}x_2 \end{cases} \\ \tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 \le \tilde{b}_1 \\ \tilde{a}_{21}x_1 + \tilde{a}_{22}x_2 \le \tilde{b}_2 \\ x \ge 0 \end{cases} \tag{14}$$

4.2. Resolution by Sakawa's Approach

4.2.1. Deterministic Equivalent Problem

By defining the level set α for $\alpha = 0.8$, we obtain the following deterministic problem:

$$\begin{cases} \text{Max } f(x) = \text{"max"} \begin{cases} f_1(x) = 9.2x_1 + 12.43x_2 \\ f_2(x) = 22.4x_1 + 41.1x_2 \\ f_3(x) = 6x_1 + 6.2x_2 \end{cases} \\ 2.6x_1 + 1.4x_2 \leq 18.6 \\ 5.6x_1 + 7.2x_2 \leq 76.4 \\ x \geq 0 \end{cases} . \tag{15}$$

By minimizing each criteria separately and maximizing them on the same way, we obtain than:

$$0 \leq f_1(x) \leq 131.9$$

$$0 \leq f_2(x) \leq 436.12$$

$$0 \leq f_3(x) \leq 68.71 .$$

4.2.2. Definition of Criteria's Memberships Functions

We should first investigate for the less interesting values and the more interesting values for each criteria. So for this, we need to define the deterministic problems as below respectively for $\alpha = 0$ and $\alpha = 1$:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 10x_1 + 14x_2 \\ f_2(x) = 24x_1 + 45x_2 \\ f_3(x) = 6x_1 + 7x_2 \end{cases} \\ 5x_1 + 1.3x_2 \leq 21 \\ 8x_1 + 8x_2 \leq 86 \\ x \geq 0 \end{cases} \tag{16}$$

and

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9x_1 + 12x_2 \\ f_2(x) = 22x_1 + 40x_2 \\ f_3(x) = 6x_1 + 6x_2 \end{cases} \\ 2x_1 + 1x_2 \leq 18 \\ 5x_1 + 7x_2 \leq 74 \\ x \geq 0 \end{cases} \tag{17}$$

Hence by solving these problems, we found: $f_{10} = 98$, $f_{20} = 341$ and $f_{30} = 49$ which are the less interesting values of criterion. And $f_{11} = 129.33$, $f_{21} = 422.86$ and $f_{31} = 73.33$ which are the more interesting value of criterion.

Now we define the membership functions as follow:

$$\mu_1(x) = \frac{129.33 - 9.2x_1 - 12.43x_2}{129.33 - 98} = 4.13 - 0.2936x_1 - 0.3967x_2 .$$

$$\mu_2(x) = \frac{422.86 - 22.4x_1 - 41.1x_2}{422.86 - 314} = 3.8844 - 0.2058x_1 - 0.377x_2 .$$

$$\mu_3(x) = \frac{73.33 - 6x_1 - 6.2x_2}{73.33 - 49} = 3.014 - 0.2466x_1 - 0.2548x_2 .$$

4.2.3. Definition of Decision Making Functions DM_i

For each criteria, we have these decision making functions:

$$DM_1(x)=\begin{cases}0, & \text{if } 4.13-0.2936x_1-0.3967x_2\le 0\\ 4.13-0.2936x_1-0.3967x_2, & \text{if } 0<4.13-0.2936x_1-0.3967x_2<1\\ 1, & \text{if } 4.13-0.2936x_1-0.3967x_2\ge 1\end{cases}$$

$$DM_2(x)=\begin{cases}0, & \text{if } 3.8844-0.2058x_1-0.377x_2\le 0\\ 3.8844-0.2058x_1-0.377x_2, & \text{if } 0<3.8844-0.2058x_1-0.377x_2<1\\ 1, & \text{if } 3.8844-0.2058x_1-0.377x_2\ge 1\end{cases}$$

$$DM_3(x)=\begin{cases}0, & \text{if } 3.014-0.2466x_1-0.2548x_2\le 0\\ 3.014-0.2466x_1-0.2548x_2, & \text{if } 0<3.014-0.2466x_1-0.2548x_2<1\\ 1, & \text{if } 3.014-0.2466x_1-0.2548x_2\ge 1\end{cases}$$

