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Picture Fuzzy Relations over Picture Fuzzy Sets

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Abstract

Nowadays, picture fuzzy set theory is a flourishing field in mathematics with uncertainty by incorporating the concept of positive, negative and neutral membership degrees of an object. A traditional crisp relation represents the satisfaction or the dissatisfaction of relationship, connection or correspondence between the objects of two or more sets. However, there are some problems that can't be solved through classical relationships, such as the relationship between two objects being vague. In those situations, picture fuzzy relation over picture fuzzy sets is an important and powerful concept which is suitable for describing correspondences between two vague objects. It represents the strength of association of the elements of picture fuzzy sets. It plays an important role in picture fuzzy modeling, inference and control system and also has important applications in relational databases, approximate reasoning, preference modeling, medical diagnosis, etc. In this article, we define picture fuzzy relations over picture fuzzy sets, including some other fundamental definitions with illustrations. The max-min and min-max compositions of picture fuzzy relations are defined in the light of picture fuzzy sets and discussed some properties related to them. The reflexivity, symmetry and transitivity of a picture fuzzy relation are described over a picture fuzzy set. Finally, various properties are explored related to the picture fuzzy relations over a picture fuzzy set.

Keywords

Picture Fuzzy Sets, Picture Fuzzy Relations, Picture Fuzzy Binary Relations, Composition of Picture Fuzzy Relations

1. Introduction

In our daily life, we face some problems where uncertainty arises that cannot be solved by classical set theory. To deal with those problems, L. A. Zadeh [1] introduced the concept of fuzzy set theory and then Atanassov [2] developed the intuitionistic fuzzy set (IFS) which was the extension of fuzzy set. However, recently many researchers keep their concentration on picture fuzzy set [3] by incorporating the concept of positive, negative and neutral membership degrees of an element which is the extension of an intuitionistic fuzzy set. After the development of picture fuzzy set, it has been considered a strong mathematical tool which is adequate in situations when human opinions involved more answers of the types yes, abstain, no and refusal.

Fuzzy relation was initially introduced by Zadeh [4] and then by Kaufmann [5]. Also, it has been studied by a number of authors, such as Klir and Yaun [6] and Zimmerman [7]. Then some scholars have used it widely in many fields, such as decision making, fuzzy reasoning, fuzzy control, medical diagnosis, clustering analysis [8] [9] [10] [11], fuzzy comprehensive evaluation [12] [13] [14]. Burillo and Bustince gave the definition of intuitionistic fuzzy relations [15] [16] and discussed some properties of them. In 2005, Lei *et al.* [17] further researched intuitionistic fuzzy relations and composition operation of intuitionistic fuzzy relations. Yang proposed the definition of kernels and closures of intuitionistic fuzzy relations and proved fourteen theorems of intuitionistic fuzzy relations [18]. B. C. Cuong [3] [19] proposed the notion of picture fuzzy relations and studied some related properties.

In this article, picture fuzzy relation over picture fuzzy set is defined. Some operations on this picture fuzzy relation are also discussed with examples. Numerous properties are explored related to picture fuzzy relation over picture fuzzy set.

This article is organized as follows: In section 2, some basic definitions and properties are described which are essential to the rest of the paper. In section 3, some structural properties of picture fuzzy relation over picture fuzzy sets are illustrated. In section 4, the compositional relations of picture fuzzy sets over picture fuzzy sets are defined and describe some related properties of them. In section 5, some properties of picture fuzzy relations in picture fuzzy sets are explained. Finally, concluding remarks are given.

2. Preliminaries

In this section, we recall some basic definitions which are used in later sections.

Definition 2.1. [1]. Let X be non-empty set. A fuzzy set A in X is given by

$$A=\{(x,\mu_A(x)):x\in X\},$$

where $\mu_A:X\to[0,1]$.

Definition 2.2. [2]. An intuitionistic fuzzy set A in X is given by

$$A=\{(x,\mu_A(x),\nu_A(x)):x\in X\},$$

where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$, with the condition

$$0\le\mu_A(x)+\nu_A(x)\le 1;\forall x\in X.$$

The values $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and non-membership degree respectively of the element *x* to the set *A*.

For any intuitionistic fuzzy set *A* on the universal set *X*, for $x \in X$

$$\pi_A(x) = 1 - (\mu_A(x) + \nu_A(x))$$

is called the hesitancy degree (or intuitionistic fuzzy index) of an element *x* in *A*. It is the degree of indeterminacy membership of the element *x* whether belonging to *A* or not.

Obviously, $0 \le \pi_A(x) \le 1$ for any $x \in X$.

Particularly, $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is always valid for any fuzzy set *A* on the universal set *X*.

The set of all intuitionistic fuzzy sets in *X* will be denoted by $IFS(X)$.

Definition 2.3. [3] [19]. A picture fuzzy set *A* on a universe of discourse *X* is of the form

$$A = \{(x, \mu_A(x), \eta_A(x), \nu_A(x)) : x \in X\},$$

where $\mu_A(x) \in [0,1]$ is called the degree of positive membership of *x* in *A*, $\eta_A(x) \in [0,1]$ is called the degree of neutral membership of *x* in *A* and $\nu_A(x) \in [0,1]$ is called the degree of negative membership of *x* in *A*, and where $\mu_A(x), \eta_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 1; \forall x \in X.$$

Here $1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)); \forall x \in X$ is called the degree of refusal membership of *x* in *A*.

The set of all picture fuzzy sets in *X* will be denoted by $PFS(X)$.

Definition 2.4. [3] [19]. Let $A, B \in PFS(X)$, then the subset, equality, the union, intersection and complement are defined as follows:

1) $A \subseteq B$ iff $\forall x \in X, \mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x)$ and $\nu_A(x) \ge \nu_B(x)$
2) $A = B$ iff $\forall x \in X, \mu_A(x) = \mu_B(x), \eta_A(x) = \eta_B(x)$ and $\nu_A(x) = \nu_B(x)$
3) $$A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \min(\nu_A(x), \nu_B(x))) : x \in X\}$$
4) $$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \max(\nu_A(x), \nu_B(x))) : x \in X\}$$
5) $A^c = \{(x, \nu_A(x), \eta_A(x), \mu_A(x)) : x \in X\}$

Definition 2.5. [19]. Let *X*, *Y* and *Z* be ordinary non-empty sets. A picture fuzzy relation *R* is a picture fuzzy subset of *X* × *Y* *i.e.* *R* given by

$$R = \{((x,y), \mu_R(x,y), \eta_R(x,y), \nu_R(x,y)) : x \in X, y \in Y\}$$

where $\mu_R : X \times Y \to [0,1]$, $\eta_R : X \times Y \to [0,1]$, $\nu_R : X \times Y \to [0,1]$ satisfying the condition

$0 \le \mu_R(x,y) + \eta_R(x,y) + \nu_R(x,y) \le 1$ for every $(x,y) \in (X \times Y)$.

The set of all picture fuzzy relations in *X* × *Y* will be denoted by $PFR(X \times Y)$.

Definition 2.6. [20]. Let $R \in PFR(X \times Y)$. We define the inverse relation R^{-1} between Y and X:

$\mu_{R^{-1}}(y,x) = \mu_R(x,y)$, $\eta_{R^{-1}}(y,x) = \eta_R(x,y)$, $\nu_{R^{-1}}(y,x) = \nu_R(x,y)$, $\forall (x,y) \in X \times Y$.

Definition 2.7. [20]. Let R and P be two picture fuzzy relations between X and Y, for every $(x,y) \in X \times Y$ we define:

1) $R \le P \Leftrightarrow (\mu_R(x,y) \le \mu_P(x,y))$ and $(\eta_R(x,y) \le \eta_P(x,y))$ and $(\nu_R(x,y) \ge \nu_P(x,y))$

2) $$R \vee P = \{(x,y), \mu_R(x,y) \vee \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y) : x \in X, y \in Y\}$$

3) $$R \wedge P = \{(x,y), \mu_R(x,y) \wedge \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \vee \nu_P(x,y) : x \in X, y \in Y\}$$

4) $R^c = \{((x,y), \nu_R(x,y), \eta_R(x,y), \mu_R(x,y)) : x \in X, y \in Y\}$

Here, $\vee$ and $\wedge$ denote the maximum and minimum operators respectively.

Definition 2.8. [19]. Let $R \in PFR(X \times Y)$ and $S \in PFR(Y \times Z)$. Then the composition of R and S is the PFR from X to Z defined as

$$R \circ S = \{((x,z), \mu_{R \circ S}(x,z), \eta_{R \circ S}(x,z), \nu_{R \circ S}(x,z)) : x \in X, z \in Z\}$$

where

$$\mu_{R \circ S}(x,z) = \vee_{y \in Y}\{\mu_S(x,y) \wedge \mu_R(y,z)\},$$

$$\eta_{R \circ S}(x,z) = \wedge_{y \in Y}\{\eta_S(x,y) \wedge \eta_R(y,z)\}$$

and $\nu_{R \circ S}(x,z) = \wedge_{y \in Y}\{\nu_S(x,y) \vee \nu_R(y,z)\}$.

Definition 2.9. [20]. The relation $R \in PFR(X \times X)$ is called:

1) Reflexive if $\mu_R(x,x) = 1$, $\eta_R(x,x) = 0$ and $\nu_R(x,x) = 0$; $\forall x \in X$.

2) Anti-reflexive if $\mu_R(x,x) = 0$, $\eta_R(x,x) = 0$ and $\nu_R(x,x) = 1$; $\forall x \in X$.

Definition 2.10. [20]. A *PFR*, $R(X \times X)$ is reflexive of order (α, γ, β) if $\mu_R(x,x) = \alpha$, $\eta_R(x,x) = \gamma$ and $\nu_R(x,x) = \beta$; $\forall x \in X$ and $\alpha + \gamma + \beta \le 1$.

Definition 2.11. [20]. A picture fuzzy relation $R(X \times X)$ is symmetric if $\mu_R(x,y) = \mu_R(y,x)$, $\eta_R(x,y) = \eta_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)$; $\forall x, y \in X$.

