

Quantization of the Kinetic Energy of Deterministic Chaos

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Abstract

In previous works, the theoretical and experimental deterministic scalar kinematic structures, the theoretical and experimental deterministic vector kinematic structures, the theoretical and experimental deterministic scalar dynamic structures, and the theoretical and experimental deterministic vector dynamic structures have been developed to compute the exact solution for deterministic chaos of the exponential pulsions and oscillons that is governed by the nonstationary three-dimensional Navier-Stokes equations. To explore properties of the kinetic energy, rectangular, diagonal, and triangular summations of a matrix of the kinetic energy and general terms of various sums have been used in the current paper to develop quantization of the kinetic energy of deterministic chaos. Nested structures of a cumulative energy pulsion, an energy pulsion of propagation, an internal energy oscillon, a diagonal energy oscillon, and an external energy oscillon have been established. In turn, the energy pulsions and oscillons include group pulsions of propagation, internal group oscillons, diagonal group oscillons, and external group oscillons. Sequentially, the group pulsions and oscillons contain wave pulsions of propagation, internal wave oscillons, diagonal wave oscillons, and external wave oscillons. Consecutively, the wave pulsions and oscillons are composed of elementary pulsions of propagation, internal elementary oscillons, diagonal elementary oscillons, and external elementary oscillons. Topology, periodicity, and integral properties of the exponential pulsions and oscillons have been studied using the novel method of the inhomogeneous Fourier expansions via eigenfunctions in coordinates and time. Symbolic computations of the exact expansions have been performed using the experimental and theoretical programming in Maple. Results of the symbolic computations have been justified by probe visualizations.

Keywords

The Navier-Stokes Equations, Quantization of Kinetic Energy, Deterministic

Chaos, Elementary Pulson of Propagation, Internal Elementary Oscillon, Diagonal Elementary Oscillon, External Elementary Oscillon, Wave Pulson of Propagation, Internal Wave Oscillon, Diagonal Wave Oscillon, External Wave Oscillon, Group Pulson of Propagation, Internal Group Oscillon, Diagonal Group Oscillon, External Group Oscillon, Energy Pulson of Propagation, Internal Energy Oscillon, Diagonal Energy Oscillon, External Energy Oscillon, Cumulative Energy Pulson

1. Introduction

The initial work on the exact solution for N nonlinear internal waves governed by the nonstationary three-dimensional (3-d) Navier-Stokes equations [1] revealed an extreme sophistication of the exact solution derived with the help of experimental and theoretical programming in Maple via the Stationary Kinematic Euler-Fourier functions. To overcome this challenge, theoretical and experimental Deterministic Scalar Kinematic (DSK) structures together with theoretical and experimental Deterministic Vector Kinematic (DVK) structures have been developed in [2] to solve the Helmholtz problem for various wave systems with the help of the method of Decomposition in Invariant Structures (DIS).

Consequently, the DSK and DVK structures have been complemented by the theoretical and experimental Deterministic Scalar Dynamic (DSD) structures accompanied by the theoretical and experimental Deterministic Vector Dynamic (DVD) structures in [3] to find the exact nonstationary 3-d solution for deterministic chaos of M internal waves from J wave groups controlled by the Navier-Stokes equations. In [3], the Dirichlet problem for the Navier-Stokes equations with the help of the Helmholtz decomposition is reduced to computation of the Archimedean, Stokes, and Navier fields, whereas the kinetic energy of deterministic chaos of the exponential pulsons and oscillons turned out to be the scalar Helmholtz potential of the Navier field.

To explore properties of the kinetic energy, rectangular, diagonal, and triangular summations of the matrix of the kinetic energy and the general terms of various sums have been used in the current paper to develop quantization of the kinetic energy of deterministic chaos. Nested structures of the cumulative energy pulson, the energy pulson of propagation, the internal energy oscillon, the diagonal energy oscillon, and the external energy oscillon have been established. In turn, the energy pulsons and oscillons include the group pulsons of propagation, the internal group oscillons, the diagonal group oscillons, and the external group oscillons. Sequentially, the group pulsons and oscillons contain the wave pulsons of propagation, the internal wave oscillons, the diagonal wave oscillons, and the external wave oscillons. Consecutively, the wave pulsons and oscillons are composed of the elementary pulsons of propagation, the internal elementary oscillons, the diagonal elementary oscillons, and the external elementary oscillons.

Topology, periodicity, and integral properties of the exponential pulsons and

oscillons are studied with the help of the inhomogeneous Fourier expansions via eigenfunctions in coordinates and time. This novel method allows to replace the (homogeneous) Fourier series including an unbounded number of terms with the inhomogeneous Fourier expansions containing a bounded number of terms that deliver exact solutions. Symbolic computations of the exact expansions have been performed using the experimental and theoretical programming in Maple since the functional bases include 16 inhomogeneous eigenfunctions for each variable. Results of the symbolic computations have justified by probe visualizations in x -, y -, t -directions.

The contents of this paper are following. Quantization of the kinetic energy of deterministic chaos of the exponential oscillons and pulsons is developed in **Section 2**, using rectangular, diagonal, and triangular summations of the matrix of the kinetic energy and the general terms of various sums. The section concludes with a summary of nested structures of energy, group, wave, elementary oscillons and pulsons.

Section 3 deals with the elementary oscillons of propagation, the elementary pulsons of propagation, the internal elementary oscillons, the diagonal elementary oscillons, and the external elementary oscillons. Eigenfunctions of the inhomogeneous Fourier expansions in x -, y -, t - are constructed in this section. The inhomogeneous Fourier expansions, periods, and averages of elementary oscillons and pulsons are computed and illustrated.

The wave oscillons of propagation, the wave pulsons of propagation, the internal wave oscillons, the diagonal wave oscillons, and the external wave oscillons are computed in **Section 4** together with their periods and averages. The wave pulsons and oscillons are also displayed in the x -, y -, t -directions. In **Section 5**, we consider topology, periodicity, integral properties, and visualizations of the group oscillons of propagation, the group pulsons of propagation, the internal group oscillons, the diagonal group oscillons, and the external group oscillons.

In **Section 6**, the inhomogeneous Fourier expansions of the energy pulson of propagation, the internal energy oscillon, the diagonal energy oscillon, the external energy oscillon, and the cumulative energy pulson are developed and visualized. **Section 7** contains a short discussion of main results and further developments.

2. Quantization of the Kinetic Energy

2.1. Decomposition of the Matrix of the Kinetic Energy

The kinetic energy K_e of the deterministic chaos of internal waves in a Newtonian fluid with a constant density ρ_c and a global velocity field \mathbf{u} is defined by

$$K_e = \frac{\rho_c}{2} \mathbf{u} \cdot \mathbf{u}, \quad (1)$$

where the global velocity field is formed by velocity fields of four wave groups

$$\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c + \mathbf{u}_d. \tag{2}$$

Substitution of (2) in (1) yields the kinetic energy of the deterministic chaos of the exponential oscillons and pulsions (118) of [3] in terms of the velocity fields of four wave groups

$$K_e = \frac{\rho_c}{2} (\mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c + \mathbf{u}_d) \cdot (\mathbf{u}_a + \mathbf{u}_b + \mathbf{u}_c + \mathbf{u}_d). \tag{3}$$

For clarification of summation, we define a matrix of the kinetic energy \mathbf{M}_e by

$$K_e = \frac{\rho_c}{2} \{ \mathbf{M}_e \}, \mathbf{M}_e = \begin{bmatrix} \mathbf{u}_a \cdot \mathbf{u}_a & \mathbf{u}_a \cdot \mathbf{u}_b & \mathbf{u}_a \cdot \mathbf{u}_c & \mathbf{u}_a \cdot \mathbf{u}_d \\ \mathbf{u}_b \cdot \mathbf{u}_a & \mathbf{u}_b \cdot \mathbf{u}_b & \mathbf{u}_b \cdot \mathbf{u}_c & \mathbf{u}_b \cdot \mathbf{u}_d \\ \mathbf{u}_c \cdot \mathbf{u}_a & \mathbf{u}_c \cdot \mathbf{u}_b & \mathbf{u}_c \cdot \mathbf{u}_c & \mathbf{u}_c \cdot \mathbf{u}_d \\ \mathbf{u}_d \cdot \mathbf{u}_a & \mathbf{u}_d \cdot \mathbf{u}_b & \mathbf{u}_d \cdot \mathbf{u}_c & \mathbf{u}_d \cdot \mathbf{u}_d \end{bmatrix}, \tag{4}$$

where the braces notation $\{ \mathbf{M} \}$ denotes the rectangular summation of all elements of the summation matrix \mathbf{M} .

Since the velocity fields of four wave groups are expanded in the weighted experimental DVK structures $[\mathbf{a}_m, \mathbf{b}_m, \mathbf{c}_m, \mathbf{d}_m]$ (20) of [3],

$$\begin{aligned} \mathbf{u}_a &= \sum_{m=1}^M \mathbf{a}_m = \sum_{n=1}^M \mathbf{a}_n, \quad \mathbf{u}_b = \sum_{m=1}^M \mathbf{b}_m = \sum_{n=1}^M \mathbf{b}_n, \\ \mathbf{u}_c &= \sum_{m=1}^M \mathbf{c}_m = \sum_{n=1}^M \mathbf{c}_n, \quad \mathbf{u}_d = \sum_{m=1}^M \mathbf{d}_m = \sum_{n=1}^M \mathbf{d}_n, \end{aligned} \tag{5}$$

matrix (4) takes the following form:

$$\mathbf{M}_e = \begin{bmatrix} \{ \mathbf{a}_m \cdot \mathbf{a}_n \} & \{ \mathbf{a}_m \cdot \mathbf{b}_n \} & \{ \mathbf{a}_m \cdot \mathbf{c}_n \} & \{ \mathbf{a}_m \cdot \mathbf{d}_n \} \\ \{ \mathbf{b}_m \cdot \mathbf{a}_n \} & \{ \mathbf{b}_m \cdot \mathbf{b}_n \} & \{ \mathbf{b}_m \cdot \mathbf{c}_n \} & \{ \mathbf{b}_m \cdot \mathbf{d}_n \} \\ \{ \mathbf{c}_m \cdot \mathbf{a}_n \} & \{ \mathbf{c}_m \cdot \mathbf{b}_n \} & \{ \mathbf{c}_m \cdot \mathbf{c}_n \} & \{ \mathbf{c}_m \cdot \mathbf{d}_n \} \\ \{ \mathbf{d}_m \cdot \mathbf{a}_n \} & \{ \mathbf{d}_m \cdot \mathbf{b}_n \} & \{ \mathbf{d}_m \cdot \mathbf{c}_n \} & \{ \mathbf{d}_m \cdot \mathbf{d}_n \} \end{bmatrix}, \tag{6}$$

where the summation braces $\{ \mathbf{a}_m \cdot \mathbf{b}_n \}$ signify the rectangular summation of all elements of a summation matrix with the general term $\mathbf{a}_m \cdot \mathbf{b}_n$ for $n = 1, 2, \dots, M$ and $m = 1, 2, \dots, M$, *i.e.*

$$\{ \mathbf{a}_m \cdot \mathbf{b}_n \} = \sum_{m=1}^M \sum_{n=1}^M \mathbf{a}_m \cdot \mathbf{b}_n \tag{7}$$

since each wave group is composed of M waves.

Primarily, we decompose \mathbf{M}_e as follows

$$\mathbf{M}_e = \mathbf{M}_d + \mathbf{M}_{u,l}. \tag{8}$$

Here,

$$\mathbf{M}_d = \begin{bmatrix} \{ \mathbf{a}_m \cdot \mathbf{a}_n \} & 0 & 0 & 0 \\ 0 & \{ \mathbf{b}_m \cdot \mathbf{b}_n \} & 0 & 0 \\ 0 & 0 & \{ \mathbf{c}_m \cdot \mathbf{c}_n \} & 0 \\ 0 & 0 & 0 & \{ \mathbf{d}_m \cdot \mathbf{d}_n \} \end{bmatrix} \tag{9}$$

is a diagonal matrix, which includes all diagonal elements of \mathbf{M}_e , and

$$\mathbf{M}_{u,l} = \begin{bmatrix} 0 & \{\mathbf{a}_m \cdot \mathbf{b}_n\} & \{\mathbf{a}_m \cdot \mathbf{c}_n\} & \{\mathbf{a}_m \cdot \mathbf{d}_n\} \\ \{\mathbf{b}_m \cdot \mathbf{a}_n\} & 0 & \{\mathbf{b}_m \cdot \mathbf{c}_n\} & \{\mathbf{b}_m \cdot \mathbf{d}_n\} \\ \{\mathbf{c}_m \cdot \mathbf{a}_n\} & \{\mathbf{c}_m \cdot \mathbf{b}_n\} & 0 & \{\mathbf{c}_m \cdot \mathbf{d}_n\} \\ \{\mathbf{d}_m \cdot \mathbf{a}_n\} & \{\mathbf{d}_m \cdot \mathbf{b}_n\} & \{\mathbf{d}_m \cdot \mathbf{c}_n\} & 0 \end{bmatrix} \quad (10)$$

is a complementary matrix, which is composed of the upper and lower triangular matrices of \mathbf{M}_e .

The kinetic energy is correspondingly decomposed as

$$K_e = K_d + K_{u,l}, \quad (11)$$

where the first sum K_d is produced by the elements of \mathbf{M}_d :

$$K_d = K_{a,a} + K_{b,b} + K_{c,c} + K_{d,d}, \quad (12)$$

and the second sum $K_{u,l}$ by the elements of $\mathbf{M}_{u,l}$:

$$K_{u,l} = K_{a,b} + K_{a,c} + K_{a,d} + K_{b,c} + K_{b,d} + K_{c,d}. \quad (13)$$

In (12)-(13), the constituents of K_d are

$$\begin{aligned} K_{a,a} &= \frac{\rho_c}{2} \{\mathbf{a}_m \cdot \mathbf{a}_n\}, & K_{b,b} &= \frac{\rho_c}{2} \{\mathbf{b}_m \cdot \mathbf{b}_n\}, \\ K_{c,c} &= \frac{\rho_c}{2} \{\mathbf{c}_m \cdot \mathbf{c}_n\}, & K_{d,d} &= \frac{\rho_c}{2} \{\mathbf{d}_m \cdot \mathbf{d}_n\}, \end{aligned} \quad (14)$$

and the constituents of $K_{u,l}$ are

$$\begin{aligned} K_{a,b} &= \frac{\rho_c}{2} \{\mathbf{a}_m \cdot \mathbf{b}_n + \mathbf{b}_m \cdot \mathbf{a}_n\}, & K_{a,c} &= \frac{\rho_c}{2} \{\mathbf{a}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{a}_n\}, \\ K_{a,d} &= \frac{\rho_c}{2} \{\mathbf{a}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{a}_n\}, & K_{b,c} &= \frac{\rho_c}{2} \{\mathbf{b}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{b}_n\}, \\ K_{b,d} &= \frac{\rho_c}{2} \{\mathbf{b}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{b}_n\}, & K_{c,d} &= \frac{\rho_c}{2} \{\mathbf{c}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{c}_n\}. \end{aligned} \quad (15)$$

Secondly, we expand all sums into internal sums with $n = m$, which correspond to internal interaction of elementary oscillons of propagation from the m th family, and external sums with $n \neq m$, which describe external interaction of elementary oscillons of propagation from the m th and n th families (see Section 2.6).

The summation matrix of the diagonal constituent $K_{a,a}$ of K_d

$$\mathbf{M}_{a,m,a,n} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_m & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_n & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_m \cdot \mathbf{a}_1 & \cdots & \mathbf{a}_m \cdot \mathbf{a}_m & \cdots & \mathbf{a}_m \cdot \mathbf{a}_n & \cdots & \mathbf{a}_m \cdot \mathbf{a}_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_n \cdot \mathbf{a}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{a}_m & \cdots & \mathbf{a}_n \cdot \mathbf{a}_n & \cdots & \mathbf{a}_n \cdot \mathbf{a}_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{a}_M \cdot \mathbf{a}_1 & \cdots & \mathbf{a}_M \cdot \mathbf{a}_m & \cdots & \mathbf{a}_M \cdot \mathbf{a}_n & \cdots & \mathbf{a}_M \cdot \mathbf{a}_M \end{bmatrix} \quad (16)$$

due to the commutative property of the dot products

$$\mathbf{a}_n \cdot \mathbf{a}_m = \mathbf{a}_m \cdot \mathbf{a}_n \quad (17)$$

yields the following reduction of the rectangular summation to the diagonal and triangular summations:

$$\{a_m \cdot a_n\} = \sum_{m=1}^M a_m \cdot a_m + 2 \sum_{m=1}^{M-1} \sum_{n=m+1}^M a_m \cdot a_n. \tag{18}$$

Similar reductions of the remaining diagonal constituents $K_{b,b}, K_{c,c}, K_{d,d}$ yield

$$\begin{aligned} \{b_m \cdot b_n\} &= \sum_{m=1}^M b_m \cdot b_m + 2 \sum_{m=1}^{M-1} \sum_{n=m+1}^M b_m \cdot b_n, \\ \{c_m \cdot c_n\} &= \sum_{m=1}^M c_m \cdot c_m + 2 \sum_{m=1}^{M-1} \sum_{n=m+1}^M c_m \cdot c_n, \\ \{d_m \cdot d_n\} &= \sum_{m=1}^M d_m \cdot d_m + 2 \sum_{m=1}^{M-1} \sum_{n=m+1}^M d_m \cdot d_n. \end{aligned} \tag{19}$$

The summation matrices of the non-diagonal constituent $K_{a,b}$ of $K_{a,l}$

$$M_{a,m,b,n} = \begin{bmatrix} a_1 \cdot b_1 & \cdots & a_1 \cdot b_m & \cdots & a_1 \cdot b_n & \cdots & a_1 \cdot b_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_m \cdot b_1 & \cdots & a_m \cdot b_m & \cdots & a_m \cdot b_n & \cdots & a_m \cdot b_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_n \cdot b_1 & \cdots & a_n \cdot b_m & \cdots & a_n \cdot b_n & \cdots & a_n \cdot b_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_M \cdot b_1 & \cdots & a_M \cdot b_m & \cdots & a_M \cdot b_n & \cdots & a_M \cdot b_M \end{bmatrix} \tag{20}$$

and

$$M_{b,m,a,n} = \begin{bmatrix} b_1 \cdot a_1 & \cdots & b_1 \cdot a_m & \cdots & b_1 \cdot a_n & \cdots & b_1 \cdot a_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_m \cdot a_1 & \cdots & b_m \cdot a_m & \cdots & b_m \cdot a_n & \cdots & b_m \cdot a_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_n \cdot a_1 & \cdots & b_n \cdot a_m & \cdots & b_n \cdot a_n & \cdots & b_n \cdot a_M \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_M \cdot a_1 & \cdots & b_M \cdot a_m & \cdots & b_M \cdot a_n & \cdots & b_M \cdot a_M \end{bmatrix} \tag{21}$$

because of the commutative properties of the dot products:

$$\begin{aligned} b_m \cdot a_m &= a_m \cdot b_m, \\ a_n \cdot b_m + b_n \cdot a_m &= a_m \cdot b_n + b_m \cdot a_n, \end{aligned} \tag{22}$$

produce the following reduction of the rectangular summation to the diagonal and triangular summations:

$$\frac{1}{2} \{a_m \cdot b_n + b_m \cdot a_n\} = \sum_{m=1}^M a_m \cdot b_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (a_m \cdot b_n + b_m \cdot a_n). \tag{23}$$

If $n = m$, the asymmetric reduction (23) is converted into the symmetric reduction (18).

Proceeding the same way for the rest of the summation matrices of the non-diagonal constituents $K_{a,c}, K_{a,d}, K_{b,c}, K_{b,d}, K_{c,d}$ returns

$$\begin{aligned}
\frac{1}{2}\{\mathbf{a}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{a}_n\} &= \sum_{m=1}^M \mathbf{a}_m \cdot \mathbf{c}_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (\mathbf{a}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{a}_n), \\
\frac{1}{2}\{\mathbf{a}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{a}_n\} &= \sum_{m=1}^M \mathbf{a}_m \cdot \mathbf{d}_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (\mathbf{a}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{a}_n), \\
\frac{1}{2}\{\mathbf{b}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{b}_n\} &= \sum_{m=1}^M \mathbf{b}_m \cdot \mathbf{c}_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (\mathbf{b}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{b}_n), \\
\frac{1}{2}\{\mathbf{b}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{b}_n\} &= \sum_{m=1}^M \mathbf{b}_m \cdot \mathbf{d}_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (\mathbf{b}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{b}_n), \\
\frac{1}{2}\{\mathbf{c}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{c}_n\} &= \sum_{m=1}^M \mathbf{c}_m \cdot \mathbf{d}_m + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (\mathbf{c}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{c}_n).
\end{aligned} \tag{24}$$

2.2. Wave, Group, and Energy Pulsons of Propagation

Thirdly, we define wave, group, and energy pulsons and oscillons via various combinations of dot products of the DVK structures and express the dot products in terms of the weighted experimental DSD structures (29) of [3].

General terms of the internal sums of (18)-(19) produce wave pulsons of propagation

$$\begin{aligned}
K_{w,a,m,a,m} &= \frac{\rho_c}{2} \mathbf{a}_m \cdot \mathbf{a}_m, & K_{w,b,m,b,m} &= \frac{\rho_c}{2} \mathbf{b}_m \cdot \mathbf{b}_m, \\
K_{w,c,m,c,m} &= \frac{\rho_c}{2} \mathbf{c}_m \cdot \mathbf{c}_m, & K_{w,d,m,d,m} &= \frac{\rho_c}{2} \mathbf{d}_m \cdot \mathbf{d}_m.
\end{aligned} \tag{25}$$

Since the DVK structures are related with the DSK structures by (20) of [3]

$$\begin{aligned}
\mathbf{a}_m &= \left[+\kappa_m \mathbf{b}_m, +\lambda_m \mathbf{c}_m, (-1)^\eta \mu_m \mathbf{a}_m \right], & \mathbf{b}_m &= \left[-\kappa_m \mathbf{a}_m, +\lambda_m \mathbf{d}_m, (-1)^\eta \mu_m \mathbf{b}_m \right], \\
\mathbf{c}_m &= \left[+\kappa_m \mathbf{d}_m, -\lambda_m \mathbf{a}_m, (-1)^\eta \mu_m \mathbf{c}_m \right], & \mathbf{d}_m &= \left[-\kappa_m \mathbf{c}_m, -\lambda_m \mathbf{b}_m, (-1)^\eta \mu_m \mathbf{d}_m \right],
\end{aligned} \tag{26}$$

the wave pulsons of propagation in the DSD structures take the following form:

$$\begin{aligned}
K_{w,a,m,a,m} &= \frac{\rho_c}{2} (\mu_m^2 a_m^2 + \kappa_m^2 b_m^2 + \lambda_m^2 c_m^2), & K_{w,b,m,b,m} &= \frac{\rho_c}{2} (\kappa_m^2 a_m^2 + \mu_m^2 b_m^2 + \lambda_m^2 d_m^2), \\
K_{w,c,m,c,m} &= \frac{\rho_c}{2} (\lambda_m^2 a_m^2 + \mu_m^2 c_m^2 + \kappa_m^2 d_m^2), & K_{w,d,m,d,m} &= \frac{\rho_c}{2} (\lambda_m^2 b_m^2 + \kappa_m^2 c_m^2 + \mu_m^2 d_m^2).
\end{aligned} \tag{27}$$

A superposition of a group of the wave pulsons of propagation

$$\begin{aligned}
K_{g,i,m,i,m} &= K_{w,a,m,a,m} + K_{w,b,m,b,m} + K_{w,c,m,c,m} + K_{w,d,m,d,m} \\
&= \rho_c \mu_m^2 (a_m^2 + b_m^2 + c_m^2 + d_m^2)
\end{aligned} \tag{28}$$

is termed a group pulson of propagation $K_{g,i,m,i,m}$. Here, $K_{g,i,m,i,m}$ is simplified by the Pythagorean identity for the wave numbers

$$\kappa_m^2 + \lambda_m^2 = \mu_m^2. \tag{29}$$

Eventually, the diagonal summation of all group pulsons of propagation results in an energy pulson of propagation

$$K_{e,i,m,i,m} = \sum_{m=1}^M K_{g,i,m,i,m}. \tag{30}$$

which shows a cumulative energy of all group pulsons of propagation.

2.3. Wave, Group, and Energy Oscillons of Diagonal External Interaction

General terms of the external sums of (18)-(19) generate wave oscillons of diagonal external interaction (diagonal wave oscillons for brevity)

$$\begin{aligned}
 K_{w,a,m,a,n} &= \frac{\rho_c}{2} \mathbf{a}_m \cdot \mathbf{a}_n, & K_{w,b,m,b,n} &= \frac{\rho_c}{2} \mathbf{b}_m \cdot \mathbf{b}_n, \\
 K_{w,c,m,c,n} &= \frac{\rho_c}{2} \mathbf{c}_m \cdot \mathbf{c}_n, & K_{w,d,m,d,n} &= \frac{\rho_c}{2} \mathbf{d}_m \cdot \mathbf{d}_n.
 \end{aligned}
 \tag{31}$$

Using (26) and Equation (26) with $m = n$, we obtain the diagonal wave oscillons in the DSD structures

$$\begin{aligned}
 K_{w,a,m,a,m} &= \rho_c (\mu_m \mu_n a_m a_n + \kappa_m \kappa_n b_m b_n + \lambda_m \lambda_n c_m c_n), \\
 K_{w,b,m,b,m} &= \rho_c (\kappa_m \kappa_n a_m a_n + \mu_m \mu_n b_m b_n + \lambda_m \lambda_n d_m d_n), \\
 K_{w,c,m,c,m} &= \rho_c (\lambda_m \lambda_n a_m a_n + \mu_m \mu_n c_m c_n + \kappa_m \kappa_n d_m d_n), \\
 K_{w,d,m,d,m} &= \rho_c (\lambda_m \lambda_n b_m b_n + \kappa_m \kappa_n c_m c_n + \mu_m \mu_n d_m d_n).
 \end{aligned}
 \tag{32}$$

Summation of (32) yields a diagonal group oscillon

$$\begin{aligned}
 K_{g,i,m,i,n} &= K_{w,a,m,a,n} + K_{w,b,m,b,n} + K_{w,c,m,c,n} + K_{w,d,m,d,n} \\
 &= \rho_c M_{m,n} (a_m a_n + b_m b_n + c_m c_n + d_m d_n),
 \end{aligned}
 \tag{33}$$

where an amplitude produced by wave numbers

$$M_{m,n} = \kappa_m \kappa_n + \lambda_m \lambda_n + \mu_m \mu_n.
 \tag{34}$$

The triangular summation of the diagonal group oscillons results in a diagonal energy oscillon

$$K_{e,i,m,i,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^M K_{g,i,m,i,n},
 \tag{35}$$

which gives a cumulative energy of all diagonal group oscillons.

So, summation of constituents $K_{a,a}, K_{b,b}, K_{c,c}, K_{d,d}$ (12) of the kinetic energy is completed with the following result:

$$K_d = K_{e,i,m,i,m} + K_{e,i,m,i,n}.
 \tag{36}$$

If $n = m$, then the diagonal wave oscillons (32) are converted into the doubled wave pulsions of propagation (27) and the diagonal group oscillon (33) becomes equal to the doubled group pulsion of propagation (28).