4.2.4. Maximisation of Functions DM_i

We solve the problem below:

$$\begin{cases}\text{"max" } \mu_D(\mu_1,\mu_2,\mu_3)\\ 0<\mu_1<1\\ 0<\mu_2<1\\ 0<\mu_3<1\\ 2.6x_1+1.4x_2\le 18.6\\ 5.6x_1+7.2x_2\le 76.4\\ x_1,x_2\ge 0\end{cases}. \tag{18}$$

By the use of aggregation function $\min(\mu_1,\mu_2,\mu_3)$, the Problem (18) becomes:

$$\begin{cases}\text{"max" } \min(\mu_1,\mu_2,\mu_3)\\ 0<\mu_1<1\\ 0<\mu_2<1\\ 0<\mu_3<1\\ 2.6x_1+1.4x_2\le 18.6\\ 5.6x_1+7.2x_2\le 76.4\\ x_1,x_2\ge 0\end{cases}. \tag{19}$$

Applying the techniques for solving the max-min problem, the Problem (19) can be transformed into:

$$\begin{cases}\max \lambda\\ \lambda\le\mu_1\\ \lambda\le\mu_2\\ \lambda\le\mu_3\\ 0<\mu_1<1\\ 0<\mu_2<1\\ 0<\mu_3<1\\ 2.6x_1+1.4x_2\le 18.6\\ 5.6x_1+7.2x_2\le 76.4\\ 0\le\lambda\le 1\\ x_1,x_2\ge 0\end{cases}. \tag{20}$$

Replacing each criteria by the Sakawa's membership functions, we have the following Problem:

$$\begin{cases} \max \lambda \\ \lambda \le 4.13 - 0.2936x_1 - 0.3967x_2 \\ \lambda \le 3.8844 - 0.2058x_1 - 0.377x_2 \\ \lambda \le 3.014 - 0.2466x_1 - 0.2548x_2 \\ 0 < 4.13 - 0.2936x_1 - 0.3967x_2 < 1 \\ 0 < 3.8844 - 0.2058x_1 - 0.377x_2 < 1 \\ 0 < 3.014 - 0.2466x_1 - 0.2548x_2 < 1 \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ 0 \le \lambda \le 1 \\ x_1, x_2 \ge 0 \end{cases} \quad (21)$$

After solving the Problem (21), we get the following solution:

$$\lambda = \frac{5165551}{536870}; x_1 = \frac{28063}{39281} \approx 0.7144 \text{ and } x_2 = \frac{569735}{77396} \approx 7.3613.$$

In this case, the compromise solution is $(x_1, x_2) = (0.7144; 7.3613)$ with

$$f_1 \approx 98.0734;\ f_2 \approx 318.552 \text{ and } f_3 \approx 49.9265.$$

In the case where the solution does not meet the investor's interest, we should change the membership functions otherwise the founded solution is the compromise solution of Problem (14).

4.3. Resolution by Reardon's Approach

4.3.1. Deterministic Equivalent Problem

For $\alpha = 0.8$; the deterministic equivalent problem of Problem (14) is given below:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9.2x_1 + 12.43x_2 \\ f_2(x) = 22.4x_1 + 41.1x_2 \\ f_3(x) = 6x_1 + 6.2x_2 \end{cases} \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0 \end{cases} . \quad (22)$$

We set that the fuzzy scale factors max and min are given respectively by: $S_{\max} = 0.8$ and $S_{\min} = 0.5$. The minimal and maximal values of criterion are given by:

$$f_{1\min} = 0,\ f_{1\max} = 131.9;$$
$$f_{2\min} = 0,\ f_{2\max} = 436.12;$$
$$f_{3\min} = 0,\ f_{3\max} = 68.71.$$

After that, we define the experimental values of each criteria:

$$O_1 = 131.9;\ O_2 = 436.12 \text{ and } O_3 = 68.71.$$

We define also the margin errors for each experimental value of criterion:

$$E_1 = 2; E_2 = 5; E_3 = 5.$$

4.3.2. Definition of Reardon's Membership Functions

By defining these, we want that:

$f_1(x) \le (131.9 - 2)$; $f_2(x) \le (436.12 - 5)$ and $(68.71 - 5) \le f_3(x) \le (68.71 + 5)$.