Definition 2.12. [20]. Let $R(X \times X)$ be a picture fuzzy relation. Then R is transitive if $R \circ R \subseteq R$.

3. Some Structures of Picture Fuzzy Relations

Definition 3.1. Let X be the universal set and $A = (\mu_A, \eta_A, \nu_A)$ and $B = (\mu_B, \eta_B, \nu_B)$ be two PFSs of X. Define the Cartesian product $A \times B$ as the PFS of $X \times X$ by $A \times B = (\mu_{A \times B}, \eta_{A \times B}, \nu_{A \times B})$ where for all $x, y \in X$, $\mu_{A \times B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$, $\eta_{A \times B}(x,y) = \min\{\eta_A(x), \eta_B(y)\}$ and $\nu_{A \times B}(x,y) = \max\{\nu_A(x), \nu_B(y)\}$.

Definition 3.2. Let R be a PFS of $X \times X$ with $R \subseteq A \times B$ *i.e.* $\forall (x,y) \in X \times X$;

1) $\mu_R(x,y) \le \mu_{A \times B}(x,y)$;

2) $\eta_R(x,y) \le \eta_{A\times B}(x,y)$;

3) $\nu_R(x,y) \ge \nu_{A\times B}(x,y)$;

4) $\mu_R(x,y)+\eta_R(x,y)+\nu_R(x,y) \le 1$.

Then we say that R is a picture fuzzy relation from A to B. In particular, if $A=B$ then R is said to be a picture fuzzy relation on A.

We denote the set of all picture fuzzy relations from A to B by $PFR(A\times B)$.

From now on, we assume that, the set X is finite; say $X=\{x_1,x_2,\cdots,x_n\}$. The picture fuzzy relation R from A to B, $R(A\times B)$ can be represented as a matrix $[R]=[\mu_{ij},\eta_{ij},\nu_{ij}]$, where $\mu_{ij}=\mu_R(x_i,x_j)$, $\eta_{ij}=\eta_R(x_i,x_j)$ and $\nu_{ij}=\nu_R(x_i,x_j)$, $i,j=1,2,\cdots,n$. We write $[R_\mu]=[\mu_{ij}]$, $[R_\eta]=[\eta_{ij}]$ and $[R_\nu]=[\nu_{ij}]$.

Example 3.2. Let $X=\{x_1,x_2,x_3\}$ be a non-empty set.

Let $A=\{(x_1,0.7,0.1,0.2),(x_2,0.5,0.2,0.2),(x_3,0.4,0.3,0.1)\}$ and $B=\{(x_1,0.5,0.2,0.3),(x_2,0.8,0.1,0.1),(x_3,0.4,0.5,0.1)\}$ be two picture fuzzy sets on X. Then

$$\begin{aligned}A\times B=\{&\langle(x_1,x_1),0.5,0.1,0.3\rangle,\langle(x_1,x_2),0.7,0.1,0.2\rangle,\langle(x_1,x_3),0.4,0.1,0.2\rangle,\\&\langle(x_2,x_1),0.5,0.2,0.3\rangle,\langle(x_2,x_2),0.5,0.1,0.2\rangle,\langle(x_2,x_3),0.4,0.2,0.2\rangle,\\&\langle(x_3,x_1),0.4,0.2,0.3\rangle,\langle(x_3,x_2),0.4,0.1,0.1\rangle,\langle(x_3,x_3),0.4,0.3,0.1\rangle\}\end{aligned}$$

$$[A\times B]=\begin{array}{c}\\x_1\\x_2\\x_3\end{array}\begin{array}{c}\begin{array}{ccc}x_1&x_2&x_3\end{array}\\\begin{bmatrix}(0.5,0.1,0.3)&(0.7,0.1,0.2)&(0.4,0.1,0.2)\\(0.5,0.2,0.3)&(0.5,0.1,0.2)&(0.4,0.2,0.2)\\(0.4,0.2,0.3)&(0.4,0.1,0.1)&(0.4,0.3,0.1)\end{bmatrix}\end{array}.$$

Let $[R]=\begin{bmatrix}(0.4,0.1,0.5)&(0.5,0.0,0.4)&(0.4,0.1,0.4)\\(0.3,0.2,0.5)&(0.5,0.1,0.4)&(0.2,0.1,0.6)\\(0.3,0.2,0.4)&(0.2,0.1,0.6)&(0.3,0.3,0.4)\end{bmatrix}$, while

$$[R_\mu]=\begin{bmatrix}0.4&0.5&0.4\\0.3&0.5&0.2\\0.3&0.2&0.3\end{bmatrix},\ [R_\eta]=\begin{bmatrix}0.1&0.0&0.1\\0.2&0.1&0.1\\0.2&0.1&0.3\end{bmatrix}\text{ and }[R_\nu]=\begin{bmatrix}0.5&0.4&0.4\\0.5&0.4&0.6\\0.4&0.6&0.4\end{bmatrix}.$$

Definition 3.3. Let R be a picture fuzzy relation on A. The complement of the relation R is a picture fuzzy relation R^c, where $\mu_{R^c}=\nu_R$, $\eta_{R^c}=\eta_R$ and $\nu_{R^c}=\mu_R$.

We can write

$$R^c=\{((x,y),\nu_R(x,y),\eta_R(x,y),\mu_R(x,y)):(x,y)\in X\times X\}.$$

Example 3.3. Consider the relation R from **example 3.2**

$$[R]=\begin{bmatrix}(0.4,0.1,0.5)&(0.5,0.0,0.4)&(0.4,0.1,0.4)\\(0.3,0.2,0.5)&(0.5,0.1,0.4)&(0.2,0.1,0.6)\\(0.3,0.2,0.4)&(0.2,0.1,0.6)&(0.3,0.3,0.4)\end{bmatrix},\text{ then}$$

$$[R^c]=\begin{bmatrix}(0.5,0.1,0.4)&(0.4,0.0,0.5)&(0.4,0.1,0.4)\\(0.5,0.2,0.3)&(0.4,0.1,0.5)&(0.6,0.1,0.2)\\(0.4,0.2,0.3)&(0.6,0.1,0.2)&(0.4,0.3,0.3)\end{bmatrix}.$$

Definition 3.4. Let R be picture fuzzy relation from A to B. Then we define the inverse relation R^{-1} as $\mu_{R^{-1}}(y,x)=\mu_R(x,y)$, $\eta_{R^{-1}}(y,x)=\eta_R(x,y)$, $\nu_{R^{-1}}(y,x)=\nu_R(x,y)$.

Example 3.4. Consider the relation R from **example 3.2**

$$[R]=\begin{bmatrix}(0.4,0.1,0.5) & (0.5,0.0,0.4) & (0.4,0.1,0.4)\\(0.3,0.2,0.5) & (0.5,0.1,0.4) & (0.2,0.1,0.6)\\(0.3,0.2,0.4) & (0.2,0.1,0.6) & (0.3,0.3,0.4)\end{bmatrix}\text{, then}$$

$$\left[R^{-1}\right]=\begin{bmatrix}(0.4,0.1,0.5) & (0.3,0.2,0.5) & (0.3,0.2,0.4)\\(0.5,0.0,0.4) & (0.5,0.1,0.4) & (0.2,0.1,0.6)\\(0.4,0.1,0.4) & (0.2,0.1,0.6) & (0.3,0.3,0.4)\end{bmatrix}.$$

Definition 3.5. Let $R_1,R_2\in PFR(A,B)$. Then we say $R_1\subseteq R_2$ if for all $x,y\in X$;

1) $\mu_{R_1}(x,y)\le\mu_{R_2}(x,y)$;
2) $\eta_{R_1}(x,y)\le\eta_{R_2}(x,y)$;
3) $\nu_{R_1}(x,y)\ge\nu_{R_2}(x,y)$.

If $R_1\subseteq R_2$ and $R_2\subseteq R_1$ then $R_1=R_2$.

Example 3.5. Consider the **example 3.2**

Let $R_1=\begin{bmatrix}(0.4,0.1,0.5) & (0.3,0.0,0.6) & (0.3,0.1,0.5)\\(0.3,0.2,0.5) & (0.4,0.1,0.4) & (0.2,0.1,0.6)\\(0.3,0.1,0.4) & (0.2,0.1,0.6) & (0.3,0.3,0.4)\end{bmatrix}$ and

$R_2=\begin{bmatrix}(0.5,0.1,0.3) & (0.5,0.1,0.4) & (0.3,0.1,0.5)\\(0.5,0.2,0.3) & (0.5,0.1,0.3) & (0.3,0.1,0.5)\\(0.4,0.1,0.4) & (0.3,0.1,0.5) & (0.4,0.3,0.3)\end{bmatrix}$ be two picture fuzzy relations from A to B.

Clearly, $R_1\subseteq R_2$.