2.4. Wave, Group, and Energy Oscillons of Internal Interaction

General terms of the internal sums of (23)-(24) correspond to wave oscillons of internal interaction (internal wave oscillons for brevity)

$$\begin{aligned}
 K_{w,a,m,b,m} &= \frac{\rho_c}{2} \mathbf{a}_m \cdot \mathbf{b}_m, & K_{w,a,m,c,m} &= \frac{\rho_c}{2} \mathbf{a}_m \cdot \mathbf{c}_m, & K_{w,a,m,d,m} &= \frac{\rho_c}{2} \mathbf{a}_m \cdot \mathbf{d}_m, \\
 K_{w,b,m,c,m} &= \frac{\rho_c}{2} \mathbf{b}_m \cdot \mathbf{c}_m, & K_{w,b,m,d,m} &= \frac{\rho_c}{2} \mathbf{b}_m \cdot \mathbf{d}_m, & K_{w,c,m,d,m} &= \frac{\rho_c}{2} \mathbf{c}_m \cdot \mathbf{d}_m.
 \end{aligned}
 \tag{37}$$

In the DSD structures, the internal wave oscillons become

$$\begin{aligned}
 K_{w,a,m,b,m} &= \rho_c \lambda_m^2 (a_m b_m + c_m d_m) = +K_{w,c,m,d,m}, \\
 K_{w,a,m,c,m} &= \rho_c \kappa_m^2 (a_m c_m + b_m d_m) = +K_{w,b,m,d,m}, \\
 K_{w,a,m,d,m} &= \rho_c \mu_m^2 (a_m d_m - b_m c_m) = -K_{w,b,m,c,m}.
 \end{aligned}
 \tag{38}$$

Adding the internal wave oscillons, we get an internal group oscillon

$$\begin{aligned}
 K_{g,i,m,j,m} &= K_{w,a,m,b,m} + K_{w,a,m,c,m} + K_{w,a,m,d,m} + K_{w,b,m,c,m} + K_{w,b,m,d,m} + K_{w,c,m,d,m} \\
 &= 2\rho_c (\lambda_m^2 (a_m b_m + c_m d_m) + \kappa_m^2 (a_m c_m + b_m d_m)).
 \end{aligned}
 \tag{39}$$

The diagonal summation of the internal group oscillon results in an internal energy oscillon

$$K_{e,i,m,j,m} = \sum_{m=1}^M K_{g,i,m,j,m},
 \tag{40}$$

which returns a cumulative energy of all internal group oscillons.

2.5. Wave, Group, and Energy Oscillons of Non-Diagonal External Interaction

General terms of the external sums of (23)-(24) correspond to wave oscillons of non-diagonal external interaction (external wave oscillons for brevity)

$$\begin{aligned}
 K_{w,a,m,b,n} &= \rho_c (\mathbf{a}_m \cdot \mathbf{b}_n + \mathbf{b}_m \cdot \mathbf{a}_n), \quad K_{w,a,m,c,n} = \rho_c (\mathbf{a}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{a}_n), \\
 K_{w,a,m,d,n} &= \rho_c (\mathbf{a}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{a}_n), \quad K_{w,b,m,c,n} = \rho_c (\mathbf{b}_m \cdot \mathbf{c}_n + \mathbf{c}_m \cdot \mathbf{b}_n), \\
 K_{w,b,m,d,n} &= \rho_c (\mathbf{b}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{b}_n), \quad K_{w,c,m,d,n} = \rho_c (\mathbf{c}_m \cdot \mathbf{d}_n + \mathbf{d}_m \cdot \mathbf{c}_n).
 \end{aligned}
 \tag{41}$$

Using (26) and Equation (26) with $m = n$, we compute the external wave oscillons via the DSD structures

$$\begin{aligned}
 K_{w,a,m,b,n} &= \rho_c (-(\kappa_m \kappa_n - \mu_m \mu_n)(a_m b_n + b_m a_n) + \lambda_m \lambda_n (c_m d_n + d_m c_n)), \\
 K_{w,c,m,d,n} &= \rho_c (+\lambda_m \lambda_n (a_m b_n + b_m a_n) - (\kappa_m \kappa_n - \mu_m \mu_n)(c_m d_n + d_m c_n)), \\
 K_{w,a,m,c,n} &= \rho_c (-(\lambda_m \lambda_n - \mu_m \mu_n)(a_m c_n + c_m a_n) + \kappa_m \kappa_n (b_m d_n + d_m b_n)), \\
 K_{w,b,m,d,n} &= \rho_c (+\kappa_m \kappa_n (a_m c_n + c_m a_n) - (\lambda_m \lambda_n - \mu_m \mu_n)(b_m d_n + d_m b_n)), \\
 K_{w,a,m,d,n} &= \rho_c (+\mu_m \mu_n (a_m d_n + d_m a_n) - (\kappa_m \kappa_n + \lambda_m \lambda_n)(b_m c_n + c_m b_n)), \\
 K_{w,b,m,c,n} &= \rho_c (-(\kappa_m \kappa_n + \lambda_m \lambda_n)(a_m d_n + d_m a_n) + \mu_m \mu_n (b_m c_n + c_m b_n)).
 \end{aligned}
 \tag{42}$$

An external group oscillon takes the following form:

$$\begin{aligned}
 K_{g,i,m,j,n} &= K_{w,a,m,b,n} + K_{w,a,m,c,n} + K_{w,a,m,d,n} + K_{w,b,m,c,n} + K_{w,b,m,d,n} + K_{w,c,m,d,n} \\
 &= \rho_c (\Lambda_{m,n} (a_m b_n + b_m a_n + c_m d_n + d_m c_n) \\
 &\quad + K_{m,n} (a_m c_n + c_m a_n + b_m d_n + d_m b_n) \\
 &\quad - N_{m,n} (a_m d_n + d_m a_n + b_m c_n + c_m b_n)).
 \end{aligned}
 \tag{43}$$

where amplitudes generated by the wave numbers

$$\begin{aligned}
 K_{m,n} &= \kappa_m \kappa_n - \lambda_m \lambda_n + \mu_m \mu_n, \quad \Lambda_{m,n} = -\kappa_m \kappa_n + \lambda_m \lambda_n + \mu_m \mu_n, \\
 M_{m,n} &= \kappa_m \kappa_n + \lambda_m \lambda_n + \mu_m \mu_n, \quad N_{m,n} = +\kappa_m \kappa_n + \lambda_m \lambda_n - \mu_m \mu_n.
 \end{aligned}
 \tag{44}$$

We then imply the triangular summation of the external group oscillons to

find an external energy oscillon

$$K_{e,i,m,j,n} = \sum_{m=1}^{M-1} \sum_{n=m+1}^M K_{g,i,m,j,n}, \tag{45}$$

which demonstrates a cumulative energy of all external group oscillons.

Thus, summation of constituents $K_{a,b}, K_{a,c}, K_{a,d}, K_{b,c}, K_{b,d}, K_{c,d}$ (13) of the kinetic energy is finished as follows:

$$K_{u,l} = K_{e,i,m,j,m} + K_{e,i,m,j,n}. \tag{46}$$

If $n = m$, then the external wave oscillons (42) are transformed into the doubled internal wave oscillons (38) and the external group oscillon (43) becomes equal to the doubled internal group oscillon (39).

2.6. Elementary Oscillons and Pulsons

Members of the m th family of elementary oscillons of propagation of the velocity potential

$$K_{o,a,m} = a_m, K_{o,b,m} = b_m, K_{o,c,m} = c_m, K_{o,d,m} = d_m \tag{47}$$

are defined via the weighted experimental DSK structures (1) of [3]

$$\begin{aligned} a_m &= +Av_m sse_m + Bv_m cse_m + Cv_m sce_m + Dv_m cce_m, \\ b_m &= -Bv_m sse_m + Av_m cse_m - Dv_m sce_m + Cv_m cce_m, \\ c_m &= -Cv_m sse_m - Dv_m cse_m + Av_m sce_m + Bv_m cce_m, \\ d_m &= +Dv_m sse_m - Cv_m cse_m - Bv_m sce_m + Av_m cce_m, \end{aligned} \tag{48}$$

where $m = 1, 2, \dots, M$, $[Av_m, Bv_m, Cv_m, Dv_m]$ are functional amplitudes, 3-d DSK functions $[sse_m, cse_m, sce_m, cce_m]$ are products

$$sse_m = sx_m sy_m ez_m, cse_m = cx_m sy_m ez_m, sce_m = sx_m cy_m ez_m, cce_m = cx_m cy_m ez_m \tag{49}$$

of the following 1-d (one-dimensional) DSK functions $[sx_m, cx_m, sy_m, cy_m]$ and an exponential function ez_m :

$$\begin{aligned} sx_m &= \sin(\kappa_m X_m), cx_m = \cos(\kappa_m X_m), \\ sy_m &= \sin(\lambda_m Y_m), cy_m = \cos(\lambda_m Y_m), ez_m = \exp\left((-1)^\eta \mu_m z\right), \end{aligned} \tag{50}$$

$$X_m = x - U_m t + X_{m,0}, Y_m = y - V_m t + Y_{m,0}. \tag{51}$$

In Equations (47)-(51), (x, y, z) are the Cartesian coordinates of a motionless frame of reference, t is time, $[X_m, Y_m, z]$ are the Cartesian coordinates of a frame of reference moving with the m th elementary oscillon of propagation, $[U_m, V_m, 0]$ is the celerity of propagation, and $[X_{m,0}, Y_{m,0}]$ are reference values of $[X_m, Y_m]$ at $t = 0$, $x = 0$, $y = 0$. A sign parameter $\eta = 0$ for $z < 0$ and $\eta = 1$ for $z > 0$.

A wave oscillon of propagation of the velocity potential

$$K_{w,a,m} = K_{o,a,m} + K_{o,b,m} + K_{o,c,m} + K_{o,d,m} = a_m + b_m + c_m + d_m \tag{52}$$

consists of the elementary oscillons of propagation.

Similarly, a group oscillon of propagation of the velocity potential

$$K_{g,i,m} = \sum_{m=1}^M K_{w,i,m} \tag{53}$$

is composed of the wave oscillons of propagation.

The wave pulsons and oscillons and the group pulsons and oscillons of the kinetic energy are constructed of elementary pulsons and oscillons of the kinetic energy.

Four elementary pulsons of propagation in the DSD structures for $m = 1, 2, \dots, M$ are defined by

$$K_{p,a,m,a,m} = \frac{\rho_c}{2} a_m^2, K_{p,b,m,b,m} = \frac{\rho_c}{2} b_m^2, K_{p,c,m,c,m} = \frac{\rho_c}{2} c_m^2, K_{p,d,m,d,m} = \frac{\rho_c}{2} d_m^2. \tag{54}$$

The wave pulsons of propagation (27) and the group pulson of propagation (28) then become the following superpositions of the elementary pulsons of propagation:

$$\begin{aligned} K_{w,a,m,a,m} &= \mu_m^2 K_{p,a,m,a,m} + \kappa_m^2 K_{p,b,m,b,m} + \lambda_m^2 K_{p,c,m,c,m}, \\ K_{w,b,m,b,m} &= \kappa_m^2 K_{p,a,m,a,m} + \mu_m^2 K_{p,b,m,b,m} + \lambda_m^2 K_{p,d,m,d,m}, \\ K_{w,c,m,c,m} &= \lambda_m^2 K_{p,a,m,a,m} + \mu_m^2 K_{p,c,m,c,m} + \kappa_m^2 K_{p,d,m,d,m}, \\ K_{w,d,m,d,m} &= \lambda_m^2 K_{p,b,m,b,m} + \kappa_m^2 K_{p,c,m,c,m} + \mu_m^2 K_{p,d,m,d,m}, \end{aligned} \tag{55}$$

and

$$K_{g,i,m,i,m} = 2\mu_m^2 (K_{p,a,m,a,m} + K_{p,b,m,b,m} + K_{p,c,m,c,m} + K_{p,d,m,d,m}). \tag{56}$$

There are six elementary oscillons of internal interaction (internal elementary oscillons for brevity) between members of the m th family of the elementary oscillons of propagation for $m = 1, 2, \dots, M$

$$\begin{aligned} K_{o,a,m,b,m} &= \rho_c a_m b_m, K_{o,a,m,c,m} = \rho_c a_m c_m, K_{o,a,m,d,m} = \rho_c a_m d_m, \\ K_{o,b,m,c,m} &= \rho_c b_m c_m, K_{o,b,m,d,m} = \rho_c b_m d_m, K_{o,c,m,d,m} = \rho_c c_m d_m. \end{aligned} \tag{57}$$

The internal wave oscillons (38) and the internal group oscillon (39) are decomposed via the internal elementary oscillons as follows:

$$\begin{aligned} K_{w,a,m,b,m} &= +K_{w,c,m,d,m} = \lambda_m^2 (K_{o,a,m,b,m} + K_{o,c,m,d,m}), \\ K_{w,a,m,c,m} &= +K_{w,b,m,d,m} = \kappa_m^2 (K_{o,a,m,c,m} + K_{o,b,m,d,m}), \\ K_{w,a,m,d,m} &= -K_{w,b,m,c,m} = \mu_m^2 (K_{o,a,m,d,m} - K_{o,b,m,c,m}), \end{aligned} \tag{58}$$

and

$$K_{g,i,m,j,m} = 2(\lambda_m^2 (K_{o,a,m,b,m} + K_{o,c,m,d,m}) + \kappa_m^2 (K_{w,a,m,c,m} + K_{w,b,m,d,m})). \tag{59}$$

There are four elementary oscillons of diagonal external interaction (diagonal elementary oscillons for brevity) between members of the m th and n th families of the elementary oscillons of propagation from the same wave groups with $n = m + 1, m + 2, \dots, M$ and $m = 1, 2, \dots, M - 1$

$$K_{o,a,m,a,m} = \rho_c a_m a_n, K_{o,b,m,b,m} = \rho_c b_m b_n, K_{o,c,m,c,m} = \rho_c c_m c_n, K_{o,d,m,d,m} = \rho_c d_m d_n. \tag{60}$$

The diagonal wave oscillons (32) and the diagonal group oscillon (33) then are subsequent superpositions of the diagonal elementary oscillons:

$$\begin{aligned}
 K_{w,a,m,a,m} &= \mu_m \mu_n K_{o,a,m,a,m} + \kappa_m \kappa_n K_{o,b,m,b,m} + \lambda_m \lambda_n K_{o,c,m,c,m}, \\
 K_{w,b,m,b,m} &= \kappa_m \kappa_n K_{o,a,m,a,m} + \mu_m \mu_n K_{o,b,m,b,m} + \lambda_m \lambda_n K_{o,d,m,d,m}, \\
 K_{w,c,m,c,m} &= \lambda_m \lambda_n K_{o,a,m,a,m} + \mu_m \mu_n K_{o,c,m,c,m} + \kappa_m \kappa_n K_{o,d,m,d,m}, \\
 K_{w,d,m,d,m} &= \lambda_m \lambda_n K_{o,b,m,b,m} + \kappa_m \kappa_n K_{o,c,m,c,m} + \mu_m \mu_n K_{o,d,m,d,m},
 \end{aligned}
 \tag{61}$$

and

$$K_{g,i,m,i,n} = M_{m,n} (K_{o,a,m,a,n} + K_{o,b,m,b,n} + K_{o,c,m,c,n} + K_{o,d,m,d,n}). \tag{62}$$

Eventually, there are six elementary oscillons of non-diagonal external interaction (external elementary oscillons for brevity) between members of the m th and n th families of the elementary oscillons of propagation from distinct pairs of wave groups with $n = m + 1, m + 2, \dots, M$ and $m = 1, 2, \dots, M - 1$

$$\begin{aligned}
 K_{o,a,m,b,n} &= \rho_c (a_m b_n + b_m a_n), \quad K_{o,a,m,c,n} = \rho_c (a_m c_n + c_m a_n), \\
 K_{o,a,m,d,n} &= \rho_c (a_m d_n + d_m a_n), \quad K_{o,b,m,c,n} = \rho_c (b_m c_n + c_m b_n), \\
 K_{o,b,m,d,n} &= \rho_c (b_m d_n + d_m b_n), \quad K_{w,c,m,d,n} = \rho_c (c_m d_n + d_m c_n).
 \end{aligned}
 \tag{63}$$

The external wave oscillons (42) and the external group oscillon (43) are decomposed in terms of the external elementary oscillons in the following way:

$$\begin{aligned}
 K_{w,a,m,b,n} &= -(\kappa_m \kappa_n - \mu_m \mu_n) K_{o,a,m,b,n} + \lambda_m \lambda_n K_{o,c,m,d,n}, \\
 K_{w,c,m,d,n} &= +\lambda_m \lambda_n K_{o,a,m,b,n} - (\kappa_m \kappa_n - \mu_m \mu_n) K_{o,c,m,d,n}, \\
 K_{w,a,m,c,n} &= -(\lambda_m \lambda_n - \mu_m \mu_n) K_{o,a,m,c,n} + \kappa_m \kappa_n K_{o,b,m,d,n}, \\
 K_{w,b,m,d,n} &= +\kappa_m \kappa_n K_{o,a,m,c,n} - (\lambda_m \lambda_n - \mu_m \mu_n) K_{o,b,m,d,n}, \\
 K_{w,a,m,d,n} &= +\mu_m \mu_n K_{o,a,m,d,n} - (\kappa_m \kappa_n + \lambda_m \lambda_n) K_{o,b,m,c,n}, \\
 K_{w,b,m,c,n} &= -(\kappa_m \kappa_n + \lambda_m \lambda_n) K_{o,a,m,d,n} + \mu_m \mu_n K_{o,b,m,c,n},
 \end{aligned}
 \tag{64}$$

and

$$\begin{aligned}
 K_{g,i,m,j,n} &= \Lambda_{m,n} (K_{o,a,m,b,n} + K_{o,c,m,d,n}) + K_{m,n} (K_{o,a,m,c,n} + K_{o,b,m,d,n}) \\
 &\quad - N_{m,n} (K_{o,a,m,d,n} + K_{o,b,m,c,n}).
 \end{aligned}
 \tag{65}$$

If $n = m$, then the diagonal elementary oscillons (60) are reduced to the doubled elementary pulsions of propagation (54) and then the external elementary oscillons (63) are transformed into the doubled internal elementary oscillons (57).

2.7. Summary of the Deterministic Quantization

Finally, we summarize quantization of the kinetic energy of the deterministic chaos of exponential oscillons and pulsions. The cumulative pulson of the kinetic energy (11) (the cumulative energy pulson for brevity) may be decomposed as follows:

$$\begin{aligned}
 K_e &= K_{e,i,m,i,m} + K_{e,i,m,j,m} + K_{e,i,m,i,n} + K_{e,i,m,j,n} \\
 &= \sum_{m=1}^M (K_{g,i,m,i,m} + K_{g,i,m,j,m}) + \sum_{m=1}^{M-1} \sum_{n=m+1}^M (K_{g,i,m,i,n} + K_{g,i,m,j,n}),
 \end{aligned}
 \tag{66}$$

where $K_{e,i,m,i,m}$ is the energy pulson of propagation (30), $K_{e,i,m,j,m}$ is the internal energy oscillon (40), $K_{e,i,m,i,n}$ is the diagonal energy oscillon (35), and

$K_{e,i,m,j,n}$ is the external energy oscillon (45).

The energy pulson and the energy oscillons are composed of the group pulsions and the group oscillons, respectively. The group pulson of propagation $K_{g,i,m,i,m}$ (28) is composed of four wave pulsions of propagation

$$K_{w,a,m,a,m}, K_{w,b,m,b,m}, K_{w,c,m,c,m}, K_{w,d,m,d,m} \quad (27)$$

that describe a vector self-interaction of velocity fields of m th elementary oscillons of propagation (47) from four wave groups $[a_m, a_m], [b_m, b_m], [c_m, c_m], [d_m, d_m]$, respectively, for $m = 1, 2, \dots, M$.

The internal group oscillon $K_{g,i,m,j,m}$ (39) consists of six internal wave oscillons $K_{w,a,m,b,m}, K_{w,a,m,c,m}, K_{w,a,m,d,m}, K_{w,b,m,c,m}, K_{w,b,m,d,m}, K_{w,c,m,d,m}$ (38) that express a vector internal interaction of velocity fields of m th elementary oscillons of propagation from the distinct wave groups

$[a_m, b_m], [a_m, c_m], [a_m, d_m], [b_m, c_m], [b_m, d_m], [c_m, d_m]$, correspondingly, for $m = 1, 2, \dots, M$.

The diagonal group oscillon $K_{g,i,m,i,m}$ (33) is constructed of four diagonal wave oscillons $K_{w,a,m,a,n}, K_{w,b,m,b,n}, K_{w,c,m,c,n}, K_{w,d,m,d,n}$ (32) that describe a vector external interaction of velocity fields of m th and n th elementary oscillons of propagation from the selfsame wave groups $[a_m, a_n], [b_m, b_n], [c_m, c_n], [d_m, d_n]$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$.

The external group oscillon $K_{g,i,m,j,n}$ (43) includes six external wave oscillons $K_{w,a,m,b,n}, K_{w,a,m,c,n}, K_{w,a,m,d,n}, K_{w,b,m,c,n}, K_{w,b,m,d,n}, K_{w,c,m,d,n}$ (42) that express a vector external interaction of velocity fields of m th and n th elementary oscillons of propagation from the distinct pairs of wave groups $[a_m, b_n]$ & $[b_m, a_n]$, $[a_m, c_n]$ & $[c_m, a_n]$, $[a_m, d_n]$ & $[d_m, a_n]$, $[b_m, c_n]$ & $[c_m, b_n]$, $[b_m, d_n]$ & $[d_m, b_n]$, $[c_m, d_n]$ & $[d_m, c_n]$, respectively, for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$.

The wave pulsions of propagation (27) are composed of four elementary pulsions of propagation $K_{p,a,m,a,m}, K_{p,b,m,b,m}, K_{p,c,m,c,m}, K_{p,d,m,d,m}$ (54) that describe a scalar self-interaction of m th elementary oscillons of propagation from four wave groups with the velocity potentials a_m, b_m, c_m, d_m , respectively, for $m = 1, 2, \dots, M$.

The internal wave oscillons (38) consist of six internal elementary oscillons $K_{o,a,m,b,m}, K_{o,a,m,c,m}, K_{o,a,m,d,m}, K_{o,b,m,c,m}, K_{o,b,m,d,m}, K_{o,c,m,d,m}$ (57) that express a scalar internal interaction of m th elementary oscillons of propagation from the distinct wave groups with the velocity potentials a_m & b_m , a_m & c_m , a_m & d_m , b_m & c_m , b_m & d_m , c_m & d_m , respectively, for $m = 1, 2, \dots, M$.

The diagonal wave oscillons (32) are constructed of four diagonal elementary oscillons $K_{o,a,m,a,n}, K_{o,b,m,b,n}, K_{o,c,m,c,n}, K_{o,d,m,d,n}$ (60) that describe a scalar external interaction of m th and n th elementary oscillons of propagation from the selfsame wave groups with the velocity potentials a_m & a_n , b_m & b_n , c_m & c_n , d_m & d_n , correspondingly, for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$.

The external wave oscillons (42) include six external elementary oscillons $K_{o,a,m,b,n}, K_{o,a,m,c,n}, K_{o,a,m,d,n}, K_{o,b,m,c,n}, K_{o,b,m,d,n}, K_{o,c,m,d,n}$ (63) that express a scalar external interaction of m th and n th elementary oscillons of propagation

from the distinct pairs of wave groups with the velocity potentials a_m & b_n and b_m & a_n , a_m & c_n and c_m & a_n , a_m & d_n and d_m & a_n , b_m & c_n and c_m & b_n , b_m & d_n and d_m & b_n , c_m & d_n and d_m & c_n , for $n = m + 1, m + 2, \dots, M$, $m = 1, 2, \dots, M - 1$.

3. Elementary Oscillons and Pulsons

3.1. The Elementary Oscillons of Propagation

To clarify topology of the elementary oscillons of propagation in the x -direction, we separate variable x and recast (47)-(51) to the following x -form:

$$\begin{aligned} K_{o,a,m} &= ez_m Q_{y,m} \sin(\kappa_m(x + s_{\alpha,y,m})), & K_{o,c,m} &= ez_m R_{y,m} \sin(\kappa_m(x + s_{\beta,y,m})), \\ K_{o,b,m} &= ez_m Q_{y,m} \cos(\kappa_m(x + s_{\alpha,y,m})), & K_{o,d,m} &= ez_m R_{y,m} \cos(\kappa_m(x + s_{\beta,y,m})), \end{aligned} \tag{67}$$

where amplitudes

$$Q_{y,m} = \sqrt{Ay_m^2 + Cy_m^2}, \quad R_{y,m} = \sqrt{By_m^2 + Dy_m^2} \tag{68}$$

are expressed via coefficients

$$\begin{aligned} Ay_m &= Cv_m cy_m + Av_m sy_m, & By_m &= Av_m cy_m - Cv_m sy_m, \\ Cy_m &= Dv_m cy_m + Bv_m sy_m, & Dy_m &= Bv_m cy_m - Dv_m sy_m, \end{aligned} \tag{69}$$

$$s_{\alpha,y,m} = \alpha_{y,m} - U_m t + X_{m,0}, \quad s_{\beta,y,m} = \beta_{y,m} - U_m t + X_{m,0} \tag{70}$$

are x -shifts in the motionless frame, and

$$\alpha_{y,m} = \frac{1}{\kappa_m} \arcsin \frac{Cy_m}{Q_{y,m}}, \quad \beta_{y,m} = \frac{1}{\kappa_m} \arcsin \frac{Dy_m}{R_{y,m}} \tag{71}$$

are X_m -shifts in the m th moving frame.

For any frozen $y = y_0, z = z_0, t = t_0$, the elementary oscillons of propagation (47) are transformed into solo-frequency (1-f) neutral oscillons in x with wave-number κ_m , amplitudes proportional to $ez_m(z_0), Q_{y,m}(y_0, t_0), R_{y,m}(y_0, t_0)$, and x -shifts

$$s_{\alpha,y,m}(y_0, t_0), \quad s_{\beta,y,m}(y_0, t_0). \tag{72}$$

See classification of oscillons and pulsons in **Section 3.5**.

Let's consider propagation velocity U_m and period $T_{x,m}$ as independent parameters. Then a wave number

$$\kappa_m = \frac{2\pi}{L_{x,m}} = \frac{2\pi}{U_m T_{x,m}}. \tag{73}$$

and a spatial period along the x -axis (an x -wavelength)

$$L_{x,m} = U_m T_{x,m}. \tag{74}$$

Since integrals of sines and cosines of (67) over x -period $L_{x,m}$ vanish,

$$\int_0^{L_{x,m}} K_{o,i,m} dx = 0, \quad i = a, b, c, d, \tag{75}$$

i.e. the elementary oscillons of propagation are neutral as the average over the x -period vanishes.