We have then

$$f_1(x) \le 129.9;\ f_2(x) \le 431.12 \text{ and } 63.71 \le f_3(x) \le 73.71.$$

We have now the following Reardon's membership functions:

- $f'(f_1) = \dfrac{0.8}{-129.9}(9.2x_1 + 12.43x_2 - 129.9) = -0.057x_1 - 0.0771x_2 + 0.8$;
- $f'(f_2) = \dfrac{0.8}{-431.12}(22.4x_1 + 41.1x_2 - 431.12) = -0.0416x_1 - 0.0781x_2 + 0.8$;
- $f'(f_3) = 0$.

Hence, the mean of these Reardon's membership functions is given by:

$$F = \frac{1}{3}\sum_{i=1}^{3} f'(f_i) = -0.0329x_1 - 0.05173x_2 + 0.5333.$$

4.3.3. Minimization of F

The minimization problem of F is given by:

$$\begin{cases} \min \beta \\ \beta + 0.0329x_1 + 0.05173x_2 = 0.5333 \\ 0.057x_1 + 0.0771x_2 \le 0.8 \\ 0.057x_1 + 0.0771x_2 \ge -0.2 \\ 0.0416x_1 + 0.0781x_2 \le 0.8 \\ 0.0416x_1 + 0.0781x_2 \ge 0.8 \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ 9.2x_1 + 12.43x_2 \le 129.9 \\ 22.4x_1 + 41.1x_2 \le 431.12 \\ 6x_1 + 6.2x_2 \ge 63.71 \\ 6x_1 + 6.2x_2 \le 73.71 \\ x_1, x_2, \beta \ge 0 \end{cases} \tag{23}$$

After solving the above problem, we get the solution below:

$$x_1 = \frac{114273}{176630} \approx 0.647;\ x_2 = \frac{9583540}{968247} \approx 9.898$$

with

$$f_1 \approx 128.98;\ f_2 \approx 421.3006;\ f_3 \approx 65.25.$$

Thus the founded solution is the compromise solution of Problem (14).

4.4. Resolution by the New Approach Based on Errors Functions

4.4.1. Deterministic Equivalent Problem

For this approach, we will solve the problem with two different degrees of possibilities, $\alpha = 0.8$ and $\alpha = 1$.

- For $\alpha = 0.8$, we have this below deterministic equivalent problem:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9.2x_1 + 12.43x_2 \\ f_2(x) = 22.4x_1 + 41.1x_2 \\ f_3(x) = 6x_1 + 6.2x_2 \end{cases} \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0 \end{cases} \tag{24}$$

- For $\alpha = 1$, we have this below deterministic equivalent problem:

$$\begin{cases} \text{Max } f(x) \equiv \text{"max"} \begin{cases} f_1(x) = 9x_1 + 12x_2 \\ f_2(x) = 22x_1 + 40x_2 \\ f_3(x) = 6x_1 + 6x_2 \end{cases} \\ 2x_1 + 1x_2 \le 18 \\ 5x_1 + 7x_2 \le 74 \\ x \ge 0 \end{cases} \tag{25}$$

4.4.2. Determination of f_r^α

- For $\alpha = 0.8$, we solve the problems below:

$$\begin{cases} \max f_r(x) \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x \ge 0; \end{cases} \quad r = \{1,2,3\}.$$

After solving these problems, we obtain:

$f_1^{0.8} = \frac{237413}{1800} \approx 131.9$, $f_2^{0.8} = \frac{26167}{60} \approx 436.12$ and $f_3^{0.8} = \frac{46721}{680} \approx 68.71$.

With $\left(0, \frac{191}{18}\right)$ solution of problem for $r = 1$ and $r = 2$, and $\left(\frac{337}{136}, \frac{1181}{136}\right)$ solution of problem for $r = 3$.