Definition 3.6. Let R_1, R_2 be picture fuzzy relations from A to B. Then the union of R_1 and R_2, $R_1\cup R_2$ is a picture fuzzy relation whose positive membership, neutral membership and negative membership are

$$\mu_{R_1\cup R_2}(x,y)=\max\left\{\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right\},$$

$$\eta_{R_1\cup R_2}(x,y)=\min\left\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\right\}$$

$$\text{and }\ \nu_{R_1\cup R_2}(x,y)=\min\left\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right\}.$$

Definition 3.7. Let R_1, R_2 be picture fuzzy relations from A to B. Then the intersection of R_1 and R_2, $R_1\cap R_2$ is a picture fuzzy relation whose positive membership, neutral membership and negative membership are

$$\mu_{R_1\cap R_2}(x,y)=\min\left\{\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right\},$$

$$\eta_{R_1\cap R_2}(x,y)=\min\left\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\right\}$$

$$\text{and }\ \nu_{R_1\cap R_2}(x,y)=\max\left\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right\}.$$

Definition 3.8. Let R_1, R_2 be picture fuzzy relations from A to B. Then we define the arithmetic mean operator between R_1 and R_2 as follows

$$\mu_{R_1 @ R_2}(x,y) = \frac{\mu_{R_1}(x,y) + \mu_{R_2}(x,y)}{2},$$

$$\eta_{R_1 @ R_2}(x,y) = \frac{\eta_{R_1}(x,y) + \eta_{R_2}(x,y)}{2}$$

$$\text{and} \quad \nu_{R_1 @ R_2}(x,y) = \frac{\nu_{R_1}(x,y) + \nu_{R_2}(x,y)}{2}$$

Definition 3.9. Let R_1, R_2 be picture fuzzy relations from A to B. Then we define the geometric mean operator between R_1 and R_2 as follows

$$\mu_{R_1 \$ R_2}(x,y) = \sqrt{\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y)},$$

$$\eta_{R_1 \$ R_2}(x,y) = \sqrt{\eta_{R_1}(x,y) \cdot \eta_{R_2}(x,y)}$$

$$\text{and} \quad \nu_{R_1 \$ R_2}(x,y) = \sqrt{\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y)}.$$

Definition 3.10. Let R_1, R_2 be picture fuzzy relations from A to B. Then we define the harmonic mean operator between R_1 and R_2 as follows

$$\mu_{R_1 © R_2}(x,y) = \frac{2\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y)}{\mu_{R_1}(x,y) + \mu_{R_2}(x,y)},$$

$$\eta_{R_1 © R_2}(x,y) = \frac{2\eta_{R_1}(x,y) \cdot \eta_{R_2}(x,y)}{\eta_{R_1}(x,y) + \eta_{R_2}(x,y)}$$

$$\text{and} \quad \nu_{R_1 © R_2}(x,y) = \frac{2\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y)}{\nu_{R_1}(x,y) + \nu_{R_2}(x,y)}.$$

Definition 3.11. Let R_1, R_2 be picture fuzzy relations from A to B. Then we define the operator "$\boxplus$" between R_1 and R_2 as follows

$$\mu_{R_1 \boxplus R_2}(x,y) = \frac{\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y)}{2\left(\mu_{R_1}(x,y) \cdot \mu_{R_2}(x,y) + 1\right)},$$

$$\eta_{R_1 \boxplus R_2}(x,y) = \frac{\eta_{R_1}(x,y) \cdot \eta_{R_2}(x,y)}{2\left(\eta_{R_1}(x,y) \cdot \eta_{R_2}(x,y) + 1\right)}$$

$$\text{and} \quad \nu_{R_1 \boxplus R_2}(x,y) = \frac{\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y)}{2\left(\nu_{R_1}(x,y) \cdot \nu_{R_2}(x,y) + 1\right)}.$$

Example 3.11. Consider the **example 3.2**

Let $R_1 = \begin{bmatrix} (0.4,0.1,0.5) & (0.5,0.0,0.4) & (0.4,0.1,0.4) \\ (0.3,0.2,0.5) & (0.5,0.1,0.4) & (0.2,0.1,0.6) \\ (0.3,0.2,0.4) & (0.2,0.1,0.6) & (0.3,0.3,0.4) \end{bmatrix}$ and

$R_2 = \begin{bmatrix} (0.5,0.0,0.4) & (0.4,0.1,0.5) & (0.2,0.1,0.6) \\ (0.2,0.1,0.7) & (0.3,0.0,0.6) & (0.3,0.1,0.7) \\ (0.1,0.1,0.6) & (0.3,0.1,0.5) & (0.4,0.2,0.4) \end{bmatrix}$ be two picture fuzzy relations from A to B. Then,

$$R_1 \cup R_2 = \begin{bmatrix} (0.5,0.0,0.4) & (0.5,0.0,0.4) & (0.4,0.1,0.4) \\ (0.3,0.1,0.5) & (0.5,0.0,0.4) & (0.3,0.1,0.6) \\ (0.3,0.1,0.4) & (0.3,0.1,0.5) & (0.4,0.2,0.4) \end{bmatrix}$$

$$R_1 \cap R_2 = \begin{bmatrix} (0.4,0.0,0.5) & (0.4,0.0,0.5) & (0.2,0.1,0.6) \\ (0.2,0.1,0.7) & (0.3,0.0,0.6) & (0.2,0.1,0.7) \\ (0.1,0.1,0.6) & (0.2,0.1,0.6) & (0.3,0.2,0.4) \end{bmatrix}$$

$$R_1^c = \begin{bmatrix} (0.5,0.1,0.4) & (0.4,0.0,0.5) & (0.4,0.1,0.4) \\ (0.5,0.2,0.3) & (0.4,0.1,0.5) & (0.6,0.1,0.2) \\ (0.4,0.2,0.3) & (0.6,0.1,0.2) & (0.4,0.3,0.3) \end{bmatrix}$$

$$R_1 @ R_2 = \begin{bmatrix} (0.45,0.05,0.45) & (0.45,0.05,0.45) & (0.30,0.10,0.50) \\ (0.25,0.15,0.60) & (0.40,0.05,0.50) & (0.25,0.10,0.65) \\ (0.20,0.15,0.50) & (0.25,0.10,0.55) & (0.35,0.25,0.40) \end{bmatrix}$$

$$R_1 \$ R_2 = \begin{bmatrix} (0.45,0.00,0.45) & (0.45,0.00,0.45) & (0.28,0.10,0.49) \\ (0.24,0.14,0.59) & (0.38,0.00,0.49) & (0.24,0.10,0.65) \\ (0.17,0.14,0.48) & (0.24,0.10,0.55) & (0.35,0.24,0.40) \end{bmatrix}$$

$$R_1 © R_2 = \begin{bmatrix} (0.44,0.00,0.44) & (0.44,0.00,0.44) & (0.27,0.10,0.48) \\ (0.24,0.13,0.58) & (0.38,0.00,0.48) & (0.24,0.10,0.65) \\ (0.15,0.13,0.48) & (0.24,0.10,0.55) & (0.34,0.24,0.40) \end{bmatrix}$$

$$R_1 \boxplus R_2 = \begin{bmatrix} (0.12,0.00,0.12) & (0.12,0.00,0.12) & (0.04,0.005,0.15) \\ (0.03,0.01,0.24) & (0.09,0.00,0.15) & (0.03,0.005,0.30) \\ (0.02,0.01,0.15) & (0.03,0.005,0.20) & (0.07,0.03,0.09) \end{bmatrix}$$

Theorem 3.12. Let $R \in PFR(A \times B)$ be a picture fuzzy relation. Then $\left(R^{-1}\right)^{-1} = R$.

Proof. By the definition of inverse relation, we have

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y), \quad \eta_{R^{-1}}(y,x) = \eta_R(x,y), \quad \nu_{R^{-1}}(y,x) = \nu_R(x,y).$$

Now,

$$\mu_{\left(R^{-1}\right)^{-1}}(y,x) = \mu_{R^{-1}}(x,y) = \mu_R(y,x)$$

$$\eta_{\left(R^{-1}\right)^{-1}}(y,x) = \eta_{R^{-1}}(x,y) = \eta_R(y,x)$$

and

$$\nu_{\left(R^{-1}\right)^{-1}}(y,x) = \nu_{R^{-1}}(x,y) = \nu_R(y,x).$$

Hence, $\left(R^{-1}\right)^{-1} = R$.

Theorem 3.13. Let $R_1, R_2 \in PFR(A \times B)$ be two picture fuzzy relations. Then

1) $\left(R_1 \cup R_2\right)^{-1} = R_1^{-1} \cup R_2^{-1}$.
2) $\left(R_1 \cap R_2\right)^{-1} = R_1^{-1} \cap R_2^{-1}$.

Proof. 1) By the definition of inverse relation, we have $\mu_{R_1^{-1}}(y,x) = \mu_{R_1}(x,y)$, $\eta_{R_1^{-1}}(y,x) = \eta_{R_1}(x,y)$, $\nu_{R_1^{-1}}(y,x) = \nu_{R_1}(x,y)$ and $\mu_{R_2^{-1}}(y,x) = \mu_{R_2}(x,y)$,

$\eta_{R_2^{-1}}(y,x)=\eta_{R_2}(x,y)$, $\nu_{R_2^{-1}}(y,x)=\nu_{R_2}(x,y)$.

Therefore,

$$\begin{aligned}\mu_{(R_1\cup R_2)^{-1}}(y,x)&=\mu_{R_1\cup R_2}(x,y)\\&=\max\left\{\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right\}\\&=\max\left\{\mu_{R_1^{-1}}(y,x),\mu_{R_2^{-1}}(y,x)\right\}\\&=\mu_{R_1^{-1}\cup R_2^{-1}}(y,x)\end{aligned}$$

$$\begin{aligned}\eta_{(R_1\cup R_2)^{-1}}(y,x)&=\eta_{R_1\cup R_2}(x,y)\\&=\min\left\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\right\}\\&=\min\left\{\eta_{R_1^{-1}}(y,x),\eta_{R_2^{-1}}(y,x)\right\}\\&=\eta_{R_1^{-1}\cup R_2^{-1}}(y,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{(R_1\cup R_2)^{-1}}(y,x)&=\nu_{R_1\cup R_2}(x,y)\\&=\min\left\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right\}\\&=\min\left\{\nu_{R_1^{-1}}(y,x),\nu_{R_2^{-1}}(y,x)\right\}\\&=\nu_{R_1^{-1}\cup R_2^{-1}}(y,x)\end{aligned}$$

Hence, $(R_1\cup R_2)^{-1}=R_1^{-1}\cup R_2^{-1}$.

2) We have, $\mu_{R_1^{-1}}(y,x)=\mu_{R_1}(x,y)$, $\eta_{R_1^{-1}}(y,x)=\eta_{R_1}(x,y)$, $\nu_{R_1^{-1}}(y,x)=\nu_{R_1}(x,y)$ and $\mu_{R_2^{-1}}(y,x)=\mu_{R_2}(x,y)$, $\eta_{R_2^{-1}}(y,x)=\eta_{R_2}(x,y)$, $\nu_{R_2^{-1}}(y,x)=\nu_{R_2}(x,y)$.