To display topology of the elementary oscillons of propagation in the y -direction, we separate variable y and transform (47)-(51) into the following y -form:

$$\begin{aligned} K_{o,a,m} &= ez_m Q_{x,m} \sin(\lambda_m (y + s_{\alpha,x,m})), & K_{o,b,m} &= ez_m R_{x,m} \sin(\lambda_m (y + s_{\beta,x,m})), \\ K_{o,c,m} &= ez_m Q_{x,m} \cos(\lambda_m (y + s_{\alpha,x,m})), & K_{o,d,m} &= ez_m R_{x,m} \cos(\lambda_m (y + s_{\beta,x,m})), \end{aligned} \quad (76)$$

where amplitudes

$$Q_{x,m} = \sqrt{Ax_m^2 + Cx_m^2}, \quad R_{x,m} = \sqrt{Bx_m^2 + Dx_m^2} \quad (77)$$

are computed in terms of coefficients

$$Ax_m = Bv_m cx_m + Av_m sx_m, \quad Bx_m = Av_m cx_m - Bv_m sx_m, \quad (78)$$

$$Cx_m = Dv_m cx_m + Cv_m sx_m, \quad Dx_m = Cv_m cx_m - Dv_m sx_m, \quad (79)$$

$$s_{\alpha,x,m} = \alpha_{x,m} - V_m t + Y_{m,0}, \quad s_{\beta,x,m} = \beta_{x,m} - V_m t + Y_{m,0} \quad (79)$$

are y -shifts in the motionless frame, and

$$\alpha_{x,m} = \frac{1}{\lambda_m} \arcsin \frac{Cx_m}{Q_{x,m}}, \quad \beta_{x,m} = \frac{1}{\lambda_m} \arcsin \frac{Dx_m}{R_{x,m}} \quad (80)$$

are Y_m -shifts in the m th moving frame.

For any frozen $x = x_0, z = z_0, t = t_0$, the elementary oscillons of propagation (47) are reduced to 1-f neutral oscillons in y with wavenumber λ_m , amplitudes proportional to $ez_m(z_0), Q_{x,m}(x_0, t_0), R_{x,m}(x_0, t_0)$, and y -shifts

$$s_{\alpha,x,m}(x_0, t_0), \quad s_{\beta,x,m}(x_0, t_0). \quad (81)$$

Since V_m and $T_{y,m}$ are independent parameters, a wave number and a spatial period along the y -axis become

$$\lambda_m = \frac{2\pi}{L_{y,m}} = \frac{2\pi}{V_m T_{y,m}}, \quad L_{y,m} = V_m T_{y,m}. \quad (82)$$

The elementary oscillons of propagation are also neutral in the y -direction because

$$\int_0^{L_{y,m}} K_{o,i,m} dy = 0, \quad i = a, b, c, d. \quad (83)$$

Topology of the elementary oscillons of propagation in time follows from the inhomogeneous Fourier t -form of (47)-(51)

$$K_{o,a,m} = \frac{ez_m}{2} (-Q_m f_{1,m} + R_m f_{3,m}), \quad K_{o,b,m} = \frac{ez_m}{2} (+Q_m f_{2,m} - R_m f_{4,m}), \quad (84)$$

$$K_{o,d,m} = \frac{ez_m}{2} (+Q_m f_{1,m} + R_m f_{3,m}), \quad K_{o,c,m} = \frac{ez_m}{2} (+Q_m f_{2,m} + R_m f_{4,m})$$

in terms of four trigonometric functions

$$f_{1,m} = \sin A_{1,m}, \quad f_{2,m} = \cos A_{1,m}, \quad f_{3,m} = \sin A_{2,m}, \quad f_{4,m} = \cos A_{2,m} \quad (85)$$

of two arguments

$$\begin{aligned} A_{1,m} &= (\omega_{x,m} + \omega_{y,m})t + s_{x,m} + s_{y,m} + \alpha_{1,m}, \\ A_{2,m} &= (\omega_{x,m} - \omega_{y,m})t + s_{x,m} - s_{y,m} + \alpha_{2,m}. \end{aligned} \quad (86)$$

In Equations (84)-(86),

$$Q_m = \sqrt{(Av_m - Dv_m)^2 + (Bv_m + Cv_m)^2}, R_m = \sqrt{(Av_m + Dv_m)^2 + (Bv_m - Cv_m)^2} \quad (87)$$

are amplitudes,

$$\omega_{x,m} = \frac{2\pi}{T_{x,m}} = \kappa_m U_m, \quad \omega_{y,m} = \frac{2\pi}{T_{y,m}} = \lambda_m V_m \quad (88)$$

are frequencies,

$$\alpha_{1,m} = \arcsin \frac{Av_m - Dv_m}{Q_m}, \quad \alpha_{2,m} = \arcsin \frac{Av_m + Dv_m}{R_m} \quad (89)$$

are t -shifts,

$$s_{x,m} = -\kappa_m (x + X_{m,0}), \quad s_{y,m} = -\lambda_m (y + Y_{m,0}) \quad (90)$$

are x - and y -shifts, respectively.

For any frozen $x = x_0, y = y_0, z = z_0$, the elementary oscillons of propagation (47) are converted into duet-frequency (2-f) neutral oscillons in t , which are generated by superpositions of sine waves $f_{1,m}, f_{3,m}$ and cosine waves $f_{2,m}, f_{4,m}$ with frequencies $\omega_{x,m} + \omega_{y,m}, \omega_{x,m} - \omega_{y,m}$, respectively, amplitudes proportional to $Q_m, R_m, ez_m(z_0)$, and t -shifts depending on

$$\omega_{x,m}, \omega_{y,m}, \alpha_{1,m}, \alpha_{2,m}, s_{x,m}(x_0), s_{y,m}(y_0). \quad (91)$$

Since temporal periods of $f_{1,m}, f_{2,m}$ and $f_{3,m}, f_{4,m}$ are equal to, respectively,

$$T_{1,m} = \frac{2\pi}{\omega_{x,m} + \omega_{y,m}}, \quad T_{2,m} = \frac{2\pi}{\omega_{x,m} - \omega_{y,m}}, \quad (92)$$

a temporal period of the elementary oscillons of propagation

$$T_{o,m} = LCM(T_{1,m}, T_{2,m}) = k_{1,m} T_{1,m} = k_{2,m} T_{2,m}, \quad (93)$$

where $LCM(a, b)$ is a least common multiple of a, b and $k_{1,m}, k_{2,m}$ are integers.

Because integrals of sine waves $f_{1,m}, f_{3,m}$ and cosine waves $f_{2,m}, f_{4,m}$ over t -periods $T_{1,m}, T_{2,m}$ vanish, the elementary oscillons of propagation are neutral in time, as well, since

$$\int_0^{T_{o,m}} K_{o,i,m} dt = 0, \quad i = a, b, c, d. \quad (94)$$

The elementary oscillon of propagation $K_{o,b,3}$ with $L_{x,3} = 4/3, L_{y,3} = 1, T_{o,3} = 2$ is visualized in **Figure 1** for the following independent parameters:

$$\begin{aligned} \rho_c = 1, \quad M = 3, \quad \eta = 0, \quad x_0 = 0, \quad y_0 = 0, \quad z_0 = 0, \quad t_0 = 0, \\ T_{x,1} = 2, \quad T_{x,2} = 1, \quad T_{x,3} = 2/3, \quad T_{y,1} = 3, \quad T_{y,2} = 3/2, \quad T_{y,3} = 1, \\ U_1 = 4, \quad U_2 = 3, \quad U_3 = 2, \quad V_1 = 3, \quad V_2 = 2, \quad V_3 = 1, \\ X_{1,0} = 7, \quad X_{2,0} = 5, \quad X_{3,0} = 3, \quad Y_{1,0} = 5, \quad Y_{2,0} = 3, \quad Y_{3,0} = 1, \\ Av_1 = 11, \quad Av_2 = 7, \quad Av_3 = 3, \quad Bv_1 = 6, \quad Bv_2 = 4, \quad Bv_3 = 2, \\ Cv_1 = 8, \quad Cv_2 = 6, \quad Cv_3 = 4, \quad Dv_1 = 7, \quad Dv_2 = 4, \quad Dv_3 = 1. \end{aligned} \quad (95)$$

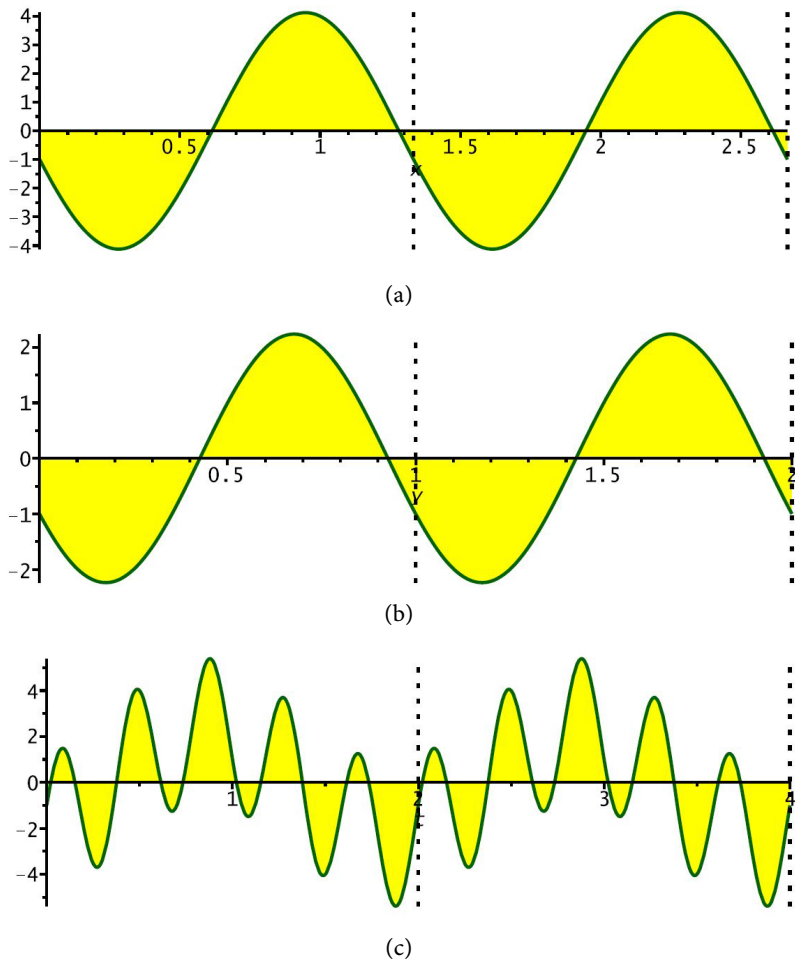


Figure 1. The elementary oscillon of propagation $K_{o,b,3}$: (a)— $K_{o,b,3}(x)$ (67), (b)— $K_{o,b,3}(y)$ (76), (c)— $K_{o,b,3}(t)$ (84) for independent parameters (95).

In **Figures 1-18**, two adjacent x -, y - and t -periods are shown by dotted vertical lines.

The elementary oscillons of propagation are displayed via 1-f neutral oscillons in x with periods $L_{x,m} = [8, 3, 4/3]$, in terms of 1-f neutral oscillons in y with periods $L_{y,m} = [9, 3, 1]$, and by 2-f neutral oscillons in t with periods $T_{o,m} = [9, 3, 2]$, five local t -maximums, and five local t -minimums, the numbers of which are specified by $k_{1,m} = [5, 5, 5]$.

In agreement with (67) and (74),

$$K_{o,i,m} \left(x + \frac{1}{4}L_{x,m}, y, z, t \right) = K_{o,j,m} (x, y, z, t), \quad [i, j] = [a, b], [c, d]. \quad (96)$$

So, x -extrema of the elementary oscillon $K_{o,a,m}$, which are equal to $\pm e z_m Q_{y,m}$, correspond to x -zeroes of the x -complementary oscillon $K_{o,b,m}$ and x -zeroes of $K_{o,a,m}$ to x -extrema of $K_{o,b,m}$, which are also equal to $\pm e z_m Q_{y,m}$. Similarly, x -extrema of $K_{o,c,m}$, which are equal to $\pm e z_m R_{y,m}$, correspond to x -zeroes of the x -complementary oscillon $K_{o,d,m}$ and x -zeroes of $K_{o,c,m}$ to x -extrema of $K_{o,d,m}$, which are equal to $\pm e z_m R_{y,m}$, as well.

Because of (76) and (82),

$$K_{o,i,m} \left(x, y + \frac{1}{4} L_{y,m}, z, t \right) = K_{o,j,m} (x, y, z, t), [i, j] = [a, c], [b, d]. \quad (97)$$

Thus, y -extrema of the elementary oscillon $K_{o,a,m}$, which are equal to $\pm e z_m Q_{x,m}$, correspond to y -zeroes of the y -complementary oscillon $K_{o,c,m}$ and y -zeroes of $K_{o,a,m}$ to y -extrema of $K_{o,c,m}$, which are also equal to $\pm e z_m Q_{x,m}$. Analogously, y -extrema of $K_{o,b,m}$, which are equal to $\pm e z_m R_{x,m}$, correspond to y -zeroes of the y -complementary oscillon $K_{o,d,m}$ and y -zeroes of $K_{o,b,m}$ to y -extrema of $K_{o,d,m}$, which are equal to $\pm e z_m R_{x,m}$, as well.

3.2. Eigenfunctions of the Inhomogeneous Fourier Expansions in x and y

Consider the m th and n th families of inhomogeneous x -eigenfunctions of propagation

$$\begin{aligned} f_{x,1,m} &= \sin A_{x,1,m}, f_{x,2,m} = \cos A_{x,1,m}, f_{x,3,m} = \sin A_{x,2,m}, f_{x,4,m} = \cos A_{x,2,m}, \\ f_{x,1,n} &= \sin A_{x,1,n}, f_{x,2,n} = \cos A_{x,1,n}, f_{x,3,n} = \sin A_{x,2,n}, f_{x,4,n} = \cos A_{x,2,n} \end{aligned} \quad (98)$$

of four arguments

$$\begin{aligned} A_{x,1,m} &= \kappa_m (x + s_{\alpha,y,m}), A_{x,2,m} = \kappa_m (x + s_{\beta,y,m}), \\ A_{x,1,n} &= \kappa_n (x + s_{\alpha,y,n}), A_{x,2,n} = \kappa_n (x + s_{\beta,y,n}), \end{aligned} \quad (99)$$

which are parametrized by two wavenumbers κ_m and κ_n with $m = 1, 2, \dots, M$ and $n = 1, 2, \dots, M$.

An x -matrix of external interaction ($n \neq m$) between x -eigenfunctions of propagation of the m th and n th families takes the following form:

$$f_{x,i,m,j,n} = \begin{bmatrix} f_{x,1,m} f_{x,1,n} & f_{x,1,m} f_{x,2,n} & f_{x,1,m} f_{x,3,n} & f_{x,1,m} f_{x,4,n} \\ f_{x,2,m} f_{x,1,n} & f_{x,2,m} f_{x,2,n} & f_{x,2,m} f_{x,3,n} & f_{x,2,m} f_{x,4,n} \\ f_{x,3,m} f_{x,1,n} & f_{x,3,m} f_{x,2,n} & f_{x,3,m} f_{x,3,n} & f_{x,3,m} f_{x,4,n} \\ f_{x,4,m} f_{x,1,n} & f_{x,4,m} f_{x,2,n} & f_{x,4,m} f_{x,3,n} & f_{x,4,m} f_{x,4,n} \end{bmatrix}. \quad (100)$$

Because of the product rules for sine and cosine waves, there are 16 inhomogeneous x -eigenfunctions of external interaction

$$\begin{aligned} f_{x,1,m,n} &= \sin A_{x,1,m,n}, f_{x,2,m,n} = \cos A_{x,1,m,n}, \\ f_{x,3,m,n} &= \sin A_{x,2,m,n}, f_{x,4,m,n} = \cos A_{x,2,m,n}, \\ f_{x,5,m,n} &= \sin A_{x,3,m,n}, f_{x,6,m,n} = \cos A_{x,3,m,n}, \\ f_{x,7,m,n} &= \sin A_{x,4,m,n}, f_{x,8,m,n} = \cos A_{x,4,m,n}, \\ f_{x,9,m,n} &= \sin A_{x,5,m,n}, f_{x,10,m,n} = \cos A_{x,5,m,n}, \\ f_{x,11,m,n} &= \sin A_{x,6,m,n}, f_{x,12,m,n} = \cos A_{x,6,m,n}, \\ f_{x,13,m,n} &= \sin A_{x,7,m,n}, f_{x,14,m,n} = \cos A_{x,7,m,n}, \\ f_{x,15,m,n} &= \sin A_{x,8,m,n}, f_{x,16,m,n} = \cos A_{x,8,m,n} \end{aligned} \quad (101)$$

of eight arguments

$$\begin{aligned}
A_{x,1,m,n} &= \kappa_{1,m,n} x + \kappa_m s_{\alpha,y,m} + \kappa_n s_{\alpha,y,n}, & A_{x,2,m,n} &= \kappa_{2,m,n} x + \kappa_m s_{\alpha,y,m} - \kappa_n s_{\alpha,y,n}, \\
A_{x,3,m,n} &= \kappa_{1,m,n} x + \kappa_m s_{\alpha,y,m} + \kappa_n s_{\beta,y,n}, & A_{x,4,m,n} &= \kappa_{2,m,n} x + \kappa_m s_{\alpha,y,m} - \kappa_n s_{\beta,y,n}, \\
A_{x,5,m,n} &= \kappa_{1,m,n} x + \kappa_m s_{\beta,y,m} + \kappa_n s_{\alpha,y,n}, & A_{x,6,m,n} &= \kappa_{2,m,n} x + \kappa_m s_{\beta,y,m} - \kappa_n s_{\alpha,y,n}, \\
A_{x,7,m,n} &= \kappa_{1,m,n} x + \kappa_m s_{\beta,y,m} + \kappa_n s_{\beta,y,n}, & A_{x,8,m,n} &= \kappa_{2,m,n} x + \kappa_m s_{\beta,y,m} - \kappa_n s_{\beta,y,n},
\end{aligned} \tag{102}$$

which are parametrized by two wavenumbers

$$\kappa_{1,m,n} = \kappa_m + \kappa_n, \quad \kappa_{2,m,n} = \kappa_m - \kappa_n \tag{103}$$

with $m = 1, 2, \dots, M$, $n = 1, 2, \dots, M$. Thus, the x -eigenfunctions of external interaction include sine and cosine waves of all possible combinations of x -wavenumbers and x -shifts.

Computing products (100) of x -eigenfunctions of propagation yields the following expansions via the x -eigenfunctions of external interaction (101):

$$\begin{aligned}
2f_{x,1,m} f_{x,1,n} &= -f_{x,2,m,n} + f_{x,4,m,n}, & 2f_{x,1,m} f_{x,2,n} &= f_{x,1,m,n} + f_{x,3,m,n}, \\
2f_{x,1,m} f_{x,3,n} &= -f_{x,6,m,n} + f_{x,8,m,n}, & 2f_{x,1,m} f_{x,4,n} &= f_{x,5,m,n} + f_{x,7,m,n}, \\
2f_{x,2,m} f_{x,1,n} &= +f_{x,1,m,n} - f_{x,3,m,n}, & 2f_{x,2,m} f_{x,2,n} &= f_{x,2,m,n} + f_{x,4,m,n}, \\
2f_{x,2,m} f_{x,3,n} &= +f_{x,5,m,n} - f_{x,7,m,n}, & 2f_{x,2,m} f_{x,4,n} &= f_{x,6,m,n} + f_{x,8,m,n}, \\
2f_{x,3,m} f_{x,1,n} &= -f_{x,10,m,n} + f_{x,12,m,n}, & 2f_{x,3,m} f_{x,2,n} &= f_{x,9,m,n} + f_{x,11,m,n}, \\
2f_{x,3,m} f_{x,3,n} &= -f_{x,14,m,n} + f_{x,16,m,n}, & 2f_{x,3,m} f_{x,4,n} &= f_{x,13,m,n} + f_{x,15,m,n}, \\
2f_{x,4,m} f_{x,1,n} &= +f_{x,9,m,n} - f_{x,11,m,n}, & 2f_{x,4,m} f_{x,2,n} &= f_{x,10,m,n} + f_{x,12,m,n}, \\
2f_{x,4,m} f_{x,3,n} &= +f_{x,13,m,n} - f_{x,15,m,n}, & 2f_{x,4,m} f_{x,4,n} &= f_{x,14,m,n} + f_{x,16,m,n}.
\end{aligned} \tag{104}$$

If $n = m$, then two wavenumbers of the x -eigenfunctions of external interaction (103) are reduced to a single degenerated wavenumber since

$$\kappa_{1,m,m} = 2\kappa_m, \quad \kappa_{2,m,m} = 0. \tag{105}$$

Consequently, eight arguments (102) of the x -eigenfunctions of external interaction are transformed into four arguments as

$$\begin{aligned}
A_{x,1,m,m} &= 2\kappa_m (x + s_{\alpha,y,m}), & A_{x,2,m,m} &= 0, \\
A_{x,3,m,m} &= \kappa_m (2x + s_{\alpha,y,m} + s_{\beta,y,m}), & A_{x,5,m,m} &= +A_{x,3,m,m}, \\
A_{x,4,m,m} &= \kappa_m (s_{\alpha,y,m} - s_{\beta,y,m}), & A_{x,6,m,m} &= -A_{x,4,m,m}, \\
A_{x,7,m,m} &= 2\kappa_m (x + s_{\beta,y,m}), & A_{x,8,m,m} &= 0.
\end{aligned} \tag{106}$$

Therefore, we define the following four arguments of x -eigenfunctions of internal interaction:

$$\begin{aligned}
B_{x,1,m} &= 2\kappa_m (x + s_{\alpha,y,m}), & B_{x,2,m} &= \kappa_m (2x + s_{\alpha,y,m} + s_{\beta,y,m}), \\
B_{x,3,m} &= 2\kappa_m (x + s_{\beta,y,m}), & B_{y,1,m} &= \kappa_m (s_{\alpha,y,m} - s_{\beta,y,m}).
\end{aligned} \tag{107}$$

If $n = m$, x -eigenfunctions of external interaction are converted into eight x -eigenfunctions because

$$\begin{aligned}
f_{x,1,m,m} &= +\sin B_{x,1,m}, & f_{x,2,m,m} &= \cos B_{x,1,m}, \\
f_{x,3,m,m} &= 0, & f_{x,4,m,m} &= 1, \\
f_{x,5,m,m} &= +\sin B_{x,2,m}, & f_{x,6,m,m} &= \cos B_{x,2,m}, \\
f_{x,7,m,m} &= +\sin B_{y,1,m}, & f_{x,8,m,m} &= \cos B_{y,1,m},
\end{aligned}$$

$$\begin{aligned}
 f_{x,9,m,m} &= +f_{x,5,m,m}, & f_{x,10,m,m} &= f_{x,6,m,m}, \\
 f_{x,11,m,m} &= -f_{x,7,m,m}, & f_{x,12,m,m} &= f_{x,8,m,m}, \\
 f_{x,13,m,m} &= +\sin B_{x,3,m}, & f_{x,14,m,m} &= \cos B_{x,3,m}, \\
 f_{x,15,m,m} &= 0, & f_{x,16,m,m} &= 1.
 \end{aligned}
 \tag{108}$$

Thus, six x -eigenfunctions of internal interaction $g_{x,1,m}, g_{x,2,m}, \dots, g_{x,6,m}$ and two vertical shifts $h_{x,1,m}, h_{x,2,m}$ are defined as follows:

$$\begin{aligned}
 g_{x,1,m} &= +\sin B_{x,1,m}, & g_{x,2,m} &= \cos B_{x,1,m}, & g_{x,3,m} &= +\sin B_{x,2,m}, & g_{x,4,m} &= \cos B_{x,2,m}, \\
 g_{x,5,m} &= +\sin B_{x,3,m}, & g_{x,6,m} &= \cos B_{x,3,m}, & h_{y,1,m} &= +\sin B_{y,1,m}, & h_{y,2,m} &= \cos B_{y,1,m}.
 \end{aligned}
 \tag{109}$$

The x -matrix of internal interaction ($n = m$) becomes a symmetrical one, namely,

$$f_{x,i,m,j,m} = \begin{bmatrix} f_{x,1,m}^2 & f_{x,1,m}f_{x,2,m} & f_{x,1,m}f_{x,3,m} & f_{x,1,m}f_{x,4,m} \\ f_{x,1,m}f_{x,2,m} & f_{x,2,m}^2 & f_{x,2,m}f_{x,3,m} & f_{x,2,m}f_{x,4,m} \\ f_{x,1,m}f_{x,3,m} & f_{x,2,m}f_{x,3,m} & f_{x,3,m}^2 & f_{x,3,m}f_{x,4,m} \\ f_{x,1,m}f_{x,4,m} & f_{x,2,m}f_{x,4,m} & f_{x,3,m}f_{x,4,m} & f_{x,4,m}^2 \end{bmatrix}.
 \tag{110}$$

Computing elements of the x -matrix of internal interaction via the x -eigenfunctions of internal interaction (109) returns

$$\begin{aligned}
 2f_{x,1,m}^2 &= -g_{x,2,m} + 1, & 2f_{x,1,m}f_{x,2,m} &= +g_{x,1,m}, \\
 2f_{x,1,m}f_{x,3,m} &= -g_{x,4,m} + h_{y,2,m}, & 2f_{x,1,m}f_{x,4,m} &= +g_{x,3,m} + h_{y,1,m}, \\
 2f_{x,2,m}^2 &= +g_{x,2,m} + 1, & 2f_{x,2,m}f_{x,3,m} &= +g_{x,3,m} - h_{y,1,m}, \\
 2f_{x,2,m}f_{x,4,m} &= +g_{x,4,m} + h_{y,2,m}, & 2f_{x,3,m}^2 &= -g_{x,6,m} + 1, \\
 2f_{x,3,m}f_{x,4,m} &= +g_{x,5,m}, & 2f_{x,4,m}^2 &= +g_{x,6,m} + 1.
 \end{aligned}
 \tag{111}$$

Computation of eigenfunctions of the inhomogeneous Fourier series in y coincides with the computation of the eigenfunctions of the inhomogeneous Fourier series in x up to the following substitutions:

$$x = y, \kappa_m = \lambda_m, \kappa_n = \lambda_n, y = x, \kappa_{1,m,m} = \lambda_{1,m,m}, \kappa_{2,m,m} = \lambda_{2,m,m}.
 \tag{112}$$

3.3. Eigenfunctions of the Inhomogeneous Fourier Expansions in t

Consider the m th (85) and n th families of inhomogeneous t -eigenfunctions of propagation

$$\begin{aligned}
 f_{1,m} &= \sin A_{1,m}, & f_{2,m} &= \cos A_{1,m}, & f_{3,m} &= \sin A_{2,m}, & f_{4,m} &= \cos A_{2,m}, \\
 f_{1,n} &= \sin A_{1,n}, & f_{2,n} &= \cos A_{1,n}, & f_{3,n} &= \sin A_{2,n}, & f_{4,n} &= \cos A_{2,n}
 \end{aligned}
 \tag{113}$$

of four arguments

$$\begin{aligned}
 A_{1,m} &= (\omega_{x,m} + \omega_{y,m})t + s_{x,m} + s_{y,m} + \alpha_{1,m}, \\
 A_{1,n} &= (\omega_{x,n} + \omega_{y,n})t + s_{x,n} + s_{y,n} + \alpha_{1,n}, \\
 A_{2,m} &= (\omega_{x,m} - \omega_{y,m})t + s_{x,m} - s_{y,m} + \alpha_{2,m}, \\
 A_{2,n} &= (\omega_{x,n} - \omega_{y,n})t + s_{x,n} - s_{y,n} + \alpha_{2,n},
 \end{aligned}
 \tag{114}$$

which are parametrized by four frequencies $\omega_{x,m} + \omega_{y,m}, \omega_{x,n} + \omega_{y,n}, \omega_{x,m} - \omega_{y,m},$

and $\omega_{x,n} - \omega_{y,n}$ with $m = 1, 2, \dots, M$, $n = 1, 2, \dots, M$.