- For $\alpha = 1$, we solve the problems below:

$$\begin{cases} \max f_r(x) \\ 2x_1 + 1x_2 \le 18 \\ 5x_1 + 7x_2 \le 74 \\ x \ge 0; \end{cases} \quad r = \{1,2,3\}.$$

After solving these problems, we obtain:

$$f_1^1 = \frac{388}{3} \approx 129.33,\quad f_2^1 = \frac{2960}{7} \approx 422.86 \quad \text{and}$$

$$f_3^1 = \frac{220}{3} \approx 73.33.$$

With $\left(\frac{52}{9}, \frac{58}{9}\right)$ solution of problem for $r = 1$ and $r = 3$, and $\left(0, \frac{74}{7}\right)$ solution of problem for $r = 2$.

4.4.3. Definition of Errors Functions $u_r(x)$ and Minimisation of Their Sum

We do this for two different values of α. In these two cases, we assume that $\varepsilon = 1$.

- For $\alpha = 0.8$, we have:

$$\begin{aligned} u_1(x) &= \frac{131.9 - 9.2x_1 - 12.43x_2 + E_1}{132.9} \\ &= 0.9925 - 0.0692x_1 - 0.0935x_2 + 0.0075E_1 \\ u_2(x) &= \frac{436.12 - 22.4x_1 - 41.1x_2 + E_2}{437.12} \\ &= 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2 \\ u_3(x) &= \frac{68.71 - 6x_1 - 6.2x_2 + E_3}{69.71} \\ &= 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_3 \end{aligned}.$$

The sum gives:

$$\sum_{r=1}^{3} u_r(x) = 2.9759 - 0.2065x_1 - 0.2764x_2 + 0.0075E_1 + 0.0023E_2 + 0.0143E_3.$$

Hence, we have this problem which minimize the sum of $u_r(x)$:

$$\begin{cases} \text{"min"}\ \beta \\ \beta + 0.2065x_1 + 0.2764x_2 - 0.0075E_1 - 0.0023E_2 - 0.0143E_3 = 2.9759 \\ 0.9925 - 0.0692x_1 - 0.0935x_2 + 0.0075E_1 \ge 0 \\ 0.9925 - 0.0692x_1 - 0.0935x_2 + 0.0075E_1 \le 1 \\ 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2 \ge 0 \\ 0.9977 - 0.0512x_1 - 0.0940x_2 + 0.0023E_2 \le 1 \\ 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_2 \ge 0 \\ 0.9857 - 0.0861x_1 - 0.0889x_2 + 0.0143E_2 \le 1 \\ 9.2x_1 + 12.43x_2 + E_1 \ge 131.9 \\ 9.2x_1 + 12.43x_2 - E_1 \le 131.9 \\ 22.4x_1 + 41.1x_2 + E_2 \ge 436.12 \\ 22.4x_1 + 41.1x_2 - E_2 \le 436.12 \\ 6x_1 + 6.2x_2 + E_3 \ge 68.71 \\ 6x_1 + 6.2x_2 - E_3 \le 68.71 \\ 2.6x_1 + 1.4x_2 \le 18.6 \\ 5.6x_1 + 7.2x_2 \le 76.4 \\ x_1, x_2 \ge 0 \end{cases} \tag{26}$$

After solving this problem, we obtain the following solution:

$$(\beta, x_1, x_2, E_1, E_2, E_3) = \left(\frac{455239}{5368700}; 0; \frac{71020}{6693}; \frac{67133}{13985800}; \frac{1}{391}; \frac{2629}{900}\right).$$

Thus the compromise solution of the Problem (14) is given by:

$(x_1, x_2) = \left(0; \frac{71020}{6693}\right)$ with

$$f_1 \approx 131.896;\ f_2 \approx 436.11;\ f_3 \approx 65.79.$$

- For $\alpha = 1$, we have:

$$\begin{aligned} u_1(x) &= \frac{129.33 - 9x_1 - 12x_2 + E_1}{130.33} \\ &= 0.9923 - 0.0691x_1 - 0.0921x_2 + 0.0077E_1; \\ u_2(x) &= \frac{422.86 - 22x_1 - 40x_2 + E_2}{423.86} \\ &= 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2; \\ u_3(x) &= \frac{73.33 - 6x_1 - 6x_2 + E_3}{74.33} \\ &= 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_3. \end{aligned}$$