Therefore,

$$\begin{aligned}\mu_{(R_1\cap R_2)^{-1}}(y,x)&=\mu_{R_1\cap R_2}(x,y)\\&=\min\left\{\mu_{R_1}(x,y),\mu_{R_2}(x,y)\right\}\\&=\min\left\{\mu_{R_1^{-1}}(y,x),\mu_{R_2^{-1}}(y,x)\right\}.\\&=\mu_{R_1^{-1}\cap R_2^{-1}}(y,x)\end{aligned}$$

$$\begin{aligned}\eta_{(R_1\cap R_2)^{-1}}(y,x)&=\eta_{R_1\cap R_2}(x,y)\\&=\min\left\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\right\}\\&=\min\left\{\eta_{R_1^{-1}}(y,x),\eta_{R_2^{-1}}(y,x)\right\}.\\&=\eta_{R_1^{-1}\cap R_2^{-1}}(y,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{(R_1\cap R_2)^{-1}}(y,x)&=\nu_{R_1\cap R_2}(x,y)\\&=\max\left\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\right\}\\&=\max\left\{\nu_{R_1^{-1}}(y,x),\nu_{R_2^{-1}}(y,x)\right\}.\\&=\nu_{R_1^{-1}\cap R_2^{-1}}(y,x)\end{aligned}$$

Hence, $\left(R_1 \cap R_2\right)^{-1} = R_1^{-1} \cap R_2^{-1}$.

Theorem 3.14. Let R_1, R_2 be picture fuzzy relations from *A* to *B*. Then $R_1 @ R_2$, $R_1 \$ R_2$, $R_1 © R_2$ are also picture fuzzy relations from *A* to *B*. But the relation $R_1 * R_2$ may not be closed *i.e.*, $R_1 \boxplus R_2$ may not be picture fuzzy relation from *A* to *B*.

Proof. It is easy to check that for Let $R_1, R_2 \in PFR(A \times B)$, $R_1 @ R_2$, $R_1 \$ R_2$, $R_1 © R_2$ are picture fuzzy relations from *A* to *B*.

Now we will show that by an example that the operation $\boxplus$ is not closed.

Let $X = \{x_1, x_2, x_3\}$ be a non-empty set.

Let $A = \{(x_1, 0.7, 0.1, 0.2), (x_2, 0.5, 0.2, 0.2), (x_3, 0.4, 0.3, 0.1)\}$ and

$B = \{(x_1, 0.5, 0.2, 0.3), (x_2, 0.8, 0.1, 0.1), (x_3, 0.4, 0.5, 0.1)\}$ be two picture fuzzy sets on *X*. Then

$$[A \times B] = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} (0.5,0.1,0.3) & (0.7,0.1,0.2) & (0.4,0.1,0.2) \\ (0.5,0.2,0.3) & (0.5,0.1,0.2) & (0.4,0.2,0.2) \\ (0.4,0.2,0.3) & (0.4,0.1,0.1) & (0.4,0.3,0.1) \end{bmatrix} \end{array}.$$

Let $R_1 = \begin{bmatrix} (0.4,0.1,0.5) & (0.5,0.0,0.4) & (0.4,0.1,0.4) \\ (0.3,0.2,0.5) & (0.5,0.1,0.4) & (0.2,0.1,0.6) \\ (0.3,0.2,0.4) & (0.2,0.1,0.6) & (0.3,0.3,0.4) \end{bmatrix}$ and

$R_2 = \begin{bmatrix} (0.5,0.0,0.4) & (0.4,0.1,0.5) & (0.2,0.1,0.6) \\ (0.2,0.1,0.7) & (0.3,0.0,0.6) & (0.3,0.1,0.7) \\ (0.1,0.1,0.6) & (0.3,0.1,0.5) & (0.4,0.2,0.4) \end{bmatrix}$ be two picture fuzzy relations from *A* to *B*. Then

$$R_1 \boxplus R_2 = \begin{bmatrix} (0.12,0.00,0.12) & (0.12,0.00,0.12) & (0.04,0.005,0.15) \\ (0.03,0.01,0.24) & (0.09,0.00,0.15) & (0.03,0.005,0.30) \\ (0.02,0.01,0.15) & (0.03,0.005,0.20) & (0.07,0.03,0.09) \end{bmatrix}$$

According to the definition of picture fuzzy relation over picture fuzzy sets, we have $\mu_{R_1 \boxplus R_2}(x_1, x_1) = 0.12 \le \mu_{A \times B}(x_1, x_1) = 0.5$,
$\eta_{R_1 \boxplus R_2}(x_1, x_1) = 0.0 \le \eta_{A \times B}(x_1, x_1) = 0.1$ but
$\nu_{R_1 \boxplus R_2}(x_1, x_1) = 0.12 \ngeq \nu_{A \times B}(x_1, x_1) = 0.3$.

Hence $R_1 \boxplus R_2$ is not picture fuzzy relation from *A* to *B*.

4. Composition of Picture Fuzzy Relations

Definition 4.1. Let $R \in PFR(A \times B)$ and $S \in PFR(B \times C)$, then the **max-min composition** of *R* and *S* is the picture fuzzy relation from *A* to *C* defined as

$$R \circ S = \left\{\left((x, y), \mu_{R \circ S}(x, y), \eta_{R \circ S}(x, y), \nu_{R \circ S}(x, y)\right) : x, y \in X\right\},$$

where

$$\mu_{R \circ S}(x, y) = \vee_{z \in X}\left\{\mu_S(x, z) \wedge \mu_R(z, y)\right\},$$

$$\eta_{R\circ S}(x,y)=\wedge_{z\in X}\{\eta_S(x,z)\wedge\eta_R(z,y)\},$$

and

$$\nu_{R\circ S}(x,y)=\wedge_{z\in X}\{\nu_S(x,z)\vee\nu_R(z,y)\}.$$

when ever $0\le\mu_{R\circ S}(x,y)+\eta_{R\circ S}(x,y)+\nu_{R\circ S}(x,y)\le 1$.

Proposition 4.2. Let $R_1\in PFR(A\times B)$ and $R_2\in PFR(B\times C)$, then $R_1\circ R_2\in PFR(A\times C)$.

Proof. For all $(x,z)\in X\times X$, let proof

$$\mu_{R_1\circ R_2}(x,z)+\eta_{R_1\circ R_2}(x,z)+\nu_{R_1\circ R_2}(x,z)\le 1.$$

For all $\epsilon>0$, there exists $y^*\in X$:

$$\mu_{R_1\circ R_2}(x,z)<\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\epsilon.$$

It is easily seen that

$$\eta_{R_1\circ R_2}(x,z)\le\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)$$

$$\text{and }\ \nu_{R_1\circ R_2}(x,z)\le\nu_{R_2}(x,y^*)\vee\nu_{R_1}(y^*,z).$$

Now,

$$\begin{aligned}&\mu_{R_1\circ R_2}(x,z)+\eta_{R_1\circ R_2}(x,z)+\nu_{R_1\circ R_2}(x,z)\\&<\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)\\&\quad+\nu_{R_2}(x,y^*)\vee\nu_{R_2}(y^*,z)+\epsilon\end{aligned}$$

Case 1: $\nu_{R_2}(x,y^*)\vee\nu_{R_1}(y^*,z)=\nu_{R_2}(x,y^*)$. Then

$$\begin{aligned}&\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)+\nu_{R_2}(x,y^*)\vee\nu_{R_1}(y^*,z)+\epsilon\\&=\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)+\nu_{R_2}(x,y^*)+\epsilon\\&\le\mu_{R_2}(x,y^*)+\eta_{R_2}(x,y^*)+\nu_{R_2}(x,y^*)+\epsilon\\&\le 1+\epsilon.\end{aligned}$$

Case 2: $\nu_{R_2}(x,y^*)\vee\nu_{R_1}(y^*,z)=\nu_{R_1}(y^*,z)$. Then

$$\begin{aligned}&\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)+\nu_{R_2}(x,y^*)\vee\nu_{R_1}(y^*,z)+\epsilon\\&=\mu_{R_2}(x,y^*)\wedge\mu_{R_1}(y^*,z)+\eta_{R_2}(x,y^*)\wedge\eta_{R_1}(y^*,z)+\nu_{R_1}(y^*,z)+\epsilon\\&\le\mu_{R_1}(y^*,z)+\eta_{R_1}(y^*,z)+\nu_{R_1}(y^*,z)+\epsilon\\&\le 1+\epsilon.\end{aligned}$$

Then $\mu_{R_1\circ R_2}(x,z)+\eta_{R_1\circ R_2}(x,z)+\nu_{R_1\circ R_2}(x,z)\le 1+\epsilon$ for all $\epsilon>0$.

Hence $\mu_{R_1\circ R_2}(x,z)+\eta_{R_1\circ R_2}(x,z)+\nu_{R_1\circ R_2}(x,z)\le 1$.

Theorem 4.3. For each $R\in PFR(A\times B)$ and $S\in PFR(B\times C)$, $(S\circ R)^{-1}=R^{-1}\circ S^{-1}$ is fulfilled.

Proof.

$$\begin{aligned}\mu_{(S\circ R)^{-1}}(z,x) &= \mu_{(S\circ R)}(x,z)\\ &= \vee_{y\in X}\{\mu_R(x,y)\wedge\mu_S(y,z)\}\\ &= \vee_{y\in X}\{\mu_{R^{-1}}(y,x)\wedge\mu_{S^{-1}}(z,y)\}\\ &= \vee_{y\in X}\{\mu_{S^{-1}}(z,y)\wedge\mu_{R^{-1}}(y,x)\}\\ &= \mu_{R^{-1}\circ S^{-1}}(z,x)\end{aligned}$$

$$\begin{aligned}\eta_{(S\circ R)^{-1}}(z,x) &= \eta_{(S\circ R)}(x,z)\\ &= \wedge_{y\in X}\{\eta_R(x,y)\wedge\eta_S(y,z)\}\\ &= \wedge_{y\in X}\{\eta_{R^{-1}}(y,x)\wedge\eta_{S^{-1}}(z,y)\}\\ &= \wedge_{y\in X}\{\eta_{S^{-1}}(z,y)\wedge\eta_{R^{-1}}(y,x)\}\\ &= \eta_{R^{-1}\circ S^{-1}}(z,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{(S\circ R)^{-1}}(z,x) &= \eta_{(S\circ R)}(x,z)\\ &= \wedge_{y\in X}\{\nu_R(x,y)\vee\nu_S(y,z)\}\\ &= \wedge_{y\in X}\{\nu_{R^{-1}}(y,x)\vee\nu_{S^{-1}}(z,y)\}.\\ &= \wedge_{y\in X}\{\nu_{S^{-1}}(z,y)\vee\nu_{R^{-1}}(y,x)\}\\ &= \nu_{R^{-1}\circ S^{-1}}(z,x).\end{aligned}$$

Theorem 4.4. Let $P,Q,R\in PFR(A\times B)$. Then

1) $P\le Q\Rightarrow P\circ R\le Q\circ R$.
2) $P\le Q\Rightarrow R\circ P\le R\circ Q$.