A t -matrix of external interaction ($n \neq m$) between t -eigenfunctions of propagation of the m th and n th families takes the following form:

$$f_{i,m,j,n} = \begin{bmatrix} f_{1,m}f_{1,n} & f_{1,m}f_{2,n} & f_{1,m}f_{3,n} & f_{1,m}f_{4,n} \\ f_{2,m}f_{1,n} & f_{2,m}f_{2,n} & f_{2,m}f_{3,n} & f_{2,m}f_{4,n} \\ f_{3,m}f_{1,n} & f_{3,m}f_{2,n} & f_{3,m}f_{3,n} & f_{3,m}f_{4,n} \\ f_{4,m}f_{1,n} & f_{4,m}f_{2,n} & f_{4,m}f_{3,n} & f_{4,m}f_{4,n} \end{bmatrix}. \quad (115)$$

Due to the product rules for sine and cosine waves, there are eight arguments of t -eigenfunctions of external interaction

$$\begin{aligned} A_{1,m,n} &= A_{1,m} + A_{1,n}, & A_{2,m,n} &= A_{1,m} - A_{1,n}, \\ A_{3,m,n} &= A_{1,m} + A_{2,n}, & A_{4,m,n} &= A_{1,m} - A_{2,n}, \\ A_{5,m,n} &= A_{2,m} + A_{1,n}, & A_{6,m,n} &= A_{2,m} - A_{1,n}, \\ A_{7,m,n} &= A_{2,m} + A_{2,n}, & A_{8,m,n} &= A_{2,m} - A_{2,n}, \end{aligned} \quad (116)$$

which include all possible sums and differences of the arguments of t -eigenfunctions of propagation (114).

Consequently, there are 16 inhomogeneous t -eigenfunctions of external interaction

$$\begin{aligned} f_{1,m,n} &= \sin A_{1,m,n}, f_{2,m,n} = \cos A_{1,m,n}, f_{3,m,n} = \sin A_{2,m,n}, f_{4,m,n} = \cos A_{2,m,n}, \\ f_{5,m,n} &= \sin A_{3,m,n}, f_{6,m,n} = \cos A_{3,m,n}, f_{7,m,n} = \sin A_{4,m,n}, f_{8,m,n} = \cos A_{4,m,n}, \\ f_{9,m,n} &= \sin A_{5,m,n}, f_{10,m,n} = \cos A_{5,m,n}, f_{11,m,n} = \sin A_{6,m,n}, f_{12,m,n} = \cos A_{6,m,n}, \\ f_{13,m,n} &= \sin A_{7,m,n}, f_{14,m,n} = \cos A_{7,m,n}, f_{15,m,n} = \sin A_{8,m,n}, f_{16,m,n} = \cos A_{8,m,n} \end{aligned} \quad (117)$$

of eight arguments

$$\begin{aligned} A_{1,m,n} &= \omega_{1,m,n} t + s_{x,m} + s_{y,m} + s_{x,n} + s_{y,n} + \alpha_{1,m} + \alpha_{1,n}, \\ A_{2,m,n} &= \omega_{2,m,n} t + s_{x,m} + s_{y,m} - s_{x,n} - s_{y,n} + \alpha_{1,m} - \alpha_{1,n}, \\ A_{3,m,n} &= \omega_{3,m,n} t + s_{x,m} + s_{y,m} + s_{x,n} - s_{y,n} + \alpha_{1,m} + \alpha_{2,n}, \\ A_{4,m,n} &= \omega_{4,m,n} t + s_{x,m} + s_{y,m} - s_{x,n} + s_{y,n} + \alpha_{1,m} - \alpha_{2,n}, \\ A_{5,m,n} &= \omega_{5,m,n} t + s_{x,m} - s_{y,m} + s_{x,n} + s_{y,n} + \alpha_{2,m} + \alpha_{1,n}, \\ A_{6,m,n} &= \omega_{6,m,n} t + s_{x,m} - s_{y,m} - s_{x,n} - s_{y,n} + \alpha_{2,m} - \alpha_{1,n}, \\ A_{7,m,n} &= \omega_{7,m,n} t + s_{x,m} - s_{y,m} + s_{x,n} - s_{y,n} + \alpha_{2,m} + \alpha_{2,n}, \\ A_{8,m,n} &= \omega_{8,m,n} t + s_{x,m} - s_{y,m} - s_{x,n} + s_{y,n} + \alpha_{2,m} - \alpha_{2,n}, \end{aligned} \quad (118)$$

which are parametrized by eight frequencies

$$\begin{aligned} \omega_{1,m,n} &= \omega_{x,m} + \omega_{y,m} + \omega_{x,n} + \omega_{y,n}, & \omega_{2,m,n} &= \omega_{x,m} + \omega_{y,m} - \omega_{x,n} - \omega_{y,n}, \\ \omega_{3,m,n} &= \omega_{x,m} + \omega_{y,m} + \omega_{x,n} - \omega_{y,n}, & \omega_{4,m,n} &= \omega_{x,m} + \omega_{y,m} - \omega_{x,n} + \omega_{y,n}, \\ \omega_{5,m,n} &= \omega_{x,m} - \omega_{y,m} + \omega_{x,n} + \omega_{y,n}, & \omega_{6,m,n} &= \omega_{x,m} - \omega_{y,m} - \omega_{x,n} - \omega_{y,n}, \\ \omega_{7,m,n} &= \omega_{x,m} - \omega_{y,m} + \omega_{x,n} - \omega_{y,n}, & \omega_{8,m,n} &= \omega_{x,m} - \omega_{y,m} - \omega_{x,n} + \omega_{y,n} \end{aligned} \quad (119)$$

with $m = 1, 2, \dots, M$, $n = 1, 2, \dots, M$.

Computation of elements of the t -matrix of external interaction (115) returns the following expansions in terms of the t -eigenfunctions of external interaction (117):

$$\begin{aligned}
 2f_{1,m} f_{1,n} &= -f_{2,m,n} + f_{4,m,n}, & 2f_{1,m} f_{2,n} &= f_{1,m,n} + f_{3,m,n}, \\
 2f_{1,m} f_{3,n} &= -f_{6,m,n} + f_{8,m,n}, & 2f_{1,m} f_{4,n} &= f_{5,m,n} + f_{7,m,n}, \\
 2f_{2,m} f_{1,n} &= +f_{1,m,n} - f_{3,m,n}, & 2f_{2,m} f_{2,n} &= f_{2,m,n} + f_{4,m,n}, \\
 2f_{2,m} f_{3,n} &= +f_{5,m,n} - f_{7,m,n}, & 2f_{2,m} f_{4,n} &= f_{6,m,n} + f_{8,m,n}, \\
 2f_{3,m} f_{1,n} &= -f_{10,m,n} + f_{12,m,n}, & 2f_{3,m} f_{2,n} &= f_{9,m,n} + f_{11,m,n}, \\
 2f_{3,m} f_{3,n} &= -f_{14,m,n} + f_{16,m,n}, & 2f_{3,m} f_{4,n} &= f_{13,m,n} + f_{15,m,n}, \\
 2f_{4,m} f_{1,n} &= +f_{9,m,n} - f_{11,m,n}, & 2f_{4,m} f_{2,n} &= f_{10,m,n} + f_{12,m,n}, \\
 2f_{4,m} f_{3,n} &= +f_{13,m,n} - f_{15,m,n}, & 2f_{4,m} f_{4,n} &= f_{14,m,n} + f_{16,m,n}.
 \end{aligned} \tag{120}$$

If $n = m$, then eight frequencies of the t -eigenfunctions of external interaction are reduced to four degenerated frequencies since

$$\begin{aligned}
 \omega_{1,m,m} &= 2(\omega_{x,m} + \omega_{y,m}), & \omega_{2,m,m} &= 0, & \omega_{3,m,m} &= 2\omega_{x,m}, & \omega_{4,m,m} &= 2\omega_{y,m}, \\
 \omega_{5,m,m} &= \omega_{3,m,m}, & \omega_{6,m,m} &= -\omega_{4,m,m}, & \omega_{7,m,m} &= 2(\omega_{x,m} - \omega_{y,m}), & \omega_{8,m,m} &= 0.
 \end{aligned} \tag{121}$$

So, we define four frequencies of t -eigenfunctions of internal interaction

$$\omega_{1,m} = 2(\omega_{x,m} + \omega_{y,m}), \quad \omega_{2,m} = 2\omega_{x,m}, \quad \omega_{3,m} = 2\omega_{y,m}, \quad \omega_{4,m} = 2(\omega_{x,m} - \omega_{y,m}). \tag{122}$$

Consequently, eight arguments of the t -eigenfunctions of external interaction are transformed into four independent arguments as

$$\begin{aligned}
 A_{1,m,m} &= 2[(\omega_{x,m} + \omega_{y,m})t + s_{x,m} + s_{y,m} + \alpha_{1,m}], & A_{2,m,m} &= 0, \\
 A_{3,m,m} &= 2\omega_{x,m}t + 2s_{x,m} + \alpha_{1,m} + \alpha_{2,m}, & A_{5,m,m} &= +A_{3,m,m}, \\
 A_{4,m,m} &= 2\omega_{y,m}t + 2s_{y,m} + \alpha_{1,m} - \alpha_{2,m}, & A_{6,m,m} &= -A_{4,m,m}, \\
 A_{7,m,m} &= 2[(\omega_{x,m} - \omega_{y,m})t + s_{x,m} - s_{y,m} + \alpha_{2,m}], & A_{8,m,m} &= 0.
 \end{aligned} \tag{123}$$

Thus, we define the following four arguments of t -eigenfunctions of internal interaction:

$$\begin{aligned}
 B_{1,m} &= \omega_{1,m}t + 2s_{x,m} + 2s_{y,m} + 2\alpha_{1,m}, & B_{2,m} &= \omega_{2,m}t + 2s_{x,m} + \alpha_{1,m} + \alpha_{2,m}, \\
 B_{4,m} &= \omega_{4,m}t + 2s_{x,m} - 2s_{y,m} + 2\alpha_{2,m}, & B_{3,m} &= \omega_{3,m}t + 2s_{y,m} + \alpha_{1,m} - \alpha_{2,m}.
 \end{aligned} \tag{124}$$

If $n = m$, 16 t -eigenfunctions of external interaction are converted into eight functions because

$$\begin{aligned}
 f_{1,m,m} &= \sin B_{1,m}, & f_{2,m,m} &= \cos B_{1,m}, & f_{3,m,m} &= 0, & f_{4,m,m} &= 1, \\
 f_{5,m,m} &= \sin B_{2,m}, & f_{6,m,m} &= \cos B_{2,m}, & f_{7,m,m} &= \sin B_{3,m}, & f_{8,m,m} &= \cos B_{3,m}, \\
 f_{9,m,m} &= f_{5,m,m}, & f_{10,m,m} &= f_{6,m,m}, & f_{11,m,m} &= -f_{7,m,m}, & f_{12,m,m} &= f_{8,m,m}, \\
 f_{13,m,m} &= \sin B_{4,m}, & f_{14,m,m} &= \cos B_{4,m}, & f_{15,m,m} &= 0, & f_{16,m,m} &= 1.
 \end{aligned} \tag{125}$$

Therefore, eight t -eigenfunctions of internal interaction are defined as follows:

$$\begin{aligned}
 g_{1,m} &= \sin B_{1,m}, & g_{2,m} &= \cos B_{1,m}, & g_{3,m} &= \sin B_{2,m}, & g_{4,m} &= \cos B_{2,m}, \\
 g_{5,m} &= \sin B_{3,m}, & g_{6,m} &= \cos B_{3,m}, & g_{7,m} &= \sin B_{4,m}, & g_{8,m} &= \cos B_{4,m}.
 \end{aligned} \tag{126}$$

Substitution of (126) in (125) yields asymptotics of t -eigenfunctions of external interaction as n approaches m

$$\begin{aligned}
f_{1,m,m} &= g_{1,m}, f_{2,m,m} = g_{2,m}, f_{3,m,m} = 0, & f_{4,m,m} &= 1, \\
f_{5,m,m} &= g_{3,m}, f_{6,m,m} = g_{4,m}, f_{7,m,m} = +g_{5,m}, f_{8,m,m} = g_{6,m}, \\
f_{9,m,m} &= g_{3,m}, f_{10,m,m} = g_{4,m}, f_{11,m,m} = -g_{5,m}, f_{12,m,m} = g_{6,m}, \\
f_{13,m,m} &= g_{7,m}, f_{14,m,m} = g_{8,m}, f_{15,m,m} = 0, & f_{16,m,m} &= 1.
\end{aligned} \tag{127}$$

The t -matrix of internal interaction ($n = m$) is a symmetrical one, viz.

$$f_{i,m,j,m} = \begin{bmatrix} f_{1,m}^2 & f_{1,m}f_{2,m} & f_{1,m}f_{3,m} & f_{1,m}f_{4,m} \\ f_{1,m}f_{2,m} & f_{2,m}^2 & f_{2,m}f_{3,m} & f_{2,m}f_{4,m} \\ f_{1,m}f_{3,m} & f_{2,m}f_{3,m} & f_{3,m}^2 & f_{3,m}f_{4,m} \\ f_{1,m}f_{4,m} & f_{2,m}f_{4,m} & f_{3,m}f_{4,m} & f_{4,m}^2 \end{bmatrix}. \tag{128}$$

Computing elements of the t -matrix of internal interaction via the t -eigenfunctions of internal interaction (126) gives

$$\begin{aligned}
2f_{1,m}^2 &= -g_{2,m} + 1, & 2f_{1,m}f_{2,m} &= +g_{1,m}, \\
2f_{1,m}f_{3,m} &= -g_{4,m} + g_{6,m}, & 2f_{1,m}f_{4,m} &= +g_{3,m} + g_{5,m}, \\
2f_{2,m}^2 &= +g_{2,m} + 1, & 2f_{2,m}f_{3,m} &= +g_{3,m} - g_{5,m}, \\
2f_{2,m}f_{4,m} &= +g_{4,m} + g_{6,m}, & 2f_{3,m}^2 &= -g_{8,m} + 1, \\
2f_{3,m}f_{4,m} &= +g_{7,m}, & 2f_{4,m}^2 &= +g_{8,m} + 1.
\end{aligned} \tag{129}$$

3.4. The Elementary Pulsons of Propagation

In eigenfunctions $f_{x,l,m}$ (98) with $l = 1, 2, 3, 4$, the elementary oscillons of propagation of the velocity potential (47), (67) are represented as follows:

$$\begin{aligned}
K_{o,a,m} &= a_m = ez_m Q_{y,m} f_{x,1,m}, & K_{o,c,m} &= c_m = ez_m R_{y,m} f_{x,3,m}, \\
K_{o,b,m} &= b_m = ez_m Q_{y,m} f_{x,2,m}, & K_{o,d,m} &= d_m = ez_m R_{y,m} f_{x,4,m}.
\end{aligned} \tag{130}$$

Substitution of (130) in (54) gives

$$\begin{aligned}
K_{p,a,m,a,m} &= \frac{1}{2} \rho_c ez_m^2 Q_{y,m}^2 f_{x,1,m}^2, & K_{p,c,m,c,m} &= \frac{1}{2} \rho_c ez_m^2 R_{y,m}^2 f_{x,3,m}^2, \\
K_{p,b,m,b,m} &= \frac{1}{2} \rho_c ez_m^2 Q_{y,m}^2 f_{x,2,m}^2, & K_{p,d,m,d,m} &= \frac{1}{2} \rho_c ez_m^2 R_{y,m}^2 f_{x,4,m}^2.
\end{aligned} \tag{131}$$

Computing squares of $f_{x,l,m}$ by (111) in terms of the x -eigenfunctions of internal interaction $g_{x,l,m}$ (109) yields

$$\begin{aligned}
K_{p,a,m,a,m} &= \frac{1}{4} \rho_c ez_m^2 Q_{y,m}^2 (1 - g_{x,2,m}), & K_{p,c,m,c,m} &= \frac{1}{4} \rho_c ez_m^2 R_{y,m}^2 (1 - g_{x,6,m}), \\
K_{p,b,m,b,m} &= \frac{1}{4} \rho_c ez_m^2 Q_{y,m}^2 (1 + g_{x,2,m}), & K_{p,d,m,d,m} &= \frac{1}{4} \rho_c ez_m^2 R_{y,m}^2 (1 + g_{x,6,m}).
\end{aligned} \tag{132}$$

For any frozen $y = y_0$, $z = z_0$, $t = t_0$, the elementary pulsons of propagation are reduced to 1-f critical positive pulsons in x with wavenumber $2\kappa_m$, amplitudes proportional to

$$\rho_c, ez_m^2(z_0), Q_{y,m}^2(y_0, t_0), R_{y,m}^2(y_0, t_0), \tag{133}$$

x -shifts (72), and energy shifts proportional to (133). The elementary pulsons of propagation are critical since they are produced by the complete squares of $f_{x,l,m}$.

The x -period of the elementary pulsions of propagation

$$L_{x,m,m} = \frac{1}{2} L_{x,m} = \frac{1}{2} U_m T_{x,m}. \tag{134}$$

Because integrals of $g_{x,2,m}$ and $g_{x,6,m}$ over $L_{x,m,m}$ vanish,

$$\begin{aligned} \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{p,i,m,i,m} dx &= \frac{1}{4} \rho_c e z_m^2 Q_{y,m}^2, \quad i = a, b, \\ \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{p,i,m,i,m} dx &= \frac{1}{4} \rho_c e z_m^2 R_{y,m}^2, \quad i = c, d. \end{aligned} \tag{135}$$

Integrals (135) show that two pairs of the elementary pulsions of propagation $K_{p,a,m,a,m}, K_{p,b,m,b,m}$ and $K_{p,c,m,c,m}, K_{p,d,m,d,m}$ on average transfer equal positive amounts of the kinetic energy along the x -axis.

Similarly, the elementary oscillons of propagation of the velocity potential may be written in the y -direction as

$$\begin{aligned} K_{o,a,m} = a_m &= e z_m Q_{x,m} f_{y,1,m}, \quad K_{o,b,m} = b_m = e z_m R_{x,m} f_{y,3,m}, \\ K_{o,c,m} = c_m &= e z_m Q_{x,m} f_{y,2,m}, \quad K_{o,d,m} = d_m = e z_m R_{x,m} f_{y,4,m}. \end{aligned} \tag{136}$$

Therefore, the elementary pulsions of propagation of the kinetic energy via $f_{y,l,m}$ become

$$\begin{aligned} K_{p,a,m,a,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{x,m}^2 f_{y,1,m}^2, \quad K_{p,b,m,b,m} = \frac{1}{2} \rho_c e z_m^2 R_{x,m}^2 f_{y,2,m}^2, \\ K_{p,c,m,c,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{x,m}^2 f_{y,2,m}^2, \quad K_{p,d,m,d,m} = \frac{1}{2} \rho_c e z_m^2 R_{x,m}^2 f_{y,4,m}^2, \end{aligned} \tag{137}$$

and in terms of $g_{y,l,m}$ take the following form:

$$\begin{aligned} K_{p,a,m,a,m} &= \frac{1}{4} \rho_c e z_m^2 Q_{x,m}^2 (1 - g_{y,2,m}), \quad K_{p,b,m,b,m} = \frac{1}{4} \rho_c e z_m^2 R_{x,m}^2 (1 - g_{y,6,m}), \\ K_{p,c,m,c,m} &= \frac{1}{4} \rho_c e z_m^2 Q_{x,m}^2 (1 + g_{y,2,m}), \quad K_{p,d,m,d,m} = \frac{1}{4} \rho_c e z_m^2 R_{x,m}^2 (1 + g_{y,6,m}). \end{aligned} \tag{138}$$

For any frozen $x = x_0, z = z_0, t = t_0$, the elementary pulsions of propagation are transformed into 1-f critical positive pulsions in y with wavenumber $2\lambda_m$, amplitudes proportional to

$$\rho_c, e z_m^2(z_0), Q_{x,m}^2(x_0, t_0), R_{x,m}^2(x_0, t_0), \tag{139}$$

y -shifts (81), and energy shifts proportional to (139). The elementary pulsions of propagation are also critical since they are formed by the complete squares of $f_{y,l,m}$.

Since amplitudes and shifts do not effect periods, the y -period

$$L_{y,m,m} = \frac{1}{2} L_{y,m} = \frac{1}{2} V_m T_{y,m}. \tag{140}$$

Because integrals of $g_{y,2,m}$ and $g_{y,6,m}$ over $L_{y,m,m}$ vanish,

$$\begin{aligned} \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{p,i,m,i,m} dy &= \frac{1}{4} \rho_c e z_m^2 Q_{x,m}^2, \quad i = a, c, \\ \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{p,i,m,i,m} dy &= \frac{1}{4} \rho_c e z_m^2 R_{x,m}^2, \quad i = b, d. \end{aligned} \tag{141}$$

Therefore, two pairs of the elementary pulsions of propagation

$K_{p,a,m,a,m}$, $K_{p,c,m,c,m}$ and $K_{p,b,m,b,m}$, $K_{p,d,m,d,m}$ on average transfer equal positive amounts of the kinetic energy along the y -axis.

With the help of (54), (47), and (84), the elementary pulsions of propagation become the complete squares of trigonometric polynomials

$$\begin{aligned} K_{p,a,m,a,m} &= \frac{ez_m^2}{8} (-Q_m f_{1,m} + R_m f_{3,m})^2, K_{p,b,m,b,m} = \frac{ez_m^2}{8} (+Q_m f_{2,m} - R_m f_{4,m})^2, \\ K_{p,d,m,d,m} &= \frac{ez_m^2}{8} (+Q_m f_{1,m} + R_m f_{3,m})^2, K_{p,c,m,c,m} = \frac{ez_m^2}{8} (+Q_m f_{2,m} + R_m f_{4,m})^2. \end{aligned} \quad (142)$$

Expanding the trigonometric polynomials and substituting elements of the t -matrix of internal interaction (129) via the t -eigenfunctions of internal interaction (126) returns

$$\begin{aligned} K_{p,a,m,a,m} &= \frac{1}{16} \rho_c ez_m^2 [Q_m^2 + R_m^2 - Q_m^2 g_{2,m} + 2Q_m R_m (g_{4,m} - g_{6,m}) - R_m^2 g_{8,m}], \\ K_{p,d,m,d,m} &= \frac{1}{16} \rho_c ez_m^2 [Q_m^2 + R_m^2 - Q_m^2 g_{2,m} - 2Q_m R_m (g_{4,m} - g_{6,m}) - R_m^2 g_{8,m}], \\ K_{p,b,m,b,m} &= \frac{1}{16} \rho_c ez_m^2 [Q_m^2 + R_m^2 + Q_m^2 g_{2,m} - 2Q_m R_m (g_{4,m} + g_{6,m}) + R_m^2 g_{8,m}], \\ K_{p,c,m,c,m} &= \frac{1}{16} \rho_c ez_m^2 [Q_m^2 + R_m^2 + Q_m^2 g_{2,m} + 2Q_m R_m (g_{4,m} + g_{6,m}) + R_m^2 g_{8,m}], \end{aligned} \quad (143)$$

where

$$Q_m^2 + R_m^2 = 2(Av_m^2 + Bv_m^2 + Cv_m^2 + Dv_m^2). \quad (144)$$

For any frozen $x = x_0, y = y_0, z = z_0$, the elementary pulsions of propagation are converted into at most quartet-frequency (4-f) critical positive pulsions in t , which are formed by superpositions of cosine waves $g_{2,m}, g_{4,m}, g_{6,m}, g_{8,m}$ with frequencies $2(\omega_{x,m} + \omega_{y,m}), 2\omega_{x,m}, 2\omega_{y,m}, 2(\omega_{x,m} - \omega_{y,m})$, correspondingly, amplitudes proportional to

$$\rho_c, Q_m, R_m, ez_m^2(z_0), \quad (145)$$

t -shifts depending on (91), and energy shifts proportional to

$$\rho_c, Q_m^2 + R_m^2, ez_m^2(z_0). \quad (146)$$

The elementary pulsions of propagation are critical as they are described by the complete squares of trigonometric polynomials (142).

Temporal periods $T_{1,m,m}, T_{2,m,m}, T_{3,m,m}, T_{4,m,m}$ of $g_{2,m}, g_{4,m}, g_{6,m}, g_{8,m}$, respectively, are

$$T_{1,m,m} = \frac{2\pi}{\omega_{x,m} + \omega_{y,m}}, T_{2,m,m} = \frac{2\pi}{\omega_{x,m}}, T_{3,m,m} = \frac{2\pi}{\omega_{y,m}}, T_{4,m,m} = \frac{2\pi}{\omega_{x,m} - \omega_{y,m}}. \quad (147)$$

Therefore, a temporal period of the elementary pulsions of propagation

$$\begin{aligned} T_{p,m,m} &= LCM(T_{1,m,m}, T_{2,m,m}, T_{3,m,m}, T_{4,m,m}) \\ &= k_{p,1,m,m} T_{1,m,m} = k_{p,2,m,m} T_{2,m,m} = k_{p,3,m,m} T_{3,m,m} = k_{p,4,m,m} T_{4,m,m}, \end{aligned} \quad (148)$$

where $k_{p,1,m,m}, k_{p,2,m,m}, k_{p,3,m,m}, k_{p,4,m,m}$ are integers.

Because integrals of cosine waves $g_{2,m}, g_{4,m}, g_{6,m}, g_{8,m}$ over

$T_{1,m,m}, T_{2,m,m}, T_{3,m,m}, T_{4,m,m}$ vanish, averages of the elementary pulsons of propagation over $T_{p,m,m}$ become

$$\frac{1}{T_{p,m,m}} \int_0^{T_{p,m,m}} K_{p,i,m,i,m} dt = \frac{1}{16} \rho_c e z_m^2 (Q_m^2 + R_m^2), \quad i = a, b, c, d. \quad (149)$$

Therefore, integral (149) validates that the elementary pulsons of propagation on average transfer in time the same positive amount of the kinetic energy.

The elementary pulson of propagation $K_{p,b,3,b,3}$ with $L_{x,3,3} = 2/3, L_{y,3,3} = 1/2$, and $T_{p,3,3} = 1$ is shown in **Figure 2** for independent parameters (95). The elementary pulsons $K_{p,a,m,a,m}, K_{p,b,m,b,m}, K_{p,c,m,c,m}, K_{p,d,m,d,m}$ are visualized by 1-f critical pulsons in x with periods $L_{x,m,m} = [4, 3/2, 2/3]$ and in terms of 1-f critical pulsons in y with periods $L_{y,m,m} = [9/2, 3/2, 1/2]$.

The elementary pulsons $K_{p,a,m,a,m}, K_{p,b,m,b,m}$ are displayed as 4-f critical pulsons in t with periods $T_{p,m,m} = [3, 3/2, 1], [4, 5, 5]$ local t -maximums, and $[4, 5, 5]$ local t -minimums, the numbers of which do not exceed or equal to $k_{p,1,m,m} = [5, 5, 5]$ due to $[1, 0, 0]$ merged t -maximums and t -minimums. The elementary pulsons $K_{p,c,m,c,m}, K_{p,d,m,d,m}$ as 4-f critical pulsons in t with periods $T_{p,m,m}, [4, 4, 5]$ local t -maximums, and $[4, 4, 5]$ local t -minimums, the numbers of which do not exceed or equal to $k_{p,1,m,m}$ because of $[1, 1, 0]$ merged t -maximums and t -minimums.