The sum gives:

$$\sum_{r=1}^{3} u_r(x) = 2.9765 - 0.2017x_1 - 0.2672x_2 + 0.0077E_1 + 0.0024E_2 + 0.0135E_3.$$

Hence, we have this Problem which minimize the sum of $u_r(x)$:

$$\begin{cases} \text{"min" } \beta \\ \beta + 0.2017x_1 + 0.2672x_2 - 0.0077E_1 - 0.0024E_2 - 0.0135E_3 = 2.9765 \\ 0.9923 - 0.0691x_1 - 0.0921x_2 + 0.0077E_1 \ge 0 \\ 0.9923 - 0.0691x_1 - 0.0921x_2 + 0.0077E_1 \le 1 \\ 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2 \ge 0 \\ 0.9976 - 0.0519x_1 - 0.0944x_2 + 0.0024E_2 \le 1 \\ 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_2 \ge 0 \\ 0.9866 - 0.0807x_1 - 0.0807x_2 + 0.0135E_2 \le 1 \\ 9x_1 + 12x_2 + E_1 \ge 129.33 \\ 9x_1 + 12x_2 - E_1 \le 129.33 \\ 22x_1 + 40x_2 + E_2 \ge 422.86 \\ 22x_1 + 40x_2 - E_2 \le 422.86 \\ 6x_1 + 6x_2 + E_3 \ge 73.33 \\ 6x_1 + 6x_2 - E_3 \le 73.33 \\ 2x_1 + x_2 \le 18 \\ 5x_1 + 7x_2 \le 74 \\ x_1, x_2 \ge 0 \end{cases} \tag{27}$$

After solving this problem, we obtain the following solution:

$$\begin{aligned} &(\beta, x_1, x_2, E_1, E_2, E_3) \\ &= \left(\frac{194607}{1073740}; \frac{2115190}{370977}; \frac{1871510}{287977}; \frac{60881}{2079710}; \frac{8495170}{228277}; \frac{368073}{2899100}\right). \end{aligned}$$

Hence the compromise solution of Problem (14) is given by:

$$(x_1, x_2) = \left(\frac{2115190}{370977}; \frac{1871510}{287977}\right) \text{ with}$$

$$f_1 \approx 129.318;\ f_2 \approx 385.44;\ f_3 \approx 73.212.$$

By this numerical example, we see clearly that the Reardon's approach is easily applied and provides an optimal solution by comparing with Sakawa's approach.

The Reardon's approach is less complex than Sakawa approach of the limited number of steps for reaching the solution. The common point is that all solutions provide by these two approaches depend on the level α chosen by the investor.

We can see clearly also that the solution obtained by the new approach is more effective (improved) than the solution obtained by Reardon approach without invoking the one obtained by Sakawa approach. As Sakawa and Reardon approaches, the solution of this new approach depends also on the choice of a level α by the investor. In addition, this new approach is less complex than others.

5. Concluding Remarks

In this article, we analyzed Sakawa and Reardon approaches in order to solve a fuzzy multi criterion linear problem. This analysis leads us to a new approach more effective in terms of solutions obtained compared with Sakawa and Reardon approaches.

Indeed, in the Sakawa and Reardon approaches, one defines the membership functions for each criterion. Sakawa's approach wants that the defined membership functions being maximal inversely to the Reardon's approach. But, in the new approach we define systematically the *error functions* in which their sum should be minimized on the sequel of the method for finding the desired compromise solution. And we see that the solution obtained by this new approach improves the one obtained by the Sakawa and Reardon approaches.

As Sakawa and Reardon approach, this new approach takes into account of fuzzy data in the formulation of the problem inside of constraints and criterion. The algorithm of this new approach is summarized in three important steps: first of all, we transform the fuzzy problem into its deterministic equivalent using level set α of a fuzzy number. Secondly, after maximization of each criterion separately, we compute the error functions on each criterion. Finally, we minimize the sum of error functions for obtaining the compromise solution.

The Sakawa approach gives a no satisfactory solution (less effective) compared with solutions from others approaches studied in this paper. The new approach gives the best solution (more effective) than the solutions from the previous one. The new approach becomes more complex when we have many criterion (at least 4).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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