Proof. 1) $P\le Q$, then

$$\mu_P(y,z)\le\mu_Q(y,z),\ \ \eta_P(y,z)\le\eta_Q(y,z)\ \text{ and }\ \nu_P(y,z)\ge\nu_Q(y,z).$$

Now,

$$\begin{aligned}\mu_{P\circ R}(x,z) &= \vee_{y\in X}\{\mu_R(x,y)\wedge\mu_P(y,z)\}\\ &\le \vee_{y\in X}\{\mu_R(x,y)\wedge\mu_Q(y,z)\}\ \text{ as } \mu_P(y,z)\le\mu_Q(y,z)\\ &= \mu_{Q\circ R}(x,z)\end{aligned}$$

Similarly, we can prove that,

$$\eta_{P\circ R}(x,z)\le\eta_{Q\circ R}(x,z).$$

Again,

$$\begin{aligned}\nu_{P\circ R}(x,z) &= \wedge_{y\in X}\{\nu_R(x,y)\vee\nu_P(y,z)\}\\ &\ge \wedge_{y\in X}\{\nu_R(x,y)\vee\nu_Q(y,z)\}\ \text{ as } \nu_P(y,z)\ge\nu_Q(y,z)\\ &\ge \nu_{Q\circ R}(x,z)\end{aligned}$$

The property 2) can be proved in similar manner.

Theorem 4.5. Let $P, R \in PFR(B\times C), Q \in PFR(A\times B)$, then

1) $(R\vee P)\circ Q = (R\circ Q)\vee(P\circ Q)$.
2) $(R\wedge P)\circ Q = (R\circ Q)\wedge(P\circ Q)$.

Proof. 1) In order to proof 1), we have to show that

a) $\mu_{(R\vee P)\circ Q}(x,z) = \mu_{R\circ Q}(x,z)\vee\mu_{P\circ Q}(x,z)$.

b) $\eta_{(R\vee P)\circ Q}(x,z) = \eta_{R\circ Q}(x,z)\wedge\eta_{P\circ Q}(x,z)$.

c) $\nu_{(R\vee P)\circ Q}(x,z) = \nu_{R\circ Q}(x,z)\wedge\nu_{P\circ Q}(x,z)$.

Now,

a)
$$\begin{aligned}\mu_{(R\vee P)\circ Q}(x,z) &= \vee_{y\in X}\{\mu_Q(x,y)\wedge(\mu_R(y,z)\vee\mu_P(y,z))\}\\ &= \vee_{y\in X}\{(\mu_Q(x,y)\wedge\mu_R(y,z))\vee(\mu_Q(x,y)\wedge\mu_P(y,z))\}\\ &= \vee_{y\in X}\{(\mu_Q(x,y)\wedge\mu_R(y,z))\}\vee\vee_{y\in X}\{(\mu_Q(x,y)\wedge\mu_P(y,z))\}\\ &= \mu_{R\circ Q}(x,z)\vee\mu_{P\circ Q}(x,z)\end{aligned}$$

b)
$$\begin{aligned}\eta_{(R\vee P)\circ Q}(x,z) &= \wedge_{y\in X}\{\eta_Q(x,y)\wedge(\eta_R(y,z)\wedge\eta_P(y,z))\}\\ &= \wedge_{y\in X}\{(\eta_Q(x,y)\wedge\eta_R(y,z))\wedge(\eta_Q(x,y)\wedge\eta_P(y,z))\}\\ &= \wedge_{y\in X}\{(\eta_Q(x,y)\wedge\eta_R(y,z))\}\wedge\wedge_{y\in X}\{(\eta_Q(x,y)\wedge\eta_P(y,z))\}\\ &= \eta_{R\circ Q}(x,z)\wedge\eta_{P\circ Q}(x,z)\end{aligned}$$

c)
$$\begin{aligned}\nu_{(R\vee P)\circ Q}(x,z) &= \wedge_{y\in X}\{\nu_Q(x,y)\vee(\nu_R(y,z)\wedge\nu_P(y,z))\}\\ &= \wedge_{y\in X}\{(\nu_Q(x,y)\vee\nu_R(y,z))\wedge(\nu_Q(x,y)\vee\nu_P(y,z))\}\\ &= \wedge_{y\in X}\{(\nu_Q(x,y)\vee\nu_R(y,z))\}\wedge\wedge_{y\in X}\{(\nu_Q(x,y)\vee\nu_P(y,z))\}\\ &= \nu_{R\circ Q}(x,z)\wedge\nu_{P\circ Q}(x,z);\ \forall(x,y)\in A\times B\end{aligned}$$

The property 2) can be proved in similar manner.

Theorem 4.6. Let $Q\in PFR(A\times B), P\in PFR(B\times C), R\in PFR(C\times D)$ then $(R\circ P)\circ Q = R\circ(P\circ Q)$.

Proof. It is sufficient to prove that

a) $\mu_{(R\circ P)\circ Q}(x,z) = \mu_{R\circ(P\circ Q)}(x,z)$.

b) $\eta_{(R\circ P)\circ Q}(x,z) = \eta_{R\circ(P\circ Q)}(x,z)$.

c) $\nu_{(R\circ P)\circ Q}(x,z) = \nu_{R\circ(P\circ Q)}(x,z)$.

We have,

$$\begin{aligned}\mu_{(R\circ P)\circ Q}(x,z) &= \vee_{y\in X}\{\mu_Q(x,y)\wedge\mu_{R\circ P}(y,z)\}\\ &= \vee_y\{\mu_Q(x,y)\wedge\{\vee_t\{\mu_P(y,t)\wedge\mu_R(t,z)\}\}\}\\ &= \vee_y\{\vee_t\{\mu_Q(x,y)\wedge\{\mu_P(y,t)\wedge\mu_R(t,z)\}\}\}\\ &= \vee_t\{\vee_y\{[\mu_Q(x,y)\wedge\mu_P(y,t)]\wedge\mu_R(t,z)\}\}\\ &= \vee_t\{\{\mu_{P\circ Q}(x,t)\wedge\mu_R(t,z)\}\}\\ &= \mu_{R\circ(P\circ Q)}(x,z);\ \forall(x,z)\in A\times C\end{aligned}$$

$$\begin{aligned}\eta_{(R\circ P)\circ Q}(x,z) &= \wedge_{y\in X}\{\eta_Q(x,y)\wedge\eta_{R\circ P}(y,z)\}\\ &= \wedge_y\{\eta_Q(x,y)\wedge\{\wedge_t\{\eta_P(y,t)\wedge\eta_R(t,z)\}\}\}\\ &= \wedge_y\{\wedge_t\{\eta_Q(x,y)\wedge\{\eta_P(y,t)\wedge\eta_R(t,z)\}\}\}\\ &= \wedge_t\{\wedge_y\{[\eta_Q(x,y)\wedge\eta_P(y,t)]\wedge\eta_R(t,z)\}\}\\ &= \wedge_t\{\eta_{P\circ Q}(x,t)\wedge\eta_R(t,z)\}\\ &= \eta_{R\circ(P\circ Q)}(x,z);\ \forall(x,z)\in A\times C\end{aligned}$$

and

$$\begin{aligned}\nu_{(R\circ P)\circ Q}(x,z) &= \wedge_{y\in X}\{\nu_Q(x,y)\vee\nu_{R\circ P}(y,z)\}\\ &= \wedge_y\{\nu_Q(x,y)\vee\{\wedge_t\{\nu_P(y,t)\vee\nu_R(t,z)\}\}\}\\ &= \wedge_y\{\wedge_t\{\nu_Q(x,y)\vee\{\nu_P(y,t)\vee\nu_R(t,z)\}\}\}\\ &= \wedge_t\{\wedge_y\{[\nu_Q(x,y)\vee\nu_P(y,t)]\vee\nu_R(t,z)\}\}\\ &= \wedge_t\{\nu_{P\circ Q}(x,t)\vee\nu_R(t,z)\}\\ &= \nu_{R\circ(P\circ Q)}(x,z);\ \forall(x,z)\in A\times C.\end{aligned}$$

Definition 4.7. Let $R\in PFR(A\times B)$ and $S\in PFR(B\times C)$, then the **min-max composition** of R and S is the picture fuzzy relation from A to C defined as

$$R*S=\{((x,y),\mu_{R*S}(x,y),\eta_{R*S}(x,y),\nu_{R*S}(x,y)):x,y\in X\},$$

where

$$\mu_{R*S}(x,y)=\wedge_{z\in X}\{\mu_S(x,z)\vee\mu_R(z,y)\},$$

$$\eta_{R*S}(x,y)=\wedge_{z\in X}\{\eta_S(x,z)\wedge\eta_R(z,y)\},$$

and $\nu_{R*S}(x,y)=\vee_{z\in X}\{\nu_S(x,z)\wedge\nu_R(z,y)\}$,
when ever $0\le\mu_{R*S}(x,y)+\eta_{R*S}(x,y)+\nu_{R*S}(x,y)\le1$.

Theorem 4.8. Let R, P be two elements of $PFR(A\times A)$, then $(R\circ P)^c=R^c*P^c$.