In the view of (131), (132), and (134),

$$K_{p,i,m,i,m} \left(x + \frac{1}{2} L_{x,m,m}, y, z, t \right) = K_{p,j,m,j,m} (x, y, z, t), \quad [i, j] = [a, b], [c, d], \quad (150)$$

$$2A_{p,a,y} = \frac{1}{2} \rho_c e z_m^2 Q_{y,m}^2, \quad 2A_{p,c,y} = \frac{1}{2} \rho_c e z_m^2 R_{y,m}^2.$$

So, x -maximums of the elementary pulson $K_{p,a,m,a,m}$, which are equal to $2A_{p,a,y}$, correspond to x -zeroes of the x -complementary pulson $K_{p,b,m,b,m}$ and x -zeroes of $K_{p,a,m,a,m}$ to x -maximums of $K_{p,b,m,b,m}$, which are also equal to $2A_{p,a,y}$. Similarly, x -maximums of the elementary pulson $K_{p,c,m,c,m}$, which are equal to $2A_{p,c,y}$, correspond to x -zeroes of the x -complementary pulson $K_{p,d,m,d,m}$ and x -zeroes of $K_{p,c,m,c,m}$ to x -maximums of $K_{p,d,m,d,m}$, which are equal to $2A_{p,c,y}$, as well.

Due to (137), (138), and (140),

$$K_{p,i,m,i,m} \left(x, y + \frac{1}{2} L_{y,m,m}, z, t \right) = K_{p,j,m,j,m} (x, y, z, t), \quad [i, j] = [a, c], [b, d], \quad (151)$$

$$2A_{p,a,x} = \frac{1}{2} \rho_c e z_m^2 Q_{x,m}^2, \quad 2A_{p,b,x} = \frac{1}{2} \rho_c e z_m^2 R_{x,m}^2.$$

Thus, y -maximums of the elementary pulson $K_{p,a,m,a,m}$, which are equal to $2A_{p,a,x}$, correspond to y -zeroes of the y -complementary pulson $K_{p,c,m,c,m}$ and y -zeroes of $K_{p,a,m,a,m}$ to y -maximums of $K_{p,c,m,c,m}$, which are also equal to $2A_{p,a,x}$. Analogously, y -maximums of the elementary pulson $K_{p,b,m,b,m}$, which are equal to $2A_{p,b,x}$, correspond to y -zeroes of the y -complementary pulson $K_{p,d,m,d,m}$ and y -zeroes of $K_{p,b,m,b,m}$ to y -maximums of $K_{p,d,m,d,m}$, which are also equal to $2A_{p,b,x}$.

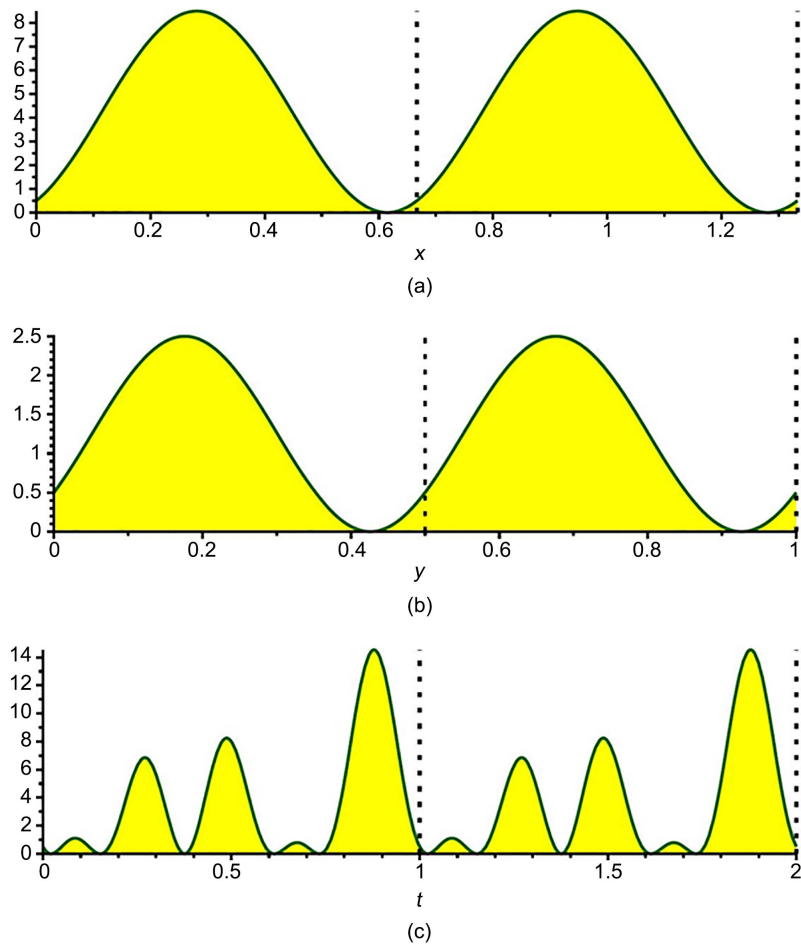


Figure 2. The elementary pulson of propagation $K_{p,b,3,b,3}$: (a)— $K_{p,b,3,b,3}(x)$ (132), (b)— $K_{p,b,3,b,3}(y)$ (138), (c)— $K_{p,b,3,b,3}(t)$ (143) for independent parameters (95).

3.5. The Internal Elementary Oscillons

Expressing definitions (57) in the x -eigenfunctions $f_{x,l,m}$ (98) with $l = 1, 2, 3, 4$ and using the x -matrix of internal interaction (111) yields the internal elementary oscillons via the x -eigenfunctions of internal interaction $g_{x,l,m}$ (109) in the following form:

$$\begin{aligned}
 K_{o,a,m,b,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{y,m}^2 g_{x,1,m}, & K_{o,c,m,d,m} &= \frac{1}{2} \rho_c e z_m^2 R_{y,m}^2 g_{x,5,m}, \\
 K_{o,a,m,c,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{y,m} R_{y,m} (h_{y,2,m} - g_{x,4,m}), \\
 K_{o,b,m,d,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{y,m} R_{y,m} (h_{y,2,m} + g_{x,4,m}), & (152) \\
 K_{o,a,m,d,m} &= + \frac{1}{2} \rho_c e z_m^2 Q_{y,m} R_{y,m} (h_{y,1,m} + g_{x,3,m}), \\
 K_{o,b,m,c,m} &= - \frac{1}{2} \rho_c e z_m^2 Q_{y,m} R_{y,m} (h_{y,1,m} - g_{x,3,m}).
 \end{aligned}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the first pair $K_{o,a,m,b,m}, K_{o,c,m,d,m}$ of internal elementary oscillons is reduced to 1-f neutral oscillons in x , which are

formed by sine waves $g_{x,1,m}, g_{x,5,m}$ with wavenumber $2\kappa_m$. The second pair $K_{o,a,m,c,m}, K_{o,b,m,d,m}$ and the third pair $K_{o,a,m,d,m}, K_{o,b,m,c,m}$ are presented by 1-f nonneutral oscillons in x , which are generated by cosine wave $g_{x,4,m}$ and sine wave $g_{x,3,m}$ with wavenumber $2\kappa_m$ and energy shifts proportional to

$$\rho_c, ez_m^2(z_0), Q_{y,m}(y_0, t_0), R_{y,m}(y_0, t_0), h_{y,1,m}(y_0, t_0), h_{y,2,m}(y_0, t_0). \tag{153}$$

For the first pair, amplitudes are proportional to (133) and x -shifts are specified by (72), for the second and third pairs amplitudes are proportional to $\rho_c, ez_m^2(z_0), Q_{y,m}(y_0, t_0), R_{y,m}(y_0, t_0)$ and x -shifts depend on (72). The x -period of the internal elementary oscillons coincides with the x -period of the elementary pulsons of propagation (134).

Since integrals of $g_{x,l,m}$ for $l = 1, 2, \dots, 6$ over $L_{x,m,m}$ vanish,

$$\begin{aligned} \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{o,i,m,j,m} dx &= 0, & [i, j] &= [a, b], [c, d], \\ \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{o,i,m,j,m} dx &= \frac{\rho_c}{2} ez_m^2 Q_{y,m} R_{y,m} h_{y,2,m}, & [i, j] &= [a, c], [b, d], \\ \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{o,a,m,d,m} dx &= -\frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{o,b,m,c,m} dx = \frac{\rho_c}{2} ez_m^2 Q_{y,m} R_{y,m} h_{y,1,m}. \end{aligned} \tag{154}$$

It is at tedious but a straightforward procedure to show that

$$\begin{aligned} Q_{y,m} R_{y,m} h_{y,1,m} &= Q_{y,m} R_{y,m} \sin \kappa_m (\alpha_{y,m} - \beta_{y,m}) = -A y_m D y_m + B y_m C y_m \\ &= A v_m D v_m - B v_m C v_m, \\ Q_{y,m} R_{y,m} h_{y,2,m} &= Q_{y,m} R_{y,m} \cos \kappa_m (\alpha_{y,m} - \beta_{y,m}) = +A y_m B y_m + C y_m D y_m \\ &= \frac{1}{2} (A v_m^2 + B v_m^2 - C v_m^2 - D v_m^2) s 2 y_m + (A v_m C y_m + B v_m D y_m) c 2 y_m, \end{aligned} \tag{155}$$

where trigonometric functions of the double argument

$$s 2 y_m = \sin(2 \lambda_m Y_m), \quad c 2 y_m = \cos(2 \lambda_m Y_m). \tag{156}$$

Integrals (154) show that the first pair of internal elementary oscillons does not transfer on average any kinetic energy, the internal elementary oscillons of the second pair transfer on average equal amounts of the kinetic energy of the same sign, and the internal elementary oscillons of the third pair transfer on average equal amounts of the kinetic energy of opposite signs along the x -axis.

For two vector fields \mathbf{u} and \mathbf{v} , a scalar field of the kinetic energy of interaction is given by

$$K_i(x, y, z, t) = \frac{\rho_c}{2} \mathbf{u}(x, y, z, t) \cdot \mathbf{w}(x, y, z, t) = \frac{\rho_c}{2} (u_x w_x + u_y w_y + u_z w_z). \tag{157}$$

By the definition of the dot product

$$K_i = \frac{\rho_c}{2} |\mathbf{u}| |\mathbf{w}| \cos A_i, \tag{158}$$

where $A_i(x, y, z, t)$ is an angle between vector fields \mathbf{u} and \mathbf{v} at a point (x, y, z, t) .

So, the maximal magnitude of K_i

$$|K_i|_{\max} = \frac{\rho_c}{2} |\mathbf{u}| |\mathbf{w}|, \tag{159}$$

does not depend on A_i , but the sign of K_i is determined by $\cos A_i$. If the vector fields \mathbf{u} and \mathbf{v} are unidirectional at (x, y, z, t) , then $K_i = |K_i|_{\max}$. If the vector fields have opposite directions, then $K_i = -|K_i|_{\max}$. The kinetic energy of interaction vanishes, if the vector fields \mathbf{u} and \mathbf{v} are orthogonal at (x, y, z, t) .

Thus, K_i becomes negative if $\pi/2 < A_i < 3\pi/2$ on the contrary to the kinetic energy of propagation of the vector field \mathbf{u}

$$K_p(x, y, z, t) = \frac{\rho_c}{2} \mathbf{u}(x, y, z, t) \cdot \mathbf{u}(x, y, z, t) = \frac{\rho_c}{2} (u_x^2 + u_y^2 + u_z^2) \tag{160}$$

that is always positively defined. To summarize, the scalar field of the kinetic energy of propagation K_p has a pulsatory nature and the scalar field of the kinetic energy of interaction K_i an oscillatory one.

Let a nonlinear wave $f(v)$ with an energy shift E_o and amplitudes $[-A_{\min}, A_{\max}]$ be decomposed into the inhomogeneous Fourier expansion via M cosine waves and sine waves with wavenumbers μ_m and v -shifts proportional to a_m, b_m

$$f(v) = E_o + \sum_{m=1}^M [A_m \cos(\mu_m v + a_m) + B_m \sin(\mu_m v + b_m)] \tag{161}$$

such that

$$-A_{\min} \leq f(v) - E_o \leq A_{\max}. \tag{162}$$

Then a period of $f(v)$ in a variable $v = [x, y, z, t]$

$$P_v = LCM \left(\frac{2\pi}{\mu_1}, \dots, \frac{2\pi}{\mu_m}, \dots, \frac{2\pi}{\mu_M} \right) \tag{163}$$

and the average of $f(v)$ over period P_v

$$\frac{1}{P_v} \int_0^{P_v} f(v) dv = E_o. \tag{164}$$

An oscillon $f(v)$ is termed a neutral oscillon, if the average of $f(v)$ over P_v vanishes, *i.e.* $E_o = 0$. So, the neutral oscillon doesn't transfer the kinetic energy along the v -direction.

An oscillon $f(v)$ is called a positive (nonneutral) oscillon, if the average of $f(v)$ over P_v is positive since $0 < E_o < A_{\min}$ and the nonneutral oscillon transfers the positive amount E_o of the kinetic energy along the v -direction.

When $E_o = A_{\min}$, the positive oscillon is transformed into a critical positive pulson, which also transfers the positive amount E_o of the kinetic energy along the v -direction. If $E_o > A_{\min}$, the critical positive pulson is converted into a supercritical pulson, which transfers the positive amount E_o of the kinetic energy along the v -direction, as well.

Similarly, an oscillon $f(v)$ is named a negative (nonneutral) oscillon, if the

average of $f(v)$ over P_v is negative because $-A_{\max} < E_o < 0$ and the non-neutral oscillon transfers the negative amount E_o of the kinetic energy along the v -direction.

When $E_o = -A_{\max}$, the negative oscillon is transformed into a critical negative pulson, which also transfers the negative amount E_o of the kinetic energy along the v -direction. If $E_o < -A_{\max}$, the critical negative pulson is converted into a subcritical pulson, which transfers the negative amount E_o of the kinetic energy along the v -direction, as well.

We then represent definitions (57) in the y -eigenfunctions $f_{y,l,m}$ with $l = 1, 2, 3, 4$ and use the y -matrix of internal interaction to compute the internal elementary oscillons in terms of the y -eigenfunctions of internal interaction $g_{y,l,m}$ as follows

$$\begin{aligned}
 K_{o,a,m,b,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{x,m} R_{x,m} (h_{x,2,m} - g_{y,4,m}), \\
 K_{o,c,m,d,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{x,m} R_{x,m} (h_{x,2,m} + g_{y,4,m}), \\
 K_{o,a,m,c,m} &= \frac{1}{2} \rho_c e z_m^2 Q_{x,m}^2 g_{y,1,m}, \quad K_{o,b,m,d,m} = \frac{1}{2} \rho_c e z_m^2 R_{x,m}^2 g_{y,5,m}, \\
 K_{o,a,m,d,m} &= + \frac{1}{2} \rho_c e z_m^2 Q_{x,m} R_{x,m} (h_{x,1,m} + g_{y,3,m}), \\
 K_{o,b,m,c,m} &= - \frac{1}{2} \rho_c e z_m^2 Q_{x,m} R_{x,m} (h_{x,1,m} - g_{y,3,m}).
 \end{aligned} \tag{165}$$

For any frozen $x = x_0, z = z_0, t = t_0$, the first pair $K_{o,a,m,b,m}, K_{o,c,m,d,m}$ and the third pair $K_{o,a,m,d,m}, K_{o,b,m,c,m}$ of internal elementary oscillons are represented by 1-f nonneutral oscillons in y , which are generated by cosine wave $g_{y,4,m}$ and sine wave $g_{y,3,m}$ with wavenumber $2\lambda_m$ and energy shifts proportional to

$$\rho_c, e z_m^2(z_0), Q_{x,m}(x_0, t_0), R_{x,m}(x_0, t_0), h_{x,1,m}(x_0, t_0), h_{x,2,m}(x_0, t_0). \tag{166}$$

The second pair $K_{o,a,m,c,m}, K_{o,b,m,d,m}$ is transformed into 1-f neutral oscillons in y , which are formed by sine waves $g_{y,1,m}, g_{y,5,m}$ with wavenumber $2\lambda_m$. For the first and third pairs, amplitudes are proportional to $\rho_c, e z_m^2(z_0), Q_{x,m}(y_0, t_0), R_{x,m}(y_0, t_0)$ and y -shifts depend on (81). For the second pair, amplitudes are proportional to (139) and y -shifts are specified by (81). The y -period of the internal elementary oscillons equals to the y -period of the elementary pulsions of propagation (140).

Because integrals of $g_{y,l,m}$ for $l = 1, 2, \dots, 6$ over the period $L_{y,m,m}$ vanish,

$$\begin{aligned}
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{o,i,m,j,m} dy &= \frac{\rho_c}{2} e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m}, \quad [i, j] = [a, b], [c, d], \\
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{o,i,m,j,m} dy &= 0, \quad [i, j] = [a, c], [b, d], \\
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{o,a,m,d,m} dy &= - \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{o,b,m,c,m} dy = \frac{\rho_c}{2} e z_m^2 Q_{x,m} R_{x,m} h_{x,1,m}.
 \end{aligned} \tag{167}$$

It may be shown that

$$\begin{aligned}
Q_{x,m} R_{x,m} h_{x,1,m} &= Q_{x,m} R_{x,m} \sin \lambda_m (\alpha_{x,m} - \beta_{x,m}) = -A x_m D x_m + B x_m C x_m \\
&= A v_m D v_m - B v_m C v_m, \\
Q_{x,m} R_{x,m} h_{x,2,m} &= Q_{x,m} R_{x,m} \cos \lambda_m (\alpha_{x,m} - \beta_{x,m}) = +A x_m B x_m + C x_m D x_m \\
&= \frac{1}{2} (A v_m^2 - B v_m^2 + C v_m^2 - D v_m^2) s 2 x_m + (A v_m B v_m + C v_m D v_m) c 2 x_m,
\end{aligned} \tag{168}$$

where trigonometric functions of the double argument

$$s 2 x_m = \sin(2 \lambda_m X_m), \quad c 2 x_m = \cos(2 \lambda_m X_m). \tag{169}$$

Thus, the internal elementary oscillons of the first pair transfer on average equal amounts of the kinetic energy of the same sign, the second pair of internal elementary oscillons on average does not transfer on average any kinetic energy, and the internal elementary oscillons of the third pair transfer on average equal amounts of the kinetic energy of opposite signs along the y -axis.

Usage of definitions (57) together with trigonometric polynomials (84) in t -eigenfunctions (113) and application of the t -matrix of internal interaction (129) yields the internal elementary oscillons in terms of the t -eigenfunctions of internal interaction (126) in the following t -form:

$$\begin{aligned}
K_{o,a,m,b,m} &= -\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{1,m} - 2 Q_m R_m g_{3,m} + R_m^2 g_{7,m}), \\
K_{o,c,m,d,m} &= +\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{1,m} + 2 Q_m R_m g_{3,m} + R_m^2 g_{7,m}), \\
K_{o,a,m,c,m} &= -\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{1,m} + 2 Q_m R_m g_{5,m} - R_m^2 g_{7,m}), \\
K_{o,b,m,d,m} &= +\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{1,m} - 2 Q_m R_m g_{5,m} - R_m^2 g_{7,m}), \\
K_{o,a,m,d,m} &= +\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{2,m} - R_m^2 g_{8,m} - Q_m^2 + R_m^2), \\
K_{o,b,m,c,m} &= +\frac{1}{8} \rho_c e z_m^2 (Q_m^2 g_{2,m} - R_m^2 g_{8,m} + Q_m^2 - R_m^2),
\end{aligned} \tag{170}$$

where

$$-Q_m^2 + R_m^2 = 4(A v_m D v_m - B v_m C v_m). \tag{171}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the first pair $K_{o,a,m,b,m}, K_{o,c,m,d,m}$ of internal elementary oscillons is presented by at most trio-frequency (3-f) neutral oscillons in t , which are produced by superpositions of sine waves $g_{1,m}, g_{3,m}, g_{7,m}$ with frequencies $2(\omega_{x,m} + \omega_{y,m}), 2\omega_{x,m}, 2(\omega_{x,m} - \omega_{y,m})$, correspondingly. The second pair $K_{o,a,m,c,m}, K_{o,b,m,d,m}$ is represented by at most 3-f neutral oscillons in t , which are generated by superpositions of sine waves $g_{1,m}, g_{5,m}, g_{7,m}$ with frequencies $2(\omega_{x,m} + \omega_{y,m}), 2\omega_{y,m}, 2(\omega_{x,m} - \omega_{y,m})$, respectively. The third pair $K_{o,a,m,d,m}, K_{o,b,m,c,m}$ is formed by 2-f nonneutral oscillons in t , which are created by superpositions of cosine waves $g_{2,m}, g_{8,m}$ with frequencies $2(\omega_{x,m} + \omega_{y,m}), 2(\omega_{x,m} - \omega_{y,m})$, correspondingly, and energy shifts proportional to $\rho_c, Q_m^2 - R_m^2, e z_m^2(z_0)$. For all pairs, amplitudes are proportional to (145) and t -shifts depend on (91).

A temporal period of the first pair

$$\begin{aligned} T_{o,1,m,m} &= LCM(T_{1,m,m}, T_{2,m,m}, T_{4,m,m}) \\ &= k_{o,1,1,m,m} T_{1,m,m} = k_{o,1,2,m,m} T_{2,m,m} = k_{o,1,4,m,m} T_{4,m,m}, \end{aligned} \tag{172}$$

where $k_{o,1,1,m,m}, k_{o,1,2,m,m}, k_{o,1,4,m,m}$ are integers and $T_{1,m,m}, T_{2,m,m}, T_{3,m,m}, T_{4,m,m}$ are given by (147).

Similarly, a temporal period of the second pair

$$\begin{aligned} T_{o,2,m,m} &= LCM(T_{1,m,m}, T_{3,m,m}, T_{4,m,m}) \\ &= k_{o,2,1,m,m} T_{1,m,m} = k_{o,2,3,m,m} T_{3,m,m} = k_{o,2,4,m,m} T_{4,m,m}, \end{aligned} \tag{173}$$

where $k_{o,2,1,m,m}, k_{o,2,3,m,m}, k_{o,2,4,m,m}$ are integers.

Analogously, a temporal period of the third pair

$$T_{o,3,m,m} = LCM(T_{1,m,m}, T_{4,m,m}) = k_{o,3,1,m,m} T_{1,m,m} = k_{o,3,4,m,m} T_{4,m,m}, \tag{174}$$

where $k_{o,3,1,m,m}, k_{o,3,4,m,m}$ are integers.

Because integrals of $g_{1,m}, g_{3,m}, g_{5,m}, g_{7,m}$ over $T_{1,m,m}, T_{2,m,m}, T_{3,m,m}, T_{4,m,m}$ vanish, averages of the internal elementary oscillons over the relevant t -periods become

$$\begin{aligned} \frac{1}{T_{o,1,m,m}} \int_0^{T_{o,1,m,m}} K_{o,i,m,j,m} dt &= 0, [i, j] = [a, b], [c, d], \\ \frac{1}{T_{o,2,m,m}} \int_0^{T_{o,2,m,m}} K_{o,i,m,j,m} dt &= 0, [i, j] = [a, c], [b, d], \\ \frac{1}{T_{o,3,m,m}} \int_0^{T_{o,3,m,m}} K_{o,a,m,d,m} dt &= -\frac{1}{T_{o,3,m,m}} \int_0^{T_{o,3,m,m}} K_{o,b,m,c,m} dt = \frac{\rho_c}{8} e z_m^2 (-Q_m^2 + R_m^2). \end{aligned} \tag{175}$$

Therefore, the first and second pairs of internal elementary oscillons do not transfer on average any kinetic energy in time, the internal elementary oscillons of the third pair on average transfer in time equal amounts of the kinetic energy of the opposite signs.

The internal elementary oscillon $K_{o,a,3,b,3}$ with

$L_{x,3,3} = 2/3, L_{y,3,3} = 1/2, T_{o,1,3,3} = 1$ is visualized in **Figure 3** for independent parameters (95). The first pair is displayed by 1-f neutral oscillons in x with periods $L_{x,m,m} = [4, 3/2, 2/3]$, the second and third pairs via 1-f nonneutral oscillons in x with periods $L_{x,m,m}$.

The first and third pairs are visualized in terms of 1-f nonneutral oscillons in y with periods $L_{y,m,m} = [9/2, 3/2, 1/2]$ and the second pair by 1-f neutral oscillons in y with periods $L_{y,m,m}$.

The first pair is shown by 3-f neutral oscillons in t with periods $T_{o,1,m,m} = [3, 3/2, 1], [3, 5, 5], [3, 3, 5]$ local t -maximums, and $[3, 5, 5], [3, 3, 5]$ local t -minimums, the numbers of which do not exceed or equal to $k_{o,1,m,m} = [5, 5, 5]$ due to $[2, 0, 0], [2, 2, 0]$ merged t -maximums and t -minimums, respectively. The second pair is visualized via 3-f neutral oscillons in t with periods $T_{o,2,m,m} = [3, 3/2, 1], [3, 4, 5]$ local t -maximums, and $[3, 4, 5]$ local t -minimums, the numbers of which do not exceed or equal to $k_{o,2,m,m} = [5, 5, 5]$ due to $[2, 1, 0]$ merged t -maximums and t -minimums. The

third pair is displayed in terms of 2-f neutral oscillons in t with periods $T_{o,3,m,m} = [3, 3/2, 1]$, $[5, 5, 5]$ local t -maximums, and $[5, 5, 5]$ local t -minimums, the numbers of which are indicated by $k_{o,3,m,m} = [5, 5, 5]$.

In agreement with (152),

$$\begin{aligned} K_{o,b,m,d,m} &= \rho_c e z_m^2 Q_{y,m} R_{y,m} h_{y,2,m} - K_{o,a,m,c,m}, \\ K_{o,a,m,d,m} &= \rho_c e z_m^2 Q_{y,m} R_{y,m} h_{y,1,m} + K_{o,b,m,c,m}. \end{aligned} \tag{176}$$

Locations of x -maximums of $K_{o,a,m,c,m}$ coincide with locations of x -minimums of $K_{o,b,m,d,m}$ and locations of x -minimums of $K_{o,a,m,c,m}$ coincide with locations of x -maximums of $K_{o,b,m,d,m}$. Locations of x -maximums and x -minimums of $K_{o,a,m,d,m}$ coincide with locations of x -maximums and x -minimums of $K_{o,b,m,c,m}$.