Proof. As $\mu_{R^c}(x,z)=\nu_R(x,z)$, $\eta_{R^c}(x,z)=\eta_R(x,z)$ and $\nu_{R^c}(x,z)=\mu_R(x,z)$; for every $(x,z)\in A\times A$.

We have,

$$R\circ P=\{(x,z),\vee_y\{\mu_P(x,y)\wedge\mu_R(y,z)\},\vee_y\{\eta_P(x,y)\wedge\eta_R(y,z)\},\\ \vee_y\{\nu_P(x,y)\vee\nu_R(y,z)\}\}.$$

Therefore,

$$\begin{aligned}(R\circ P)^c &= \{(x,z),\wedge_y\{\nu_P(x,y)\vee\nu_R(y,z)\},\wedge_y\{\eta_P(x,y)\wedge\eta_R(y,z)\},\\ &\quad \wedge_y\{\mu_P(x,y)\wedge\mu_R(y,z)\}\}\\ &= R^c*P^c.\end{aligned}$$

Theorem 4.9. For each $R \in PFR(A \times B)$ and $S \in PFR(B \times C)$, $(S * R)^{-1} = R^{-1} * S^{-1}$ is fulfilled.

Proof.

$$\begin{aligned}\mu_{(S*R)^{-1}}(z,x) &= \mu_{(S*R)}(x,z) \\ &= \wedge_{y\in X}\{\mu_R(x,y) \vee \mu_S(y,z)\} \\ &= \wedge_{y\in X}\{\mu_{R^{-1}}(y,x) \vee \mu_{S^{-1}}(z,y)\} \\ &= \wedge_{y\in X}\{\mu_{S^{-1}}(z,y) \vee \mu_{R^{-1}}(y,x)\} \\ &= \mu_{R^{-1}*S^{-1}}(z,x)\end{aligned}$$

$$\begin{aligned}\eta_{(S*R)^{-1}}(z,x) &= \eta_{(S*R)}(x,z) \\ &= \wedge_{y\in X}\{\eta_R(x,y) \wedge \eta_S(y,z)\} \\ &= \wedge_{y\in X}\{\eta_{R^{-1}}(y,x) \wedge \eta_{S^{-1}}(z,y)\} \\ &= \wedge_{y\in X}\{\eta_{S^{-1}}(z,y) \wedge \eta_{R^{-1}}(y,x)\} \\ &= \eta_{R^{-1}*S^{-1}}(z,x)\end{aligned}$$

and

$$\begin{aligned}\nu_{(S*R)^{-1}}(z,x) &= \nu_{(S*R)}(x,z) \\ &= \vee_{y\in X}\{\nu_R(x,y) \wedge \nu_S(y,z)\} \\ &= \vee_{y\in X}\{\nu_{R^{-1}}(y,x) \wedge \nu_{S^{-1}}(z,y)\} \\ &= \vee_{y\in X}\{\nu_{S^{-1}}(z,y) \wedge \nu_{R^{-1}}(y,x)\} \\ &= \nu_{R^{-1}*S^{-1}}(z,x).\end{aligned}$$

Therefore, $(S * R)^{-1} = R^{-1} * S^{-1}$.

Theorem 4.10. Let $P, Q, R \in PFR(A \times B)$. Then

1) $P \le Q \Rightarrow P * R \le Q * R$.

2) $P \le Q \Rightarrow R * P \le R * Q$.

Proof. 1) $P \le Q$, then

$$\mu_P(y,z) \le \mu_Q(y,z),\quad \eta_P(y,z) \le \eta_Q(y,z) \text{ and } \nu_P(y,z) \ge \nu_Q(y,z)$$

Now,

$$\begin{aligned}\mu_{P*R}(x,z) &= \wedge_{y\in X}\{\mu_R(x,y) \vee \mu_P(y,z)\} \\ &\le \wedge_{y\in X}\{\mu_R(x,y) \vee \mu_Q(y,z)\} \text{ as } \mu_P(y,z) \le \mu_Q(y,z) \\ &= \mu_{Q*R}(x,z)\end{aligned}$$

Similarly, we can prove that,

$$\eta_{P*R}(x,z) \le \eta_{Q*R}(x,z).$$

Again,

$$\begin{aligned}\nu_{P*R}(x,z) &= \vee_{y\in X}\{\nu_R(x,y) \wedge \nu_P(y,z)\} \\ &\ge \vee_{y\in X}\{\nu_R(x,y) \wedge \nu_Q(y,z)\} \text{ as } \nu_P(y,z) \ge \nu_Q(y,z) \\ &\ge \nu_{Q*R}(x,z)\end{aligned}$$

The property 2) can be proved in similar manner.

Theorem 4.11. Let $P, R \in PFR(B \times C), Q \in PFR(A \times B)$, then

1) $(R \vee P) * Q = (R * Q) \vee (P * Q)$.

2) $(R \wedge P) * Q = (R * Q) \wedge (P * Q)$.

Proof. 1) In order to proof 1), we have to show that

a) $\mu_{(R \vee P) * Q}(x,z) = \mu_{R * Q}(x,z) \vee \mu_{P * Q}(x,z)$.

b) $\eta_{(R \vee P) * Q}(x,z) = \eta_{R * Q}(x,z) \wedge \eta_{P * Q}(x,z)$.

c) $\nu_{(R \vee P) * Q}(x,z) = \nu_{R * Q}(x,z) \wedge \nu_{P * Q}(x,z)$.

a)
$$\begin{aligned}\mu_{(R \vee P) * Q}(x,z) &= \wedge_{y \in X}\left\{\mu_Q(x,y) \vee \left(\mu_R(y,z) \vee \mu_P(y,z)\right)\right\} \\ &= \wedge_{y \in X}\left\{\left(\mu_Q(x,y) \vee \mu_R(y,z)\right) \vee \left(\mu_Q(x,y) \vee \mu_P(y,z)\right)\right\} \\ &= \wedge_{y \in X}\left\{\left(\mu_Q(x,y) \vee \mu_R(y,z)\right)\right\} \vee \wedge_{y \in X}\left\{\left(\mu_Q(x,y) \vee \mu_P(y,z)\right)\right\} \\ &= \mu_{R * Q}(x,z) \vee \mu_{P * Q}(x,z)\end{aligned}$$

b)
$$\begin{aligned}\eta_{(R \vee P) * Q}(x,z) &= \wedge_{y \in X}\left\{\eta_Q(x,y) \wedge \left(\eta_R(y,z) \wedge \eta_P(y,z)\right)\right\} \\ &= \wedge_{y \in X}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right) \wedge \left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\} \\ &= \wedge_{y \in X}\left\{\left(\eta_Q(x,y) \wedge \eta_R(y,z)\right)\right\} \wedge \wedge_{y \in X}\left\{\left(\eta_Q(x,y) \wedge \eta_P(y,z)\right)\right\} \\ &= \eta_{R * Q}(x,z) \wedge \eta_{P * Q}(x,z)\end{aligned}$$

c)
$$\begin{aligned}\nu_{(R \vee P) * Q}(x,z) &= \vee_{y \in X}\left\{\nu_Q(x,y) \wedge \left(\nu_R(y,z) \wedge \nu_P(y,z)\right)\right\} \\ &= \vee_{y \in X}\left\{\left(\nu_Q(x,y) \wedge \nu_R(y,z)\right) \wedge \left(\nu_Q(x,y) \wedge \nu_P(y,z)\right)\right\} \\ &= \vee_{y \in X}\left\{\left(\nu_Q(x,y) \wedge \nu_R(y,z)\right)\right\} \wedge \vee_{y \in X}\left\{\left(\nu_Q(x,y) \wedge \nu_P(y,z)\right)\right\} \\ &= \nu_{R * Q}(x,z) \wedge \nu_{P * Q}(x,z);\ \forall (x,y) \in A \times B.\end{aligned}$$

The property 2) can be proved in similar manner.

Theorem 4.12. Let $Q \in PFR(A \times B), P \in PFR(B \times C), R \in PFR(C \times D)$ then $(R * P) * Q = R * (P * Q)$.

Proof. It is sufficient to prove that

a) $\mu_{(R * P) * Q}(x,z) = \mu_{R *(P * Q)}(x,z)$.

b) $\eta_{(R * P) * Q}(x,z) = \eta_{R *(P * Q)}(x,z)$.

c) $\nu_{(R * P) * Q}(x,z) = \nu_{R *(P * Q)}(x,z)$.

We have,

$$\begin{aligned}\mu_{(R * P) * Q}(x,z) &= \wedge_{y \in X}\left\{\mu_Q(x,y) \vee \mu_{(R * P)}(y,z)\right\} \\ &= \wedge_{y \in X}\left\{\mu_Q(x,y) \vee \left\{\wedge_t\left\{\mu_P(y,t) \vee \mu_R(t,z)\right\}\right\}\right\} \\ &= \wedge_{y \in X}\left\{\wedge_t\left\{\mu_Q(x,y) \vee \left\{\mu_P(y,t) \vee \mu_R(t,z)\right\}\right\}\right\} \\ &= \wedge_t\left\{\wedge_y\left\{\left[\mu_Q(x,y) \vee \mu_P(y,t)\right] \vee \mu_R(t,z)\right\}\right\} \\ &= \wedge_t\left\{\left\{\mu_{P * Q}(x,t) \vee \mu_R(t,z)\right\}\right\} \\ &= \mu_{R *(P * Q)}(x,z);\ \ \forall (x,z) \in A \times C\end{aligned}$$

$$\begin{aligned}\eta_{(R*P)*Q}(x,z) &= \wedge_{y\in X}\{\eta_Q(x,y)\wedge\eta_{R*P}(y,z)\}\\ &= \wedge_y\{\eta_Q(x,y)\wedge\{\wedge_t\{\eta_P(y,t)\wedge\eta_R(t,z)\}\}\}\\ &= \wedge_y\{\wedge_t\{\eta_Q(x,y)\wedge\{\eta_P(y,t)\wedge\eta_R(t,z)\}\}\}\\ &= \wedge_t\{\wedge_y\{[\eta_Q(x,y)\wedge\eta_P(y,t)]\wedge\eta_R(t,z)\}\}\\ &= \wedge_t\{\eta_{P\circ Q}(x,t)\wedge\eta_R(t,z)\}\\ &= \eta_{R*(P*Q)}(x,z);\ \forall(x,z)\in A\times C\end{aligned}$$

and

$$\begin{aligned}\nu_{(R*P)*Q}(x,z) &= \vee_{y\in X}\{\nu_Q(x,y)\wedge\nu_{R*P}(y,z)\}\\ &= \vee_{y\in X}\{\nu_Q(x,y)\wedge\{\vee_t\{\nu_P(y,t)\wedge\nu_R(t,z)\}\}\}\\ &= \vee_y\{\vee_t\{\nu_Q(x,y)\wedge\{\nu_P(y,t)\wedge\nu_R(t,z)\}\}\}\\ &= \vee_t\{\vee_y\{[\nu_Q(x,y)\wedge\nu_P(y,t)]\wedge\nu_R(t,z)\}\}\\ &= \vee_t\{\nu_{P*Q}(x,t)\wedge\nu_R(t,z)\}\\ &= \nu_{R*(P*Q)}(x,z),\ \forall(x,z)\in A\times C.\end{aligned}$$

5. Picture Fuzzy Relations in a Picture Fuzzy Set

Definition 5.1. The relation $R\in PFR(A\times A)$ is called:

1) Reflexive if for every $x\in X$, $\mu_R(x,x)=1$, $\eta_R(x,x)=0$ and $\nu_R(x,x)=0$.