Because of (165),

$$\begin{aligned} K_{o,c,m,d,m} &= \rho_c e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m} - K_{o,a,m,b,m}, \\ K_{o,a,m,d,m} &= \rho_c e z_m^2 Q_{x,m} R_{x,m} h_{x,1,m} + K_{o,b,m,c,m}. \end{aligned} \tag{177}$$

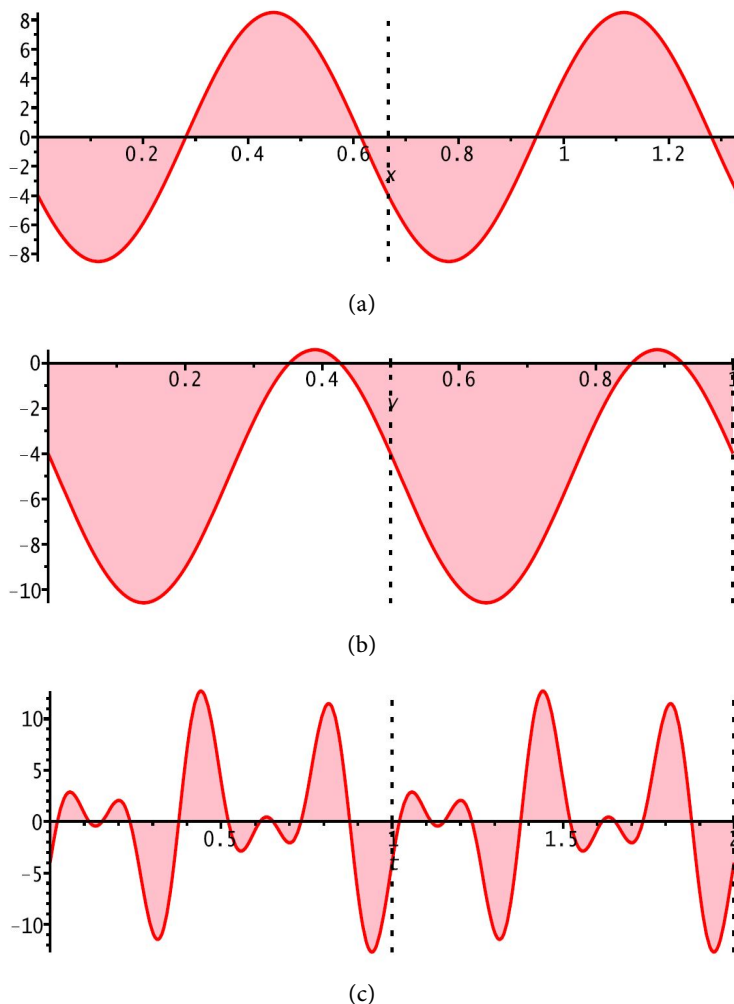


Figure 3. The internal elementary oscillon $K_{o,a,3,b,3}$: (a)— $K_{o,a,3,b,3}(x)$ (152), (b)— $K_{o,a,3,b,3}(y)$ (165), (c)— $K_{o,a,3,b,3}(t)$ (170) for independent parameters (95).

Locations of y -maximums of $K_{o,a,m,b,m}$ coincide with locations of y -minimums of $K_{o,c,m,d,m}$ and locations of y -minimums of $K_{o,a,m,b,m}$ coincide with locations of y -maximums of $K_{o,c,m,d,m}$. Locations of y -maximums and y -minimums of $K_{o,a,m,d,m}$ coincide with locations of y -maximums and y -minimums of $K_{o,b,m,c,m}$.

Due to (170),

$$K_{o,a,m,d,m} = \frac{1}{4} \rho_c e z_m^2 (-Q_m^2 + R_m^2) + K_{o,b,m,c,m}. \tag{178}$$

Locations of t -maximums and t -minimums of $K_{o,b,m,c,m}$ coincide with locations of t -maximums and t -minimums of $K_{o,a,m,d,m}$.

3.6. The Diagonal Elementary Oscillons

Using (130), we recast definitions (60) of the diagonal elementary oscillons in terms of x -eigenfunctions $f_{x,l,m}, f_{x,l,n}$ (98) and convert them with the help of the x -matrix of external interaction (104) into the following form:

$$\begin{aligned} K_{o,a,m,a,n} &= \frac{1}{2} \rho_c e z_m e z_n Q_{y,m} Q_{y,n} (-f_{x,2,m,n} + f_{x,4,m,n}), \\ K_{o,b,m,b,n} &= \frac{1}{2} \rho_c e z_m e z_n Q_{y,m} Q_{y,n} (+f_{x,2,m,n} + f_{x,4,m,n}), \\ K_{o,c,m,c,n} &= \frac{1}{2} \rho_c e z_m e z_n R_{y,m} R_{y,n} (-f_{x,14,m,n} + f_{x,16,m,n}), \\ K_{o,d,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n R_{y,m} R_{y,n} (+f_{x,14,m,n} + f_{x,16,m,n}). \end{aligned} \tag{179}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal elementary oscillons are transformed into 2-f neutral oscillons in x , which are produced by superpositions of two pairs of cosine waves $f_{x,2,m,n}, f_{x,4,m,n}$ and $f_{x,14,m,n}, f_{x,16,m,n}$ with two wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, respectively, amplitudes proportional to

$$\rho_c, e z_m(z_0), e z_n(z_0), Q_{y,m}(y_0, t_0), Q_{y,n}(y_0, t_0), R_{y,m}(y_0, t_0), R_{y,n}(y_0, t_0) \tag{180}$$

and x -shifts depending on

$$\kappa_m, \kappa_n, s_{\alpha,y,m}(y_0, t_0), s_{\alpha,y,n}(y_0, t_0), s_{\beta,y,m}(y_0, t_0), s_{\beta,y,n}(y_0, t_0). \tag{181}$$

If $n = m$, the diagonal elementary oscillons (179) are reduced to the doubled elementary pulsions of propagation (132) in the view of (108)-(109).

A spatial period of the diagonal elementary oscillons along the x -axis

$$L_{x,m,n} = LCM(L_{x,1,m,n}, L_{x,2,m,n}) = k_{x,1,m,n} L_{x,1,m,n} = k_{x,2,m,n} L_{x,2,m,n}, \tag{182}$$

where

$$L_{x,1,m,n} = \frac{2\pi}{\kappa_{1,m,n}}, L_{x,2,m,n} = \frac{2\pi}{\kappa_{2,m,n}} \tag{183}$$

are x -periods of $f_{x,2,m,n}, f_{x,4,m,n}$ and $f_{x,14,m,n}, f_{x,16,m,n}$, respectively, and $k_{x,1,m,n}, k_{x,2,m,n}$ are integers.

Because integrals of $f_{x,l,m,n}$ for $l = 1, 2, 5, 6, 9, 10, 13, 14$ over $L_{x,1,m,n}$ vanish and integrals of $f_{x,l,m,n}$ for $l = 3, 4, 7, 8, 11, 12, 15, 16$ over $L_{x,2,m,n}$ also vanish, averages of the diagonal elementary oscillons over $L_{x,m,n}$ vanish since

$$\int_0^{L_{x,m,n}} K_{o,i,m,i,n} dx = 0, \quad i = a, b, c, d \tag{184}$$

in the view of (179), (182) and the additive interval property. The diagonal elementary oscillons are neutral since on average they do not transfer any kinetic energy along the x -axis.

Similarly, representation of definitions (60) in y -eigenfunctions $f_{y,l,m}, f_{y,l,n}$ with the help of (136) and transformation into y -eigenfunctions of external interaction $f_{y,l,m,n}$ yield

$$\begin{aligned} K_{o,a,m,a,n} &= \frac{1}{2} \rho_c e z_m e z_n Q_{x,m} Q_{x,n} (-f_{y,2,m,n} + f_{y,4,m,n}), \\ K_{o,c,m,c,n} &= \frac{1}{2} \rho_c e z_m e z_n Q_{x,m} Q_{x,n} (+f_{y,2,m,n} + f_{y,4,m,n}), \\ K_{o,b,m,b,n} &= \frac{1}{2} \rho_c e z_m e z_n R_{x,m} R_{x,n} (-f_{y,14,m,n} + f_{y,16,m,n}), \\ K_{o,d,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n R_{x,m} R_{x,n} (+f_{y,14,m,n} + f_{y,16,m,n}). \end{aligned} \tag{185}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal elementary oscillons are transformed into 2-f neutral oscillons in y , which are formed by superpositions of two pairs of cosine waves $f_{y,2,m,n}, f_{y,4,m,n}$ and $f_{y,14,m,n}, f_{y,16,m,n}$ with wave-numbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, respectively, amplitudes proportional to

$$\rho_c, e z_m(z_0), e z_n(z_0), Q_{x,m}(x_0, t_0), Q_{x,n}(x_0, t_0), R_{x,m}(x_0, t_0), R_{x,n}(x_0, t_0) \tag{186}$$

and y -shifts depending on

$$\lambda_m, \lambda_n, s_{\alpha,x,m}(x_0, t_0), s_{\alpha,x,n}(x_0, t_0), s_{\beta,x,m}(x_0, t_0), s_{\beta,x,n}(x_0, t_0). \tag{187}$$

If $n = m$, the diagonal elementary oscillons (185) are reduced to the doubled elementary pulsons of propagation (138).

A y -period of the diagonal elementary oscillons

$$L_{y,m,n} = LCM(L_{y,1,m,n}, L_{y,2,m,n}) = k_{y,1,m,n} L_{y,1,m,n} = k_{y,2,m,n} L_{y,2,m,n}, \tag{188}$$

where

$$L_{y,1,m,n} = \frac{2\pi}{\lambda_{1,m,n}}, \quad L_{y,2,m,n} = \frac{2\pi}{\lambda_{2,m,n}} \tag{189}$$

are y -periods of $f_{y,2,m,n}, f_{y,4,m,n}$ and $f_{y,14,m,n}, f_{y,16,m,n}$, respectively, and $k_{y,1,m,n}, k_{y,2,m,n}$ are integers.

In the view of vanishing integrals of $f_{y,l,m,n}$ for $l = 1, 2, 5, 6, 9, 10, 13, 14$ over $L_{y,1,m,n}$ and vanishing integrals of $f_{y,l,m,n}$ for $l = 3, 4, 7, 8, 11, 12, 15, 16$ over $L_{y,2,m,n}$,

$$\int_0^{L_{y,m,n}} K_{o,i,m,i,n} dy = 0, \quad i = a, b, c, d. \tag{190}$$

The diagonal elementary oscillons are also neutral along the y -axis since on average they do not transfer any kinetic energy.

In agreement with (60), (47), (84), we use the t -matrix of external interaction

(120) to compute the inhomogeneous Fourier expansion of the diagonal elementary oscillons via the t -eigenfunctions of external interaction (117) as follows:

$$\begin{aligned}
 K_{o,a,m,a,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[-Q_m Q_n (f_{2,m,n} - f_{4,m,n}) + Q_m R_n (f_{6,m,n} - f_{8,m,n}) \right. \\
 &\quad \left. + R_m Q_n (f_{10,m,n} - f_{12,m,n}) - R_m R_n (f_{14,m,n} - f_{16,m,n}) \right], \\
 K_{o,d,m,d,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[-Q_m Q_n (f_{2,m,n} - f_{4,m,n}) - Q_m R_n (f_{6,m,n} - f_{8,m,n}) \right. \\
 &\quad \left. - R_m Q_n (f_{10,m,n} - f_{12,m,n}) - R_m R_n (f_{14,m,n} - f_{16,m,n}) \right], \\
 K_{o,b,m,b,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[+Q_m Q_n (f_{2,m,n} + f_{4,m,n}) - Q_m R_n (f_{6,m,n} + f_{8,m,n}) \right. \\
 &\quad \left. - R_m Q_n (f_{10,m,n} + f_{12,m,n}) + R_m R_n (f_{14,m,n} + f_{16,m,n}) \right], \\
 K_{o,c,m,c,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[+Q_m Q_n (f_{2,m,n} + f_{4,m,n}) + Q_m R_n (f_{6,m,n} + f_{8,m,n}) \right. \\
 &\quad \left. + R_m Q_n (f_{10,m,n} + f_{12,m,n}) + R_m R_n (f_{14,m,n} + f_{16,m,n}) \right].
 \end{aligned} \tag{191}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal elementary oscillons are transformed into at most octet-frequency (8-f) neutral oscillons in t , which are produced by superpositions of cosine waves $f_{2l,m,n}$ with frequencies $\omega_{l,m,n}$ for $l = 1, 2, \dots, 8$, amplitudes proportional to

$$\rho_c, Q_m, Q_n, R_m, R_n, e z_m(z_0), e z_n(z_0) \tag{192}$$

and t -shifts depending on

$$\omega_{x,m}, \omega_{x,n}, \omega_{y,m}, \omega_{y,n}, \alpha_{1,m}, \alpha_{1,n}, \alpha_{2,m}, \alpha_{2,n}, s_{x,m}(x_0), s_{x,n}(x_0), s_{y,m}(y_0), s_{y,n}(y_0). \tag{193}$$

If $n = m$, the diagonal elementary oscillons (191) are reduced to the doubled elementary pulsions of propagation (143) due to (127).

Since a t -period $T_{l,m,n}$ of $f_{2l,m,n}$ for $l = 1, 2, \dots, 8$ is

$$T_{l,m,n} = \frac{2\pi}{\omega_{l,m,n}}, \tag{194}$$

a t -period of the diagonal elementary oscillons

$$\begin{aligned}
 T_{d,m,n} &= LCM(T_{1,m,n}, \dots, T_{l,m,n}, \dots, T_{8,m,n}) \\
 &= k_{d,1,m,n} T_{1,m,n} = \dots = k_{d,l,m,n} T_{l,m,n} = \dots = k_{d,8,m,n} T_{8,m,n},
 \end{aligned} \tag{195}$$

where $k_{d,1,m,n}, \dots, k_{d,l,m,n}, \dots, k_{d,8,m,n}$ are integers.

In the view of vanishing integrals of $f_{2l,m,n}$ over $T_{l,m,n}$,

$$\int_0^{T_{d,m,n}} K_{o,i,m,i,n} dt = 0, \quad i = a, b, c, d. \tag{196}$$

Integral (196) demonstrates that the diagonal elementary oscillons are neutral in time, as well, as they do not transfer on average any kinetic energy.

The diagonal elementary oscillon $K_{o,b,2,b,3}$ with $L_{x,2,3} = 12, L_{y,2,3} = 3/2, T_{d,2,3} = 6$ is shown **Figure 4** for independent parameters (95).

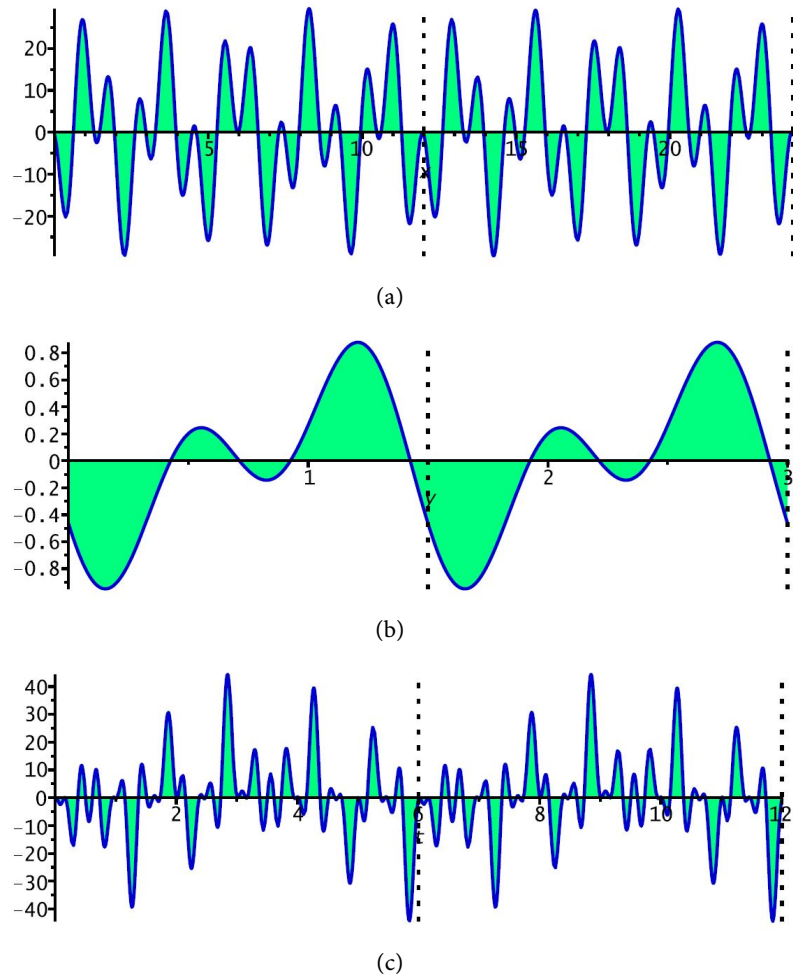


Figure 4. The diagonal elementary oscillon $K_{o,b,2,b,3}$: (a)— $K_{o,b,2,b,3}(x)$ (179), (b)— $K_{o,b,2,b,3}(y)$ (185), (c)— $K_{o,b,2,b,3}(t)$ (191) for independent parameters (95).

Along the x -axis, the diagonal elementary oscillons are visualized by 2-f neutral oscillons in x with periods $L_{x,m,n} = [L_{x,1,2}, L_{x,1,3}, L_{x,2,3}] = [24, 8, 12], [11, 7, 13]$ local x -maximums, and $[11, 7, 13]$ x -local minimums, the numbers of which are specified by $k_{x,1,m,n} = [k_{x,1,1,2}, k_{x,1,1,3}, k_{x,1,2,3}] = [11, 7, 13]$.

Along the y -axis, the diagonal elementary oscillons are displayed via 2-f neutral oscillons in y with periods $L_{y,m,n} = [9/2, 9/2, 3/2], [2, 5, 2]$ local y -maximums, and $[2, 5, 2]$ local y -minimums, the numbers of which are indicated by $k_{y,1,m,n} = [2, 5, 2]$.

The diagonal elementary oscillons $K_{o,a,m,a,n}, K_{o,b,m,b,n}$ are [7-f, 7-f, 6-f] neutral oscillons in t with periods $T_{d,m,n} = [6, 3, 6], [13, 9, 25]$ local t -maximums, and $[13, 9, 25]$ local t -minimums, the numbers of which do not exceed or equal to $k_{d,1,m,n} = [15, 10, 25]$ due to $[2, 1, 0]$ merged t -maximums and t -minimums. The diagonal elementary oscillons $K_{o,c,m,c,n}, K_{o,d,m,d,n}$ are [7-f, 7-f, 6-f] neutral oscillons in t with periods $T_{d,m,n}, [13, 9, 23]$ local t -maximums, and $[13, 9, 23]$ local t -minimums, the numbers of which do not exceed $k_{d,1,m,n}$ due to $[2, 1, 2]$ merged t -maximums and t -minimums.

3.7. The External Elementary Oscillons

With the help of (130), we transform definitions (63) of the external elementary oscillons via x -eigenfunctions $f_{x,l,m}, f_{x,l,n}$ (98) and express them using the x -matrix of external interaction (104) as follows:

$$\begin{aligned}
 K_{o,a,m,b,n} &= \rho_c e z_m e z_n Q_{y,m} Q_{y,n} f_{x,1,m,n}, \quad K_{o,c,m,d,n} = \rho_c e z_m e z_n R_{y,m} R_{y,n} f_{x,13,m,n}, \\
 K_{o,a,m,c,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{y,m} R_{y,n} \left(-f_{x,6,m,n} + f_{x,8,m,n} \right) \right. \\
 &\quad \left. + R_{y,m} Q_{y,n} \left(-f_{x,10,m,n} + f_{x,12,m,n} \right) \right], \\
 K_{o,b,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{y,m} R_{y,n} \left(+f_{x,6,m,n} + f_{x,8,m,n} \right) \right. \\
 &\quad \left. + R_{y,m} Q_{y,n} \left(+f_{x,10,m,n} + f_{x,12,m,n} \right) \right], \\
 K_{o,a,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{y,m} R_{y,n} \left(+f_{x,5,m,n} + f_{x,7,m,n} \right) \right. \\
 &\quad \left. + R_{y,m} Q_{y,n} \left(+f_{x,9,m,n} - f_{x,11,m,n} \right) \right], \\
 K_{o,b,m,c,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{y,m} R_{y,n} \left(+f_{x,5,m,n} - f_{x,7,m,n} \right) \right. \\
 &\quad \left. + R_{y,m} Q_{y,n} \left(+f_{x,9,m,n} + f_{x,11,m,n} \right) \right].
 \end{aligned} \tag{197}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the first pair $K_{o,a,m,b,n}, K_{o,c,m,d,n}$ of external elementary oscillons is reduced to 1-f neutral oscillons in x , which are formed by a pair of sine waves $f_{x,1,m,n}, f_{x,13,m,n}$ with wavenumber $\kappa_{1,m,n}$. The second pair $K_{o,a,m,c,n}, K_{o,b,m,d,n}$ is transformed into 2-f neutral oscillons in x , which are generated by superpositions of two pairs of cosine waves $f_{x,6,m,n}, f_{x,8,m,n}$ and $f_{x,10,m,n}, f_{x,12,m,n}$ with wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, respectively. The third pair $K_{o,a,m,d,n}, K_{o,b,m,c,n}$ is converted into 2-f neutral oscillons in x , which are produced by superpositions of two pairs of sine waves $f_{x,5,m,n}, f_{x,7,m,n}$ and $f_{x,9,m,n}, f_{x,11,m,n}$ with wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, correspondingly.

For all three pairs, amplitudes are proportional to parameters (180) and x -shifts depend on parameters (181). The x -period of the first elementary pair is $L_{x,1,m,n}$ and the x -period of the second and third elementary pairs is $L_{x,m,n}$ that are defined by (182)-(183). If $n = m$, the external elementary oscillons (197) are reduced to the doubled internal elementary oscillons (152) due to (108)-(109).

Averages over the x -period of six external elementary oscillons vanish since

$$\begin{aligned}
 \int_0^{L_{x,1,m,n}} K_{o,i,m,j,n} dx &= 0, \quad [i, j] = [a, b], [c, d], \\
 \int_0^{L_{x,m,n}} K_{o,i,m,j,n} dx &= 0, \quad [i, j] = [a, c], [b, d], [a, d], [b, c],
 \end{aligned} \tag{198}$$

because of the same reasons as for the diagonal elementary oscillons (184). The external elementary oscillons are also neutral because on average they do not transfer any kinetic energy along the x -axis.

Analogously, definitions (63) written in y -eigenfunctions $f_{y,l,m}, f_{y,l,n}$ with the help of (136) and the y -matrix of external interaction give

$$\begin{aligned}
 K_{o,a,m,b,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{x,m} R_{x,n} \left(-f_{y,6,m,n} + f_{y,8,m,n} \right) \right. \\
 &\quad \left. + R_{x,m} Q_{x,n} \left(-f_{y,10,m,n} + f_{y,12,m,n} \right) \right], \\
 K_{o,c,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{x,m} R_{x,n} \left(+f_{y,6,m,n} + f_{y,8,m,n} \right) \right. \\
 &\quad \left. + R_{x,m} Q_{x,n} \left(+f_{y,10,m,n} + f_{y,12,m,n} \right) \right], \\
 K_{o,a,m,c,n} &= \rho_c e z_m e z_n Q_{x,m} Q_{x,n} f_{y,1,m,n}, \quad K_{o,b,m,d,n} = \rho_c e z_m e z_n R_{x,m} R_{x,n} f_{y,13,m,n}, \\
 K_{o,a,m,d,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{x,m} R_{x,n} \left(+f_{y,5,m,n} + f_{y,7,m,n} \right) \right. \\
 &\quad \left. + R_{x,m} Q_{x,n} \left(+f_{y,9,m,n} - f_{y,11,m,n} \right) \right], \\
 K_{o,b,m,c,n} &= \frac{1}{2} \rho_c e z_m e z_n \left[Q_{x,m} R_{x,n} \left(+f_{y,5,m,n} - f_{y,7,m,n} \right) \right. \\
 &\quad \left. + R_{x,m} Q_{x,n} \left(+f_{y,9,m,n} + f_{y,11,m,n} \right) \right].
 \end{aligned} \tag{199}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the first pair $K_{o,a,m,b,n}, K_{o,c,m,d,n}$ of external elementary oscillons is transformed into 2-f neutral oscillons in y , which are generated by superpositions of two pairs of cosine waves $f_{y,6,m,n}, f_{y,8,m,n}$ and $f_{y,10,m,n}, f_{y,12,m,n}$ with wavenumbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, correspondingly. The second pair $K_{o,a,m,c,n}, K_{o,b,m,d,n}$ is reduced to 1-f neutral oscillons in y , which are formed by a pair of sine waves $f_{y,1,m,n}, f_{y,13,m,n}$ with wavenumber $\lambda_{1,m,n}$. The third pair $K_{o,a,m,d,n}, K_{o,b,m,c,n}$ is converted into 2-f neutral oscillons in y , which are produced by superpositions of two pairs of sine waves $f_{y,5,m,n}, f_{y,7,m,n}$ and $f_{y,9,m,n}, f_{y,11,m,n}$ with wavenumbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, respectively.

For all pairs, amplitudes are proportional to parameters (186) and y -shifts depend on parameters (187). The y -period of the first and third elementary pairs is $L_{y,m,n}$ and the y -period of the second elementary pair is $L_{y,1,m,n}$ that are specified by (188)-(189). If $n = m$, the external elementary oscillons (199) are transformed into the doubled internal elementary oscillons (165).

The external elementary oscillons are also neutral along the y -axis since on average they do not transfer any kinetic energy as

$$\begin{aligned}
 \int_0^{L_{y,m,n}} K_{o,i,m,j,n} dy &= 0, \quad [i, j] = [a, b], [c, d], [a, d], [b, c], \\
 \int_0^{L_{y,1,m,n}} K_{o,i,m,j,n} dy &= 0, \quad [i, j] = [a, c], [b, d].
 \end{aligned} \tag{200}$$

With the help of (63), (47), and (84), we employ the t -matrix of external interaction (120) to compute the inhomogeneous Fourier expansion of the external elementary oscillons via the t -eigenfunctions of external interaction (117) in the following t -form:

$$\begin{aligned}
 K_{o,a,m,b,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[-Q_m Q_n f_{1,m,n} + Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n} - R_m R_n f_{13,m,n} \right], \\
 K_{o,c,m,d,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[+Q_m Q_n f_{1,m,n} + Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n} + R_m R_n f_{13,m,n} \right],
 \end{aligned}$$

$$\begin{aligned}
 K_{o,a,m,c,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[-Q_m Q_n f_{1,m,n} - Q_m R_n f_{7,m,n} + R_m Q_n f_{11,m,n} + R_m R_n f_{13,m,n} \right], \\
 K_{o,b,m,d,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[+Q_m Q_n f_{1,m,n} - Q_m R_n f_{7,m,n} + R_m Q_n f_{11,m,n} - R_m R_n f_{13,m,n} \right], \\
 K_{o,a,m,d,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[+Q_m Q_n (f_{2,m,n} - f_{4,m,n}) - R_m R_n (f_{14,m,n} - f_{16,m,n}) \right], \\
 K_{o,b,m,c,n} &= \frac{1}{4} \rho_c e z_m e z_n \left[+Q_m Q_n (f_{2,m,n} + f_{4,m,n}) - R_m R_n (f_{14,m,n} + f_{16,m,n}) \right].
 \end{aligned}
 \tag{201}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the first pair $K_{o,a,m,b,n}, K_{o,c,m,d,n}$ of external elementary oscillons is reduced to at most 4-f neutral oscillons in t , which are formed by superpositions of four sine waves $f_{1,m,n}, f_{5,m,n}, f_{9,m,n}, f_{13,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{3,m,n}, \omega_{5,m,n}, \omega_{7,m,n}$, respectively. The second pair $K_{o,a,m,c,n}, K_{o,b,m,d,n}$ is transformed into at most 4-f neutral oscillons in t , which are generated by superpositions of four sine waves $f_{1,m,n}, f_{7,m,n}, f_{11,m,n}, f_{13,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{4,m,n}, \omega_{6,m,n}, \omega_{7,m,n}$, correspondingly. The third pair $K_{o,a,m,d,n}, K_{o,b,m,c,n}$ is converted into at most 4-f neutral oscillons in t , which are produced by superpositions of four cosine waves $f_{2,m,n}, f_{4,m,n}, f_{14,m,n}, f_{16,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{2,m,n}, \omega_{7,m,n}, \omega_{8,m,n}$, respectively.