2) Anti-reflexive if for every $x\in X$, $\mu_R(x,x)=0$, $\eta_R(x,x)=0$ and $\nu_R(x,x)=1$.

Definition 5.2. A *PFR R* on $A\in PFS(X)$ is reflexive of order (α,γ,β) if $\mu_R(x,x)=\alpha$, $\eta_R(x,x)=\gamma$ and $\nu_R(x,x)=\beta$; $\forall x\in X$ and $\alpha+\gamma+\beta\le 1$.

Theorem 5.3. Let $R\in PFR(A\times A)$, then R is reflexive iff R^c is anti-reflexive,

Proof. Let R is reflexive. Then we have,

$$\mu_R(x,x)=1,\ \eta_R(x,x)=0\ \text{and}\ \nu_R(x,x)=0.$$

From the definition of complement relation, we have

$$\mu_{R^c}(x,x)=\nu_R(x,x),\ \eta_{R^c}(x,x)=\eta_R(x,x)\ \text{and}\ \nu_{R^c}(x,x)=\mu_R(x,x)$$

which implies,

$$\mu_{R^c}(x,x)=0,\ \eta_{R^c}(x,x)=0\ \text{and}\ \nu_{R^c}(x,x)=1.$$

Thus, R^c is anti-reflexive.

Conversely, let R^c is anti-reflexive. Then

$$\mu_{R^c}(x,x)=0,\ \eta_{R^c}(x,x)=0\ \text{and}\ \nu_{R^c}(x,x)=1.$$

From the definition of complement relation, we have

$$\mu_{R^c}(x,x)=\nu_R(x,x),\ \eta_{R^c}(x,x)=\eta_R(x,x)\ \text{and}\ \nu_{R^c}(x,x)=\mu_R(x,x)$$

which implies,

$$\nu_R(x,x)=0,\quad \eta_R(x,x)=0 \text{ and } \mu_R(x,x)=1.$$

Theorem 5.4. Let $R_1 \in PFR(A\times A)$ is reflexive. Then

1) R_1^{-1} is reflexive.

2) $R_1 \vee R_2$ is reflexive for every $R_2 \in PFR(A\times A)$.

3) $R_1 \wedge R_2$ is reflexive if and only if $R_2 \in PFR(A\times A)$ is reflexive.

Proof. 1) Since R_1 is reflexive, so for every $x\in X$;

$$\mu_{R_1}(x,x)=1,\quad \eta_{R_1}(x,x)=0 \text{ and } \nu_{R_1}(x,x)=0.$$

From the definition of inverse relation, we have

$$\mu_{R_1^{-1}}(y,x)=\mu_{R_1}(x,y),\quad \eta_{R_1^{-1}}(y,x)=\eta_{R_1}(x,y),\quad \nu_{R_1^{-1}}(y,x)=\nu_{R_1}(x,y).$$

Therefore,

$\mu_{R_1^{-1}}(x,x)=\mu_{R_1}(x,x)=1$; as R_1 is reflexive.

$\eta_{R_1^{-1}}(x,x)=\eta_{R_1}(x,x)=0$; as R_1 is reflexive.

$\nu_{R_1^{-1}}(x,x)=\nu_{R_1}(x,x)=0$; as R_1 is reflexive.

Thus, R_1^{-1} is reflexive.

2)

$$\mu_{R_1\vee R_2}(x,x)=\mu_{R_1}(x,x)\vee\mu_{R_2}(x,x)=1\vee\mu_{R_2}(x,x)=1$$

$$\eta_{R_1\vee R_2}(x,x)=\eta_{R_1}(x,x)\wedge\eta_{R_2}(x,x)=0\wedge\eta_{R_2}(x,x)=0$$

$$\nu_{R_1\vee R_2}(x,x)=\nu_{R_1}(x,x)\wedge\nu_{R_2}(x,x)=0\wedge\nu_{R_2}(x,x)=0$$

Thus, $R_1 \vee R_2$ is reflexive.

3)

$$\mu_{R_1\wedge R_2}(x,x)=\mu_{R_1}(x,x)\wedge\mu_{R_2}(x,x)=1\wedge\mu_{R_2}(x,x)=\mu_{R_2}(x,x)=1$$

$$\eta_{R_1\wedge R_2}(x,x)=\eta_{R_1}(x,x)\wedge\eta_{R_2}(x,x)=1\wedge\eta_{R_2}(x,x)=\eta_{R_2}(x,x)=0$$

$$\nu_{R_1\wedge R_2}(x,x)=\nu_{R_1}(x,x)\vee\nu_{R_2}(x,x)=0\vee\nu_{R_2}(x,x)=\nu_{R_2}(x,x)=0$$

Thus, $R_1 \wedge R_2$ is reflexive.

Theorem 5.5. Let $R_1, R_2 \in PFR(A\times A)$ are reflexive. Then

1) $R_1 \cup R_2$ is reflexive,

2) $R_1 \cap R_2$ is reflexive.

Proof. 1) Since R_1 and R_2 are reflexive, so for every $x\in X$;

$\mu_{R_1}(x,x)=1$, $\eta_{R_1}(x,x)=0$ and $\nu_{R_1}(x,x)=0$ and

$\mu_{R_2}(x,x)=1$, $\eta_{R_2}(x,x)=0$ and $\nu_{R_2}(x,x)=0$.

Now,

$$\begin{aligned}\mu_{R_1\cup R_2}(x,x)&=\max\{\mu_{R_1}(x,x),\mu_{R_2}(x,x)\}\\&=\max\{1,1\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}\\&=1\end{aligned}$$

$$\begin{aligned}\eta_{R_1\cup R_2}(x,y)&=\min\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\}\\&=\min\{0,0\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}\\&=0\end{aligned}$$

$$\nu_{R_1 \cup R_2}(x,y) = \min\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\}$$
$$= \min\{0,0\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}$$
$$= 0$$

Therefore, $R_1 \cup R_2$ is reflexive.

2)

Since R_1 and R_2 are reflexive, so for every $x \in X$;

$$\mu_{R_1}(x,x) = 1,\ \eta_{R_1}(x,x) = 0 \text{ and } \nu_{R_1}(x,x) = 0 \text{ and}$$

$$\mu_{R_2}(x,x) = 1,\ \eta_{R_2}(x,x) = 0 \text{ and } \nu_{R_2}(x,x) = 0.$$

Now,

$$\mu_{R_1 \cap R_2}(x,y) = \min\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\}$$
$$= \min\{1,1\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}$$
$$= 1$$

$$\eta_{R_1 \cap R_2}(x,y) = \min\{\eta_{R_1}(x,y), \eta_{R_2}(x,y)\}$$
$$= \min\{0,0\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}$$
$$= 0$$

$$\nu_{R_1 \cap R_2}(x,y) = \max\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\}$$
$$= \max\{0,0\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive}$$
$$= 0$$

Therefore, $R_1 \cap R_2$ is reflexive.

Theorem 5.6. If R is reflexive of order (α,γ,β),then then so is R^{-1}.

Proof. Since R is reflexive of order (α,γ,β), so for every $x \in X$;

$\mu_R(x,x) = \alpha$, $\eta_R(x,x) = \gamma$ and $\nu_R(x,x) = \beta$.

From the definition of inverse relation, we have

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y),\ \eta_{R^{-1}}(y,x) = \eta_R(x,y),\ \nu_{R^{-1}}(y,x) = \nu_R(x,y).$$

Therefore,

$\mu_{R^{-1}}(x,x) = \mu_R(x,x) = \alpha$; as R is reflexive of order α.

$\eta_{R^{-1}}(x,x) = \eta_R(x,x) = \gamma$; as R is reflexive of order γ.

$\nu_{R^{-1}}(x,x) = \nu_R(x,x) = \beta$; as R is reflexive of order β.

Thus, R^{-1} is reflexive of order (α,γ,β).

Theorem 5.7. Let $R_1, R_2 \in PFR(A \times A)$ are reflexive of order (α,γ,β). Then

1) $R_1 \cup R_2$ is reflexive of order (α,γ,β).

2) $R_1 \cap R_2$ is reflexive of order (α,γ,β).

Proof. 1) Since R_1 and R_2 are reflexive of order (α,γ,β), so for every $x \in X$;

$$\mu_{R_1}(x,x) = \alpha,\ \eta_{R_1}(x,x) = \gamma \text{ and } \nu_{R_1}(x,x) = \beta \text{ and}$$

$$\mu_{R_2}(x,x) = \alpha,\ \eta_{R_2}(x,x) = \gamma \text{ and } \nu_{R_2}(x,x) = \beta.$$

Now,

$$\mu_{R_1 \cup R_2}(x,x) = \max\{\mu_{R_1}(x,x), \mu_{R_2}(x,x)\}$$
$$= \max\{\alpha,\alpha\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \alpha$$
$$= \alpha$$

$$\eta_{R_1 \cup R_2}(x,y) = \min\{\eta_{R_1}(x,y), \eta_{R_2}(x,y)\}$$
$$= \min\{\gamma,\gamma\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \gamma$$
$$= \gamma$$

$$\nu_{R_1 \cup R_2}(x,y) = \min\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\}$$
$$= \min\{\beta,\beta\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \beta$$
$$= \beta$$

Therefore, $R_1 \cup R_2$ is reflexive of order (α,γ,β).

2) Since R_1 and R_2 are reflexive of order (α,γ,β), so for every $x \in X$;

$$\mu_{R_1}(x,x) = \alpha,\ \eta_{R_1}(x,x) = \gamma \text{ and } \nu_{R_1}(x,x) = \beta \text{ and}$$

$$\mu_{R_2}(x,x) = \alpha,\ \eta_{R_2}(x,x) = \gamma \text{ and } \nu_{R_2}(x,x) = \beta.$$

Now,

$$\mu_{R_1 \cap R_2}(x,y) = \min\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\}$$
$$= \min\{\alpha,\alpha\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \alpha$$
$$= \alpha$$

$$\eta_{R_1 \cap R_2}(x,y) = \min\{\eta_{R_1}(x,y), \eta_{R_2}(x,y)\}$$
$$= \min\{\gamma,\gamma\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \gamma$$
$$= \gamma$$

$$\nu_{R_1 \cap R_2}(x,y) = \max\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\}$$
$$= \max\{\beta,\beta\} \text{ as } R_1 \text{ and } R_2 \text{ are reflexive of order } \beta$$
$$= \beta$$

Therefore, $R_1 \cap R_2$ is reflexive of order (α,γ,β).

Definition 5.8. A *PFR*, $R \in PFR(A \times A)$ is symmetric if and only if $\mu_R(x,y) = \mu_R(y,x)$, $\eta_R(x,y) = \eta_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)$; $\forall x, y \in X$.

Theorem 5.9. If R is symmetric, then so is R^{-1}.

Proof. We know,

$$\mu_{R^{-1}}(x,y) = \mu_R(y,x)$$
$$= \mu_R(x,y) \text{ since } R \text{ is symmetric}$$
$$= \mu_{R^{-1}}(y,x) \text{ by definition of } R^{-1}$$

$$\eta_{R^{-1}}(x,y) = \eta_R(y,x)$$
$$= \eta_R(x,y) \text{ since } R \text{ is symmetric}$$
$$= \eta_{R^{-1}}(y,x) \text{ by definition of } R^{-1}$$

and

$$\begin{aligned}\nu_{R^{-1}}(x,y) &= \nu_R(y,x)\\ &= \nu_R(x,y) \text{ since } R \text{ is symmetric}\\ &= \nu_{R^{-1}}(y,x) \text{ by definition of } R^{-1}.\end{aligned}$$

Theorem 5.10. R is symmetric if and only if $R = R^{-1}$.

Proof. Let R is symmetric, then

$\mu_{R^{-1}}(x,y) = \mu_R(y,x) = \mu_R(x,y)$; since R is symmetric.

$\mu_{R^{-1}}(x,y) = \eta_R(y,x) = \eta_R(x,y)$; since R is symmetric.

$\nu_{R^{-1}}(x,y) = \nu_R(y,x) = \nu_R(x,y)$; since R is symmetric.

So $R^{-1} = R$.

Conversely, let $R^{-1} = R$. Then

$$\mu_R(x,y) = \mu_{R^{-1}}(x,y) = \mu_R(y,x)$$

$$\eta_R(x,y) = \eta_{R^{-1}}(x,y) = \eta_R(y,x)$$

$$\nu_R(x,y) = \nu_{R^{-1}}(x,y) = \nu_R(y,x)$$

Hence, R is symmetric.

Theorem 5.11. Let $R_1, R_2 \in PFR(A \times A)$ are symmetric. Then

1) $R_1 \cup R_2$ is symmetric,

2) $R_1 \cap R_2$ is symmetric.

Proof. 1) Since R_1 and R_2 are symmetric, so $\forall x, y \in X$;

$\mu_{R_1}(x,y) = \mu_{R_1}(y,x)$, $\eta_{R_1}(x,y) = \eta_{R_1}(y,x)$ and $\nu_{R_1}(x,y) = \nu_{R_1}(y,x)$ and

$\mu_{R_2}(x,y) = \mu_{R_2}(y,x)$, $\eta_{R_2}(x,y) = \eta_{R_2}(y,x)$ and $\nu_{R_2}(x,y) = \nu_{R_2}(y,x)$.

Now,

$$\begin{aligned}\mu_{R_1 \cup R_2}(x,y) &= \max\left\{\mu_{R_1}(x,y), \mu_{R_2}(x,y)\right\}\\ &= \max\left\{\mu_{R_1}(y,x), \mu_{R_2}(y,x)\right\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\ &= \mu_{R_1 \cup R_2}(y,x)\end{aligned}$$

$$\begin{aligned}\eta_{R_1 \cup R_2}(x,y) &= \min\left\{\eta_{R_1}(x,y), \eta_{R_2}(x,y)\right\}\\ &= \min\left\{\eta_{R_1}(y,x), \eta_{R_2}(y,x)\right\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\ &= \eta_{R_1 \cup R_2}(x,y)\end{aligned}$$

$$\begin{aligned}\nu_{R_1 \cup R_2}(x,y) &= \min\left\{\nu_{R_1}(x,y), \nu_{R_2}(x,y)\right\}\\ &= \min\left\{\nu_{R_1}(y,x), \nu_{R_2}(y,x)\right\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\ &= \nu_{R_1 \cup R_2}(y,x)\end{aligned}$$

Therefore, $R_1 \cup R_2$ is symmetric.

2)

Since R_1 and R_2 are symmetric, so $\forall x, y \in X$;

$\mu_{R_1}(x,y) = \mu_{R_1}(y,x)$, $\eta_{R_1}(x,y) = \eta_{R_1}(y,x)$ and $\nu_{R_1}(x,y) = \nu_{R_1}(y,x)$ and

$$\mu_{R_2}(x,y)=\mu_{R_2}(y,x),\ \eta_{R_2}(x,y)=\eta_{R_2}(y,x) \text{ and } \nu_{R_2}(x,y)=\nu_{R_2}(y,x)$$

$$\begin{aligned}\mu_{R_1\cap R_2}(x,y)&=\min\{\mu_{R_1}(x,y),\mu_{R_2}(x,y)\}\\&=\min\{\mu_{R_1}(y,x),\mu_{R_2}(y,x)\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\&=\mu_{R_1\cap R_2}(y,x)\end{aligned}$$

$$\begin{aligned}\eta_{R_1\cap R_2}(x,y)&=\min\{\eta_{R_1}(x,y),\eta_{R_2}(x,y)\}\\&=\min\{\eta_{R_1}(y,x),\eta_{R_2}(y,x)\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\&=\eta_{R_1\cap R_2}(y,x)\end{aligned}$$

$$\begin{aligned}\nu_{R_1\cap R_2}(x,y)&=\max\{\nu_{R_1}(x,y),\nu_{R_2}(x,y)\}\\&=\max\{\nu_{R_1}(y,x),\nu_{R_2}(y,x)\} \text{ as } R_1 \text{ and } R_2 \text{ are symmetric}\\&=\nu_{R_1\cap R_2}(y,x)\end{aligned}$$

Therefore, $R_1\cap R_2$ is symmetric.

Definition 5.12. A *PFR* R on $A\in PFS(X)$ is said to be transitive if $R\circ R\subseteq R$.

Theorem 5.13. If R is transitive, then so is R^{-1}.

Proof. We know,

$$\begin{aligned}\mu_{R^{-1}}(x,y)&=\mu_R(y,x)\\&\ge\mu_{R\circ R}(y,x)\\&=\vee_{z\in X}\{\mu_R(y,z)\wedge\mu_R(z,x)\}\\&=\vee_{z\in X}\{\mu_{R^{-1}}(x,z)\wedge\mu_{R^{-1}}(z,y)\}\\&=\mu_{R^{-1}\circ R^{-1}}(x,y)\end{aligned}$$

$$\begin{aligned}\eta_{R^{-1}}(x,y)&=\eta_R(y,x)\\&\ge\eta_{R\circ R}(y,x)\\&=\wedge_{z\in X}\{\eta_R(y,z)\wedge\eta_R(z,x)\}\\&=\wedge_{z\in X}\{\eta_{R^{-1}}(x,z)\wedge\eta_{R^{-1}}(z,y)\}\\&=\eta_{R^{-1}\circ R^{-1}}(x,y)\end{aligned}$$

$$\begin{aligned}\nu_{R^{-1}}(x,y)&=\nu_R(y,x)\\&\le\nu_{R\circ R}(y,x)\\&=\wedge_{z\in X}\{\nu_R(y,z)\vee\nu_R(z,x)\}\\&=\wedge_{z\in X}\{\nu_{R^{-1}}(x,z)\vee\nu_{R^{-1}}(z,y)\}\\&=\nu_{R^{-1}\circ R^{-1}}(x,y)\end{aligned}$$

So, $R^{-1}\circ R^{-1}\subseteq R^{-1}$.

Hence, the theorem is proved.

Theorem 5.14. If R_1 and R_2 are two picture fuzzy transitive relations on a set *X*, then so is $R_1\cap R_2$.

Proof. Trivial.

6. Conclusions

Picture fuzzy relations play a vital role in real life situations where uncertainty arises with the positive, neutral and negative membership degrees of an element. Nowadays many researchers have become interested in working in this field because of the less complicated and rapid decision making strategies using picture fuzzy relations. In this work, picture fuzzy relations over picture fuzzy sets have been defined with examples. Several properties are described related to the picture fuzzy relations over picture fuzzy sets.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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