For all pairs, amplitudes are proportional to parameters (192) and t -shifts depend on parameters (193). If $n = m$, the external elementary oscillons (201) are converted to the doubled internal elementary oscillons (170) because of (127).

Temporal periods of the first pair

$$\begin{aligned}
 T_{o,1,m,n} &= LCM(T_{1,m,n}, T_{3,m,n}, T_{5,m,n}, T_{7,m,n}) \\
 &= k_{o,1,1,m,n} T_{1,m,n} = k_{o,1,3,m,n} T_{3,m,n} = k_{o,1,5,m,n} T_{5,m,n} = k_{o,1,7,m,n} T_{7,m,n},
 \end{aligned}
 \tag{202}$$

the second pair

$$\begin{aligned}
 T_{o,2,m,n} &= LCM(T_{1,m,n}, T_{4,m,n}, T_{6,m,n}, T_{7,m,n}) \\
 &= k_{o,2,1,m,n} T_{1,m,n} = k_{o,2,4,m,n} T_{4,m,n} = k_{o,2,6,m,n} T_{6,m,n} = k_{o,2,7,m,n} T_{7,m,n},
 \end{aligned}
 \tag{203}$$

and the third pair

$$\begin{aligned}
 T_{o,3,m,n} &= LCM(T_{1,m,n}, T_{2,m,n}, T_{7,m,n}, T_{8,m,n}) \\
 &= k_{o,3,1,m,n} T_{1,m,n} = k_{o,3,2,m,n} T_{2,m,n} = k_{o,3,7,m,n} T_{7,m,n} = k_{o,3,8,m,n} T_{8,m,n},
 \end{aligned}
 \tag{204}$$

where $k_{o,1,1,m,n}, k_{o,1,3,m,n}, k_{o,1,5,m,n}, k_{o,1,7,m,n}, k_{o,2,1,m,n}, k_{o,2,4,m,n}, k_{o,2,6,m,n}, k_{o,2,7,m,n}, k_{o,3,1,m,n}, k_{o,3,2,m,n}, k_{o,3,7,m,n}, k_{o,3,8,m,n}$ are integers, $T_{l,m,n}$ for $l = 1, 2, \dots, 8$ are specified by (194).

Because integrals of sine waves $f_{2l-1,m,n}$ over the relevant t -periods $T_{l,m,n}$ vanish for $l = 1, 2, \dots, 8$, averages of the external elementary oscillons also vanish for the reason that

$$\begin{aligned}
 \int_0^{T_{o,1,m,n}} K_{o,i,m,j,n} dt &= 0, [i, j] = [a, b], [c, d], \\
 \int_0^{T_{o,2,m,n}} K_{o,i,m,j,n} dt &= 0, [i, j] = [a, c], [b, d], \\
 \int_0^{T_{o,3,m,n}} K_{o,i,m,j,n} dt &= 0, [i, j] = [a, d], [b, c]
 \end{aligned}
 \tag{205}$$

due to (201)-(204) and the additive interval property. In agreement with (205), the external elementary oscillons are also neutral because on average they do not transfer in time any kinetic energy.

The diagonal elementary oscillon $K_{o,a,2,c,3}$ with $L_{x,2,3} = 12, L_{y,1,2,3} = 3/4, T_{o,2,2,3} = 6$ is shown in **Figure 5** for independent parameters (95).

Along the x -axis, the first pair is visualized by 1-f neutral oscillons in x with periods $L_{x,1,m,n} = [24/11, 8/7, 12/13]$. The second and third pairs are displayed by 2-f neutral oscillons in x with periods $L_{x,m,n} = [24, 8, 12], [11, 7, 13]$ local x -maximums, and $[11, 7, 13]$ local x -minimums, the numbers of which are specified by $k_{x,1,m,n} = [11, 7, 13]$.

Along the y -axis, the first and third pairs are depicted by 2-f neutral oscillons in y with periods $L_{y,m,n} = [9/2, 9/2, 3/2], [2, 5, 2]$ local y -maximums, and $[2, 5, 2]$ local y -minimums, the numbers of which are indicated by $k_{y,1,m,n} = [2, 5, 2]$. The second pair is shown by 1-f neutral oscillons in y with periods $L_{y,1,m,n} = [9/4, 9/10, 3/4]$.

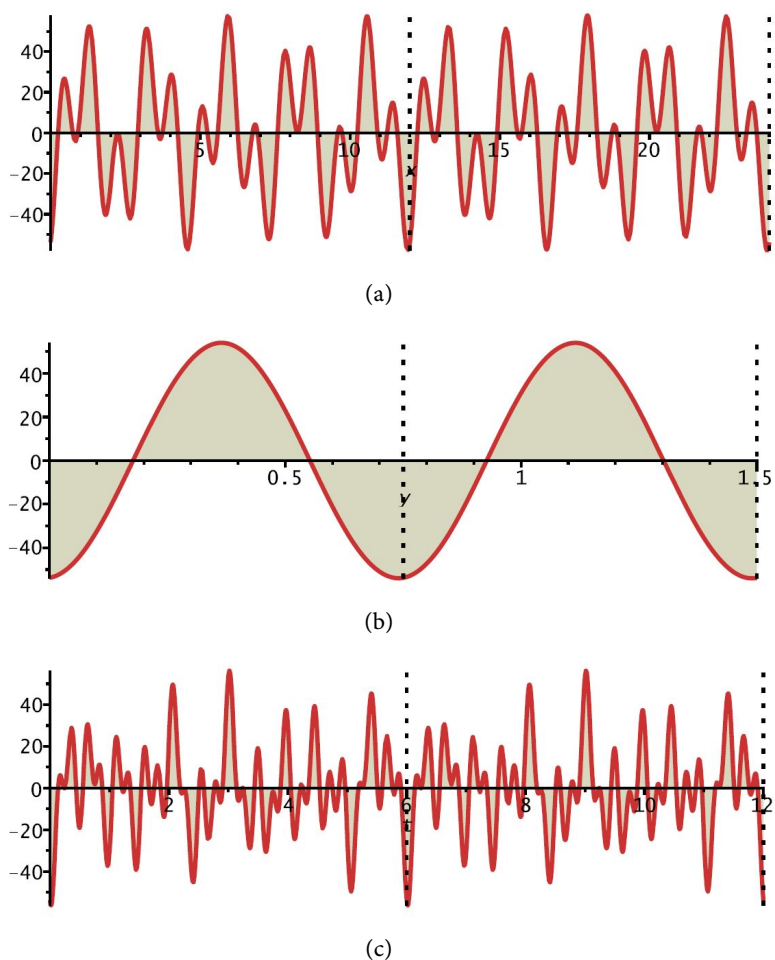


Figure 5. The external elementary oscillon $K_{o,a,2,c,3}$: (a)— $K_{o,a,2,c,3}(x)$ (197), (b)— $K_{o,a,2,c,3}(y)$ (199), (c)— $K_{o,a,2,c,3}(t)$ (201) for independent parameters (95).

The first pair is presented by 4-f neutral oscillons in t with periods $T_{o,1,m,n} = [6, 3/2, 6]$, $[15, 5, 25]$, $[11, 5, 25]$ local t -maximums, $[15, 5, 25]$, $[11, 5, 25]$ local t -minimums, the numbers of which do not exceed or equal to $k_{o,1,m,n} = [15, 5, 25]$ due to $[0, 0, 0]$, $[4, 0, 0]$ merged t -maximums and t -minimums, respectively. The second pair is represented via [3-f, 4-f, 4-f] neutral oscillons in t with periods $T_{o,2,m,n} = [2, 3, 6]$, $[5, 9, 25]$ local t -maximums, and $[5, 9, 25]$ local t -minimums, the numbers of which do not exceed or equal to $k_{o,2,m,n} = [5, 10, 25]$ because of $[0, 1, 0]$ merged t -maximums and t -minimums. The third pair is visualized by [4-f, 4-f, 3-f] neutral oscillons in t with periods $T_{o,3,m,n} = [6, 3, 6]$, $[15, 10, 25]$ local t -maximums, and $[15, 10, 25]$ local t -minimums, the numbers of which are provided by $k_{o,3,m,n} = [15, 10, 25]$.

4. Wave Oscillons and Pulsons

4.1. The Wave Oscillon of Propagation

Separation of variable x in (52) gives

$$K_{w,i,m} = e z_m Q_{w,y,m} \sin(\kappa_m (x + s_{w,\alpha,y,m})), \tag{206}$$

where

$$Q_{w,y,m} = \sqrt{A y_{w,m}^2 + C y_{w,m}^2} \tag{207}$$

is an amplitude, which is expressed via coefficients

$$\begin{aligned} A y_{w,m} &= C v_{w,m} c y_m + A v_{w,m} s y_m, & C y_{w,m} &= D v_{w,m} c y_m + B v_{w,m} s y_m, \\ A v_{w,m} &= A v_m - B v_m - C v_m + D v_m, & B v_{w,m} &= A v_m + B v_m - C v_m - D v_m, \\ C v_{w,m} &= A v_m - B v_m + C v_m - D v_m, & D v_{w,m} &= A v_m + B v_m + C v_m + D v_m. \end{aligned} \tag{208}$$

In Equation (206),

$$s_{w,\alpha,y,m} = \alpha_{w,y,m} - U_m t + X_{m,0} \tag{209}$$

is an x -shift in the motionless frame and

$$\alpha_{w,y,m} = \frac{1}{\kappa_m} \arcsin \frac{C y_{w,m}}{Q_{w,y,m}} \tag{210}$$

is an X_m -shift in the m th moving frame.

For any frozen $y = y_0, z = z_0, t = t_0$, the wave oscillon of propagation is transformed into a 1-f neutral oscillon in x with the wavenumber κ_m , an amplitude proportional to $e z_m(z_0), Q_{w,y,m}(y_0, t_0)$ and the x -shift $s_{w,\alpha,y,m}(y_0, t_0)$. The x -period $L_{x,m}$ of the wave oscillon of propagation coincides with (74).

The wave oscillon of propagation of the velocity potential is neutral in the x -direction since

$$\int_0^{L_{x,m}} K_{w,i,m} dx = 0. \tag{211}$$

Proceeding with the same approach, we separate variable y in (52) to find

$$K_{w,i,m} = e z_m Q_{w,x,m} \sin(\lambda_m (y + s_{w,\alpha,x,m})), \tag{212}$$

where an amplitude

$$Q_{w,x,m} = \sqrt{Ax_{w,m}^2 + Cx_{w,m}^2} \tag{213}$$

is computed via coefficients

$$Ax_{w,m} = Bv_{w,m} cx_m + Av_{w,m} sx_m, \quad Cx_{w,m} = Dv_{w,m} cx_m + Cv_{w,m} sx_m. \tag{214}$$

In Equation (212),

$$s_{w,\alpha,x,m} = \alpha_{w,x,m} - V_m t + Y_{m,0} \tag{215}$$

is a y -shift in the motionless frame and

$$\alpha_{w,x,m} = \frac{1}{\lambda_m} \arcsin \frac{Cx_{w,m}}{Q_{w,x,m}} \tag{216}$$

is an Y_m -shift in the m th moving frame.

For any frozen $y = y_0, z = z_0, t = t_0$, the wave oscillon of propagation is converted into a 1-f neutral oscillon in y with the wavenumber λ_m , an amplitude proportional to $ez_m(z_0), Q_{w,x,m}(x_0, t_0)$ and the y -shift $s_{w,\alpha,x,m}(x_0, t_0)$. The y -period $L_{y,m}$ of the wave oscillon of propagation is specified by (82).

The wave oscillon of propagation of the velocity potential is also neutral along the y -axis because

$$\int_0^{L_{y,m}} K_{w,i,m} dy = 0. \tag{217}$$

Topology of the wave oscillon of propagation in time is described by the inhomogeneous Fourier t -expansion of (52)

$$K_{w,i,m} = \frac{1}{2} ez_m (-Q_{w,m} f_{w,1,m} + R_{w,m} f_{w,3,m}) \tag{218}$$

expressed via two trigonometric functions

$$f_{w,1,m} = \sin A_{w,1,m}, \quad f_{w,3,m} = \sin A_{w,2,m} \tag{219}$$

of two arguments

$$\begin{aligned} A_{w,1,m} &= (\omega_{x,m} + \omega_{y,m})t + s_{x,m} + s_{y,m} + \alpha_{w,1,m}, \\ A_{w,2,m} &= (\omega_{x,m} - \omega_{y,m})t + s_{x,m} - s_{y,m} + \alpha_{w,2,m}. \end{aligned} \tag{220}$$

In Equations (218), (220),

$$\begin{aligned} Q_{w,m} &= \sqrt{(Av_{w,m} - Dv_{w,m})^2 + (Bv_{w,m} + Cv_{w,m})^2}, \\ R_{w,m} &= \sqrt{(Av_{w,m} + Dv_{w,m})^2 + (Bv_{w,m} - Cv_{w,m})^2} \end{aligned} \tag{221}$$

are amplitudes and

$$\alpha_{w,1,m} = \arcsin \frac{Av_{w,m} - Dv_{w,m}}{Q_{w,m}}, \quad \alpha_{w,2,m} = \arcsin \frac{Av_{w,m} + Dv_{w,m}}{R_{w,m}} \tag{222}$$

are t -shifts.

For any frozen $x = x_0, y = y_0, z = z_0$, the wave oscillon of propagation is reduced to a 2-f neutral oscillon in t , which is formed by superpositions of sine

waves $f_{w,1,m}, f_{w,3,m}$ with frequencies $\omega_{x,m} + \omega_{y,m}, \omega_{x,m} - \omega_{y,m}$, correspondingly, amplitudes proportional to $Q_{w,m}, R_{w,m}, ez_m(z_0)$, and t -shifts depending on $\omega_{x,m}, \omega_{y,m}, \alpha_{w,1,m}, \alpha_{w,2,m}, s_{x,m}(x_0), s_{y,m}(y_0)$. The t -period of the wave oscillon of propagation is given by (92)-(93).

Because integrals of sine waves $f_{w,1,m}, f_{w,3,m}$ over $T_{1,m}, T_{2,m}$ vanish, respectively, average of the wave oscillon of propagation over t -period $T_{o,m}$ vanishes, as well, since

$$\int_0^{T_{o,m}} K_{w,i,m} dt = 0. \tag{223}$$

The wave oscillon of propagation $K_{w,i,3}$ with $L_{x,3} = 4/3, L_{y,3} = 1, T_{o,3} = 2$ is visualized in **Figure 6** for independent parameters (95). The wave oscillons of propagation $K_{w,i,m}$ are displayed by 1-f neutral oscillons in x with periods $L_{x,m} = [8, 3, 4/3]$, via 1-f neutral oscillons in y with periods $L_{y,m} = [9, 3, 1]$, and through 2-f neutral oscillons in t with periods $T_{o,m} = [9, 3, 2]$, 5 local t -maximums, and 5 local t -minimums, the numbers of which are specified by $k_{1,m} = [5, 5, 5]$.

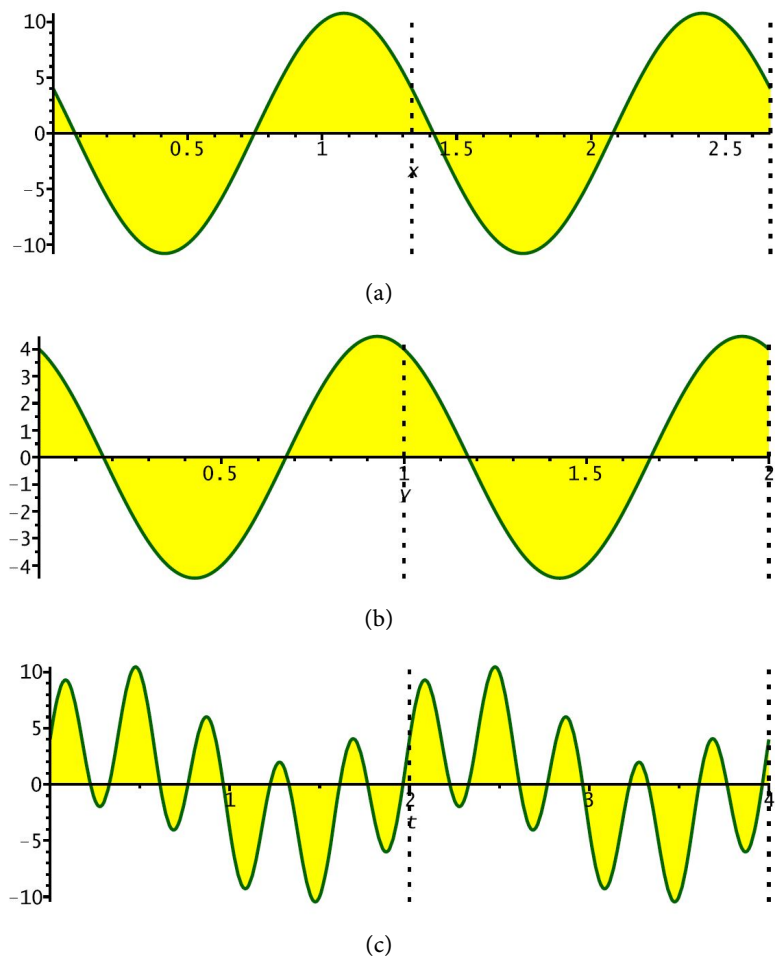


Figure 6. The wave oscillon of propagation $K_{w,i,3}$: (a)— $K_{w,i,3}(x)$ (206), (b)— $K_{w,i,3}(y)$ (212), (c)— $K_{w,i,3}(t)$ (218) for independent parameters (95).

4.2. The Wave Pulsons of Propagation

Combining (55) and (132) yields that the wave pulsons of propagation (27) in terms of the x -eigenfunctions of internal interaction $g_{x,l,m}$ (109) become

$$\begin{aligned}
 K_{w,a,m,a,m} &= \frac{1}{4} \rho_c e z_m^2 \left[(\kappa_m^2 + \mu_m^2) Q_{y,m}^2 + \lambda_m^2 R_{y,m}^2 - \lambda_m^2 (Q_{y,m}^2 g_{x,2,m} + R_{y,m}^2 g_{x,6,m}) \right], \\
 K_{w,b,m,b,m} &= \frac{1}{4} \rho_c e z_m^2 \left[(\kappa_m^2 + \mu_m^2) Q_{y,m}^2 + \lambda_m^2 R_{y,m}^2 + \lambda_m^2 (Q_{y,m}^2 g_{x,2,m} + R_{y,m}^2 g_{x,6,m}) \right], \\
 K_{w,c,m,c,m} &= \frac{1}{4} \rho_c e z_m^2 \left[\lambda_m^2 Q_{y,m}^2 + (\kappa_m^2 + \mu_m^2) R_{y,m}^2 - \lambda_m^2 (Q_{y,m}^2 g_{x,2,m} + R_{y,m}^2 g_{x,6,m}) \right], \\
 K_{w,d,m,d,m} &= \frac{1}{4} \rho_c e z_m^2 \left[\lambda_m^2 Q_{y,m}^2 + (\kappa_m^2 + \mu_m^2) R_{y,m}^2 + \lambda_m^2 (Q_{y,m}^2 g_{x,2,m} + R_{y,m}^2 g_{x,6,m}) \right].
 \end{aligned}
 \tag{224}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the wave pulsons of propagation are reduced to 1-f supercritical pulsons in x , which are formed by a superposition of cosine waves $g_{x,2,m}, g_{x,6,m}$ with wavenumber $2\kappa_m$, amplitudes proportional to λ_m^2 and (133), x -shifts (72), and energy shifts depending on κ_m^2, λ_m^2 , and (133). Therefore, the x -period of the wave pulsons of propagation is specified by (134).

Since integrals of $g_{x,2,m}$ and $g_{x,6,m}$ over $L_{x,m,m}$ vanish,

$$\begin{aligned}
 \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,i,m,i,m} dx &= \frac{\rho_c}{4} e z_m^2 \left[\lambda_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \kappa_m^2 Q_{y,m}^2 \right], \quad i = a, b, \\
 \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,i,m,i,m} dx &= \frac{\rho_c}{4} e z_m^2 \left[\lambda_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \kappa_m^2 R_{y,m}^2 \right], \quad i = c, d.
 \end{aligned}
 \tag{225}$$

Two pairs of the wave pulsons of propagation $K_{w,a,m,a,m}, K_{w,b,m,b,m}$ and $K_{w,c,m,c,m}, K_{w,d,m,d,m}$ on average transfer equal positive amounts of the kinetic energy along the x -axis.

Similarly, we combine (55) and (138) to compute the inhomogeneous Fourier expansion for the wave pulsons of propagation (27) via the y -eigenfunctions of internal interaction $g_{y,l,m}$ in the following form:

$$\begin{aligned}
 K_{w,a,m,a,m} &= \frac{1}{4} \rho_c e z_m^2 \left[(\lambda_m^2 + \mu_m^2) Q_{x,m}^2 + \kappa_m^2 R_{x,m}^2 - \kappa_m^2 (Q_{x,m}^2 g_{y,2,m} + R_{x,m}^2 g_{y,6,m}) \right], \\
 K_{w,c,m,c,m} &= \frac{1}{4} \rho_c e z_m^2 \left[(\lambda_m^2 + \mu_m^2) Q_{x,m}^2 + \kappa_m^2 R_{x,m}^2 + \kappa_m^2 (Q_{x,m}^2 g_{y,2,m} + R_{x,m}^2 g_{y,6,m}) \right], \\
 K_{w,b,m,b,m} &= \frac{1}{4} \rho_c e z_m^2 \left[\kappa_m^2 Q_{x,m}^2 + (\lambda_m^2 + \mu_m^2) R_{x,m}^2 - \kappa_m^2 (Q_{x,m}^2 g_{y,2,m} + R_{x,m}^2 g_{y,6,m}) \right], \\
 K_{w,d,m,d,m} &= \frac{1}{4} \rho_c e z_m^2 \left[\kappa_m^2 Q_{x,m}^2 + (\lambda_m^2 + \mu_m^2) R_{x,m}^2 + \kappa_m^2 (Q_{x,m}^2 g_{y,2,m} + R_{x,m}^2 g_{y,6,m}) \right].
 \end{aligned}
 \tag{226}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the wave pulsons of propagation are converted into 1-f supercritical pulsons in y , which are produced by a superposition of cosine waves $g_{y,2,m}, g_{y,6,m}$ with wavenumber $2\lambda_m$, amplitudes proportional to κ_m^2 and (139), y -shifts (81), and energy shifts depending on κ_m^2, λ_m^2 , and (139). Therefore, the y -period of the wave pulsons of propagation is specified by (140).

Because integrals of $g_{y,2,m}$ and $g_{y,6,m}$ over $L_{y,m,m}$ vanish,

$$\frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,i,m,i,m} dy = \frac{\rho_c}{4} e z_m^2 \left[\kappa_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \lambda_m^2 Q_{x,m}^2 \right], \quad i = a, c, \tag{227}$$

$$\frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,i,m,i,m} dy = \frac{\rho_c}{4} e z_m^2 \left[\kappa_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \lambda_m^2 R_{x,m}^2 \right], \quad i = b, d.$$

So, two pairs of the wave pulsions of propagation $K_{w,a,m,a,m}, K_{w,c,m,c,m}$ and $K_{w,b,m,b,m}, K_{w,d,m,d,m}$ on average transfer equal positive amounts of the kinetic energy along the y -axis.

Substitution of (143) in (55) and collection of like terms results in the inhomogeneous Fourier expansion of the wave pulsions of propagation (27) through the t -eigenfunctions of internal interaction $g_{l,m}$ (126)

$$K_{w,a,m,a,m} = \frac{1}{4} \rho_c e z_m^2 \left[\frac{1}{2} \mu_m^2 (Q_m^2 + R_m^2) + Q_m R_m (\lambda_m^2 g_{4,m} - \kappa_m^2 g_{6,m}) \right],$$

$$K_{w,d,m,d,m} = \frac{1}{4} \rho_c e z_m^2 \left[\frac{1}{2} \mu_m^2 (Q_m^2 + R_m^2) - Q_m R_m (\lambda_m^2 g_{4,m} - \kappa_m^2 g_{6,m}) \right], \tag{228}$$

$$K_{w,b,m,b,m} = \frac{1}{4} \rho_c e z_m^2 \left[\frac{1}{2} \mu_m^2 (Q_m^2 + R_m^2) - Q_m R_m (\lambda_m^2 g_{4,m} + \kappa_m^2 g_{6,m}) \right],$$

$$K_{w,c,m,c,m} = \frac{1}{4} \rho_c e z_m^2 \left[\frac{1}{2} \mu_m^2 (Q_m^2 + R_m^2) + Q_m R_m (\lambda_m^2 g_{4,m} + \kappa_m^2 g_{6,m}) \right].$$

For any frozen $x = x_0, y = y_0, z = z_0$, the wave pulsions of propagation are transformed into 2-f supercritical pulsions in t , which are generated by superpositions of cosine waves $g_{4,m}, g_{6,m}$ with frequencies $2\omega_{x,m}, 2\omega_{y,m}$, respectively, amplitudes proportional to κ_m^2, λ_m^2 , and (145), t -shifts depending on (91), and energy shifts proportional to μ_m^2 and (146).

A temporal period of the wave pulsions of propagation

$$T_{w,m,m} = LCM(T_{2,m,m}, T_{3,m,m}) = k_{w,2,m,m} T_{2,m,m} = k_{w,3,m,m} T_{3,m,m}, \tag{229}$$

where $k_{w,2,m,m}, k_{w,3,m,m}$ are integers and $T_{2,m,m}, T_{3,m,m}$ given by (147).

Average of the wave pulsions of propagation over $T_{w,m,m}$ takes the following form:

$$\frac{1}{T_{w,m,m}} \int_0^{T_{w,m,m}} K_{w,i,m,i,m} dt = \frac{1}{4} \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2), \quad i = a, b, c, d. \tag{230}$$

Integral (230) validates that the wave pulsions of propagation on average transfer the same positive amount of the kinetic energy in time.

The wave pulson of propagation $K_{w,b,3,b,3}$ with $L_{x,3,3} = 2/3, L_{y,3,3} = 1/2, T_{w,3,3} = 1$ is shown in **Figure 7** for independent parameters (95). The wave pulsions of propagation are visualized by 1-f supercritical pulsions in x with periods $L_{x,m,m} = [4, 3/2, 2/3]$, through 1-f supercritical pulsions in y with periods $L_{y,m,m} = [9/2, 3/2, 1/2]$, and via 2-f supercritical pulsions in t with periods $T_{w,m,m} = [3, 3/2, 1], [3, 3, 3]$ local t -maximums, and $[3, 3, 3]$ local t -minimums, the numbers of which are specified by $k_{w,2,m,m} = [3, 3, 3]$.

In agreement with (224),

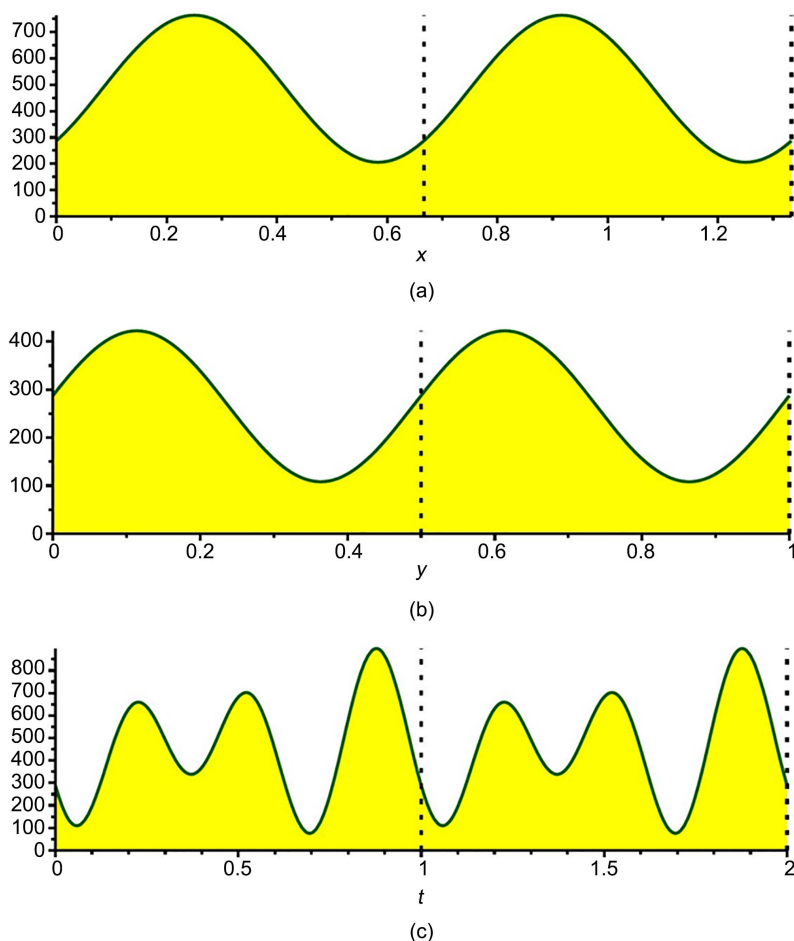


Figure 7. The wave pulson of propagation $K_{w,b3,b3}$: (a)— $K_{w,b3,b3}(x)$ (224), (b)— $K_{w,b3,b3}(y)$ (226), (c)— $K_{w,b3,b3}(t)$ (228) for independent parameters (95).

$$\begin{aligned}
 K_{w,i,m,i,m} \left(x + \frac{1}{2}L_{x,m,m}, y, z, t \right) &= K_{w,j,m,j,m} (x, y, z, t), \quad [i, j] = [a, b], [c, d], \\
 K_{w,i,m,i,m} &= \frac{1}{2} \rho_c \kappa_m^2 e z_m^2 (Q_{y,m}^2 - R_{y,m}^2) + K_{w,j,m,j,m}, \quad [i, j] = [a, c], [b, d].
 \end{aligned}
 \tag{231}$$

Therefore, x -maximums of $K_{w,b,m,b,m}$ correspond to x -minimums of $K_{w,a,m,a,m}$ and x -minimums of $K_{w,b,m,b,m}$ to x -maximums of $K_{w,a,m,a,m}$. Similarly, x -maximums of $K_{w,c,m,c,m}$ correspond to x -minimums of $K_{w,d,m,d,m}$ and x -minimums of $K_{w,c,m,c,m}$ to x -maximums of $K_{w,d,m,d,m}$. Locations of x -maximums and x -minimums of $K_{w,c,m,c,m}$ coincide with locations of x -maximums and x -minimums of $K_{w,a,m,a,m}$. Analogously, locations of x -maximums and x -minimums of $K_{w,b,m,b,m}$ match locations of x -maximums and x -minimums of $K_{w,d,m,d,m}$.

Due to (226),

$$\begin{aligned}
 K_{w,i,m,i,m} \left(x, y + \frac{1}{2}L_{y,m,m}, z, t \right) &= K_{w,j,m,j,m} (x, y, z, t), \quad [i, j] = [a, c], [b, d], \\
 K_{w,i,m,i,m} &= \frac{1}{2} \rho_c \lambda_m^2 e z_m^2 (Q_{x,m}^2 - R_{x,m}^2) + K_{w,j,m,j,m}, \quad [i, j] = [a, b], [c, d].
 \end{aligned}
 \tag{232}$$

Consequently, y -maximums of $K_{w,c,m,c,m}$ correspond to y -minimums of $K_{w,a,m,a,m}$ and y -minimums of $K_{w,c,m,c,m}$ to y -maximums of $K_{w,a,m,a,m}$. Analogously, y -maximums of $K_{w,b,m,b,m}$ correspond to y -minimums of $K_{w,d,m,d,m}$ and y -minimums of $K_{w,b,m,b,m}$ to y -maximums of $K_{w,d,m,d,m}$. Locations of y -maximums and y -minimums of $K_{w,b,m,b,m}$ match locations of y -maximums and y -minimums of $K_{w,a,m,a,m}$. Similarly, locations of y -maximums and y -minimums of $K_{w,c,m,c,m}$ coincide with locations of y -maximums and y -minimums of $K_{w,d,m,d,m}$.

Because of (228),

$$K_{w,i,m,i,m} = \frac{1}{4} \rho_c \mu_m^2 e z_m^2 (Q_m^2 + R_m^2) - K_{w,j,m,j,m}, [i, j] = [a, d], [b, c]. \quad (233)$$

So, locations of t -maximums of $K_{w,d,m,d,m}$ and $K_{w,c,m,c,m}$ correspond to locations of t -minimums of $K_{w,a,m,a,m}$ and $K_{w,b,m,b,m}$, respectively. Analogously, locations of t -minimums of $K_{w,d,m,d,m}$ and $K_{w,c,m,c,m}$ to locations of t -maximums of $K_{w,a,m,a,m}$ and $K_{w,b,m,b,m}$, correspondingly.

4.3. The Internal Wave Oscillons

Substituting (152) in (58) results in the inhomogeneous Fourier expansion of the internal wave oscillons (38) along the x -axis

$$\begin{aligned} K_{w,a,m,b,m} &= + K_{w,c,m,d,m} = \frac{\rho_c}{2} \lambda_m^2 e z_m^2 (Q_{y,m}^2 g_{x,1,m} + R_{y,m}^2 g_{x,5,m}), \\ K_{w,a,m,c,m} &= + K_{w,b,m,d,m} = \rho_c \kappa_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,2,m}, \\ K_{w,a,m,d,m} &= - K_{w,b,m,c,m} = \rho_c \mu_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,1,m}. \end{aligned} \quad (234)$$

For any frozen $y = y_0, z = z_0, t = t_0$, the first pair $K_{w,a,m,b,m}, K_{w,c,m,d,m}$ of internal wave oscillons is transformed into 1-f neutral oscillons in x , which is produced by a superposition of sine waves $g_{x,1,m}, g_{x,5,m}$ with wavenumber $2\kappa_m$, amplitudes proportional to λ_m^2 and (133), and x -shifts (72). The second pair $K_{w,a,m,c,m}, K_{w,b,m,d,m}$ and the third pair $K_{w,a,m,d,m}, K_{w,b,m,c,m}$ are reduced to 0-f (uniform) noncritical pulsons in x with energy shifts proportional to κ_m^2, μ_m^2 , and (153). The x -wavelength of the internal wave oscillons is given by (134).

Averages over the x -wavelength of six internal wave oscillons

$$\begin{aligned} \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,i,m,j,m} dx &= 0, [i, j] = [a, b], [c, d], \\ \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,i,m,j,m} dx &= \rho_c \kappa_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,2,m}, [i, j] = [a, c], [b, d], \\ \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,a,m,d,m} dx &= - \frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{w,b,m,c,m} dx = \rho_c \mu_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,1,m} \end{aligned} \quad (235)$$

prove that the first pair of internal wave oscillons on average does not transfer any kinetic energy along the x -axis, the internal wave oscillons of the second pair on average transfer along the x -axis equal amounts of the kinetic energy, and the internal wave oscillons of the third pair on average transfer along the x -axis equal amounts of the kinetic energy of opposite signs.

Analogously, we substitute (165) in (58) to find the inhomogeneous Fourier expansion of the internal wave oscillons along the y -axis

$$\begin{aligned}
 K_{w,a,m,b,m} &= + K_{w,c,m,d,m} = \rho_c \lambda_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m}, \\
 K_{w,a,m,c,m} &= + K_{w,b,m,d,m} = \frac{\rho_c}{2} \kappa_m^2 e z_m^2 (Q_{x,m}^2 g_{y,1,m} + R_{x,m}^2 g_{y,5,m}), \\
 K_{w,a,m,d,m} &= - K_{w,b,m,c,m} = \rho_c \mu_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,1,m}.
 \end{aligned}
 \tag{236}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the first pair $K_{w,a,m,b,m}, K_{w,c,m,d,m}$ and the third pair $K_{w,a,m,d,m}, K_{w,b,m,c,m}$ of internal wave oscillons are converted into 0-f noncritical pulsons in y with energy shifts proportional to λ_m^2, μ_m^2 , and (166). The second pair $K_{w,a,m,c,m}, K_{w,b,m,d,m}$ is reduced to 1-f neutral oscillons in y , which is formed by a superposition of sine waves $g_{y,1,m}, g_{y,5,m}$ with wave-number $2\lambda_m$, amplitudes proportional to κ_m^2 and (139), and y -shifts (81). The y -wavelength of the internal wave oscillons is specified by (140).

The internal wave oscillons of the first pair on average transfer along the y -axis equal amounts of the kinetic energy, the second pair of internal wave oscillons on average does not transfer any kinetic energy along the y -axis, and the internal wave oscillons of the third pair on average transfer along the y -axis equal amounts of the kinetic energy of opposite signs because

$$\begin{aligned}
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,i,m,j,m} dy &= \rho_c \lambda_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m}, \quad [i, j] = [a, b], [c, d], \\
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,i,m,j,m} dy &= 0, \quad [i, j] = [a, c], [b, d], \\
 \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,a,m,d,m} dy &= - \frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{w,b,m,c,m} dy = \rho_c \mu_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,1,m}.
 \end{aligned}
 \tag{237}$$

Substitution of (170) in (58) yields the inhomogeneous Fourier expansion of the internal wave oscillons in time

$$\begin{aligned}
 K_{w,a,m,b,m} &= + K_{w,c,m,d,m} = + \frac{\rho_c}{2} \lambda_m^2 e z_m^2 Q_m R_m g_{3,m}, \\
 K_{w,a,m,c,m} &= + K_{w,b,m,d,m} = - \frac{\rho_c}{2} \kappa_m^2 e z_m^2 Q_m R_m g_{5,m}, \\
 K_{w,a,m,d,m} &= - K_{w,b,m,c,m} = + \rho_c \mu_m^2 e z_m^2 (A v_m D v_m - B v_m C v_m).
 \end{aligned}
 \tag{238}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the first pair $K_{w,a,m,b,m}, K_{w,c,m,d,m}$ and the second pair $K_{w,a,m,c,m}, K_{w,b,m,d,m}$ of internal wave oscillons are transformed into 1-f neutral oscillons in t , which are formed by sine waves $g_{3,m}, g_{5,m}$ with frequencies $2\omega_{x,m}, 2\omega_{y,m}$, respectively, amplitudes proportional to κ_m^2, λ_m^2 , and (145), and t -shifts depending on (91). The temporal period of the first and second pairs are $T_{2,m,m}$ and $T_{3,m,m}$, which are specified by (147). The third pair $K_{w,a,m,d,m}, K_{w,b,m,c,m}$ is transformed into 0-f noncritical pulsons with energy shifts depending on $\rho_c, \mu_m^2, A v_m, B v_m, C v_m, D v_m, e z_m^2 (z_0)$.

For the same reasoning as for the internal elementary oscillons,

$$\begin{aligned}
 & + \frac{1}{T_{2,m,m}} \int_0^{T_{2,m,m}} K_{w,i,m,j,m} dt = 0, \quad [i, j] = [a, b], [c, d], \\
 & + \frac{1}{T_{3,m,m}} \int_0^{T_{3,m,m}} K_{w,i,m,j,m} dt = 0, \quad [i, j] = [a, c], [b, d], \\
 & + \frac{1}{T_{o,3,m,m}} \int_0^{T_{o,3,m,m}} K_{w,a,m,d,m} dt \\
 & = -\frac{1}{T_{o,3,m,m}} \int_0^{T_{o,3,m,m}} K_{w,b,m,c,m} dt = \rho_c \mu_m^2 e z_m^2 (A v_m D v_m - B v_m C v_m).
 \end{aligned}
 \tag{239}$$

Similar to the internal elementary oscillons (175), the first and second pairs of internal wave oscillons on average do not transfer in time any kinetic energy and the internal wave oscillons of the third pair transfer in time equal amounts of the kinetic energy of the opposite signs.

The internal wave oscillon $K_{w,a,3,b,3}$ with $L_{x,3,3} = 2/3, L_{y,3,3} = 1/2, T_{2,3,3} = 1/3$ is visualized in **Figure 8** for independent parameters (95). The first pair is displayed by 1-f neutral oscillons in x with periods $L_{x,m,m} = [4, 3/2, 2/3]$, the second and third pairs via 0-f noncritical pulsions in x with periods $L_{x,m,m}$.

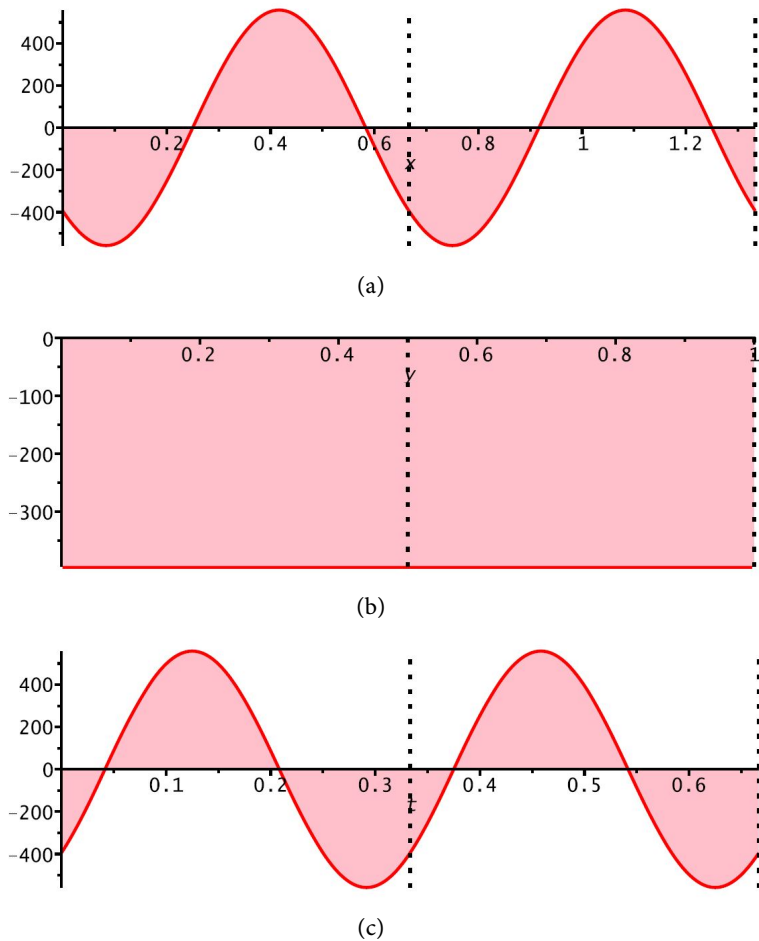


Figure 8. The internal wave oscillon $K_{w,a,3,b,3}$: (a)— $K_{w,a,3,b,3}(x)$ (234), (b)— $K_{w,a,3,b,3}(y)$ (236), (c)— $K_{w,a,3,b,3}(t)$ (238) for independent parameters (95).

In the y -direction, the first and third pairs are visualized by 0-f noncritical pulsions in y with periods $L_{y,m,m} = [9/2, 3/2, 1/2]$ and the second pair via 1-f neutral oscillons in y with periods $L_{y,m,m}$.

The first pair is represented by 1-f neutral oscillons in t with periods $T_{2,m,m} = [1, 1/2, 1/3]$, the second pair via 1-f neutral oscillons in t with periods $T_{3,m,m} = [3/2, 3/4, 1/2]$, and the third pair in terms of 0-f noncritical pulsions in t with periods $T_{o,3,m,m} = [3, 3/2, 1]$.

4.4. The Diagonal Wave Oscillons

Substituting (179) in (61), we convert definitions (32) into the inhomogeneous Fourier expansion of the diagonal wave oscillons along the x -axis

$$\begin{aligned}
 K_{w,a,m,a,n} &= A_{w,m,n} \left\{ +Q_{y,m} Q_{y,n} \left[(\kappa_m \kappa_n - \mu_m \mu_n) f_{x,2,m,n} + (\kappa_m \kappa_n + \mu_m \mu_n) f_{x,4,m,n} \right] \right. \\
 &\quad \left. - R_{y,m} R_{y,n} \lambda_m \lambda_n (f_{x,14,m,n} - f_{x,16,m,n}) \right\}, \\
 K_{w,b,m,b,n} &= A_{w,m,n} \left\{ -Q_{y,m} Q_{y,n} \left[(\kappa_m \kappa_n - \mu_m \mu_n) f_{x,2,m,n} - (\kappa_m \kappa_n + \mu_m \mu_n) f_{x,4,m,n} \right] \right. \\
 &\quad \left. + R_{y,m} R_{y,n} \lambda_m \lambda_n (f_{x,14,m,n} + f_{x,16,m,n}) \right\}, \\
 K_{w,c,m,c,n} &= A_{w,m,n} \left\{ -Q_{y,m} Q_{y,n} \lambda_m \lambda_n (f_{x,2,m,n} - f_{x,4,m,n}) \right. \\
 &\quad \left. + R_{y,m} R_{y,n} \left[(\kappa_m \kappa_n - \mu_m \mu_n) f_{x,14,m,n} + (\kappa_m \kappa_n + \mu_m \mu_n) f_{x,16,m,n} \right] \right\}, \\
 K_{w,d,m,d,n} &= A_{w,m,n} \left\{ +Q_{y,m} Q_{y,n} \lambda_m \lambda_n (f_{x,2,m,n} + f_{x,4,m,n}) \right. \\
 &\quad \left. - R_{y,m} R_{y,n} \left[(\kappa_m \kappa_n - \mu_m \mu_n) f_{x,14,m,n} - (\kappa_m \kappa_n + \mu_m \mu_n) f_{x,16,m,n} \right] \right\},
 \end{aligned} \tag{240}$$

where

$$A_{w,m,n} = \frac{1}{2} \rho_c e z_m e z_n. \tag{241}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal wave oscillons are presented by 2-f neutral oscillons in x , which are formed by superpositions of two pairs of cosine waves $f_{x,2,m,n}, f_{x,4,m,n}$ and $f_{x,14,m,n}, f_{x,16,m,n}$ with two wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, respectively, amplitudes depending on $\kappa_m, \kappa_n, \lambda_m, \lambda_n, \mu_m, \mu_n$, and (180), and x -shifts, which depend on (181). The x -wavelength of the diagonal wave oscillons is given by (182)-(183). If $n = m$, the diagonal wave oscillons (240) are transformed by (108)-(109) into the doubled wave pulsions of propagation (224).

Similar to (184), the diagonal wave oscillons on average do not transfer any kinetic energy along the x -axis since

$$\int_0^{L_{x,m,n}} K_{w,i,m,i,n} dx = 0, \quad i = a, b, c, d. \tag{242}$$

Analogously, representation of definitions (32) in y -eigenfunctions of external interaction $f_{y,l,m,n}$ with the help of (185) and (61) yields

$$\begin{aligned}
 K_{w,a,m,a,n} &= A_{w,m,n} \left\{ +Q_{x,m} Q_{x,n} \left[(\lambda_m \lambda_n - \mu_m \mu_n) f_{y,2,m,n} + (\lambda_m \lambda_n + \mu_m \mu_n) f_{y,4,m,n} \right] \right. \\
 &\quad \left. - R_{x,m} R_{x,n} \kappa_m \kappa_n (f_{y,14,m,n} - f_{y,16,m,n}) \right\},
 \end{aligned}$$

$$\begin{aligned}
 K_{w,c,m,c,n} &= A_{w,m,n} \left\{ -Q_{x,m} Q_{x,n} \left[(\lambda_m \lambda_n - \mu_m \mu_n) f_{y,2,m,n} - (\lambda_m \lambda_n + \mu_m \mu_n) f_{y,4,m,n} \right] \right. \\
 &\quad \left. + R_{x,m} R_{x,n} \kappa_m \kappa_n (f_{y,14,m,n} + f_{y,16,m,n}) \right\}, \\
 K_{w,b,m,b,n} &= A_{w,m,n} \left\{ -Q_{x,m} Q_{x,n} \kappa_m \kappa_n (f_{y,2,m,n} - f_{y,4,m,n}) \right. \\
 &\quad \left. + R_{x,m} R_{x,n} \left[(\lambda_m \lambda_n - \mu_m \mu_n) f_{y,14,m,n} + (\lambda_m \lambda_n + \mu_m \mu_n) f_{y,16,m,n} \right] \right\}, \\
 K_{w,d,m,d,n} &= A_{w,m,n} \left\{ +Q_{x,m} Q_{x,n} \kappa_m \kappa_n (f_{y,2,m,n} + f_{y,4,m,n}) \right. \\
 &\quad \left. - R_{x,m} R_{x,n} \left[(\lambda_m \lambda_n - \mu_m \mu_n) f_{y,14,m,n} - (\lambda_m \lambda_n + \mu_m \mu_n) f_{y,16,m,n} \right] \right\}.
 \end{aligned}
 \tag{243}$$

For any frozen $x = x_0, z = z_0, t = t_0$, the diagonal wave oscillons are transformed into 2-f neutral oscillons in y , which are produced by superpositions of two pairs of cosine waves $f_{y,2,m,n}, f_{y,4,m,n}$ and $f_{y,14,m,n}, f_{y,16,m,n}$ with two wavenumbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, respectively, amplitudes, which depend on $\kappa_m, \kappa_n, \lambda_m, \lambda_n, \mu_m, \mu_n$, and (186), and y -shifts depending on (187). The y -period of the diagonal wave oscillons is specified by (188)-(189). If $n = m$, the diagonal wave oscillons (243) are reduced to the doubled wave pulsions of propagation (226).

Like the diagonal elementary oscillons (190), the diagonal wave oscillons are neutral along the y -axis because

$$\int_0^{L_{y,m,n}} K_{w,i,m,i,n} dy = 0, \quad i = a, b, c, d.
 \tag{244}$$

Substitution of (191) in (61) and collection of like terms produce the inhomogeneous Fourier expansion of the diagonal wave oscillons in time

$$\begin{aligned}
 K_{o,a,m,a,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[+Q_m Q_n (N_{m,n} f_{2,m,n} + M_{m,n} f_{4,m,n}) + Q_m R_n (\Lambda_{m,n} f_{6,m,n} - K_{m,n} f_{8,m,n}) \right. \\
 &\quad \left. + R_m Q_n (\Lambda_{m,n} f_{10,m,n} - K_{m,n} f_{12,m,n}) + R_m R_n (N_{m,n} f_{14,m,n} + M_{m,n} f_{16,m,n}) \right], \\
 K_{o,d,m,d,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[+Q_m Q_n (N_{m,n} f_{2,m,n} + M_{m,n} f_{4,m,n}) - Q_m R_n (\Lambda_{m,n} f_{6,m,n} - K_{m,n} f_{8,m,n}) \right. \\
 &\quad \left. - R_m Q_n (\Lambda_{m,n} f_{10,m,n} - K_{m,n} f_{12,m,n}) + R_m R_n (N_{m,n} f_{14,m,n} + M_{m,n} f_{16,m,n}) \right], \\
 K_{o,b,m,b,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[-Q_m Q_n (N_{m,n} f_{2,m,n} - M_{m,n} f_{4,m,n}) - Q_m R_n (\Lambda_{m,n} f_{6,m,n} + K_{m,n} f_{8,m,n}) \right. \\
 &\quad \left. - R_m Q_n (\Lambda_{m,n} f_{10,m,n} + K_{m,n} f_{12,m,n}) - R_m R_n (N_{m,n} f_{14,m,n} - M_{m,n} f_{16,m,n}) \right], \\
 K_{o,c,m,c,n} &= \frac{1}{8} \rho_c e z_m e z_n \left[-Q_m Q_n (N_{m,n} f_{2,m,n} - M_{m,n} f_{4,m,n}) + Q_m R_n (\Lambda_{m,n} f_{6,m,n} + K_{m,n} f_{8,m,n}) \right. \\
 &\quad \left. + R_m Q_n (\Lambda_{m,n} f_{10,m,n} + K_{m,n} f_{12,m,n}) - R_m R_n (N_{m,n} f_{14,m,n} - M_{m,n} f_{16,m,n}) \right],
 \end{aligned}
 \tag{245}$$

where

$$\begin{aligned}
 K_{m,n} &= \kappa_m \kappa_n - \lambda_m \lambda_n + \mu_m \mu_n, \quad \Lambda_{m,n} = -\kappa_m \kappa_n + \lambda_m \lambda_n + \mu_m \mu_n, \\
 M_{m,n} &= \kappa_m \kappa_n + \lambda_m \lambda_n + \mu_m \mu_n, \quad N_{m,n} = +\kappa_m \kappa_n + \lambda_m \lambda_n - \mu_m \mu_n
 \end{aligned}
 \tag{246}$$

are nonlinear amplitudes produced by various wavenumbers.

For any frozen $x = x_0, y = y_0, z = z_0$, the diagonal wave oscillons are converted into at most 8-f neutral oscillons in t , which are generated by superposi-

tions of cosine waves $f_{2l,m,n}$ with frequencies $\omega_{l,m,n}$ for $l = 1, 2, \dots, 8$, amplitudes proportional to $K_{m,n}, \Lambda_{m,n}, M_{m,n}, N_{m,n}$, and (192), and t -shifts depending on (193). The t -period of the diagonal wave oscillons is also provided by (194)-(195). If $n = m$, the diagonal wave oscillons (245) are transformed into the doubled wave pulsons of propagation (228).

Since

$$\int_0^{T_{d,m,n}} K_{w,i,m,i,n} dt = 0, \quad i = a, b, c, d, \tag{247}$$

the diagonal wave oscillons on average do not transfer in time any kinetic energy, as well.

The diagonal wave oscillon $K_{w,b,2,b,3}$ with $L_{x,2,3} = 12, L_{y,2,3} = 3/2, T_{d,2,3} = 6$ is shown in **Figure 9** for independent parameters (95). The diagonal wave oscillons $K_{w,b,m,b,n}$ are represented by 2-f neutral oscillons in x with periods $L_{x,m,n} = [24, 8, 12], [9, 5, 13]$ local x -maximums, and $[9, 5, 13]$ local x -minimums, the numbers of which do not exceed or equal to $k_{x,1,m,n} = [11, 7, 13]$ due to $[2, 2, 0]$ merged x -maximums and x -minimums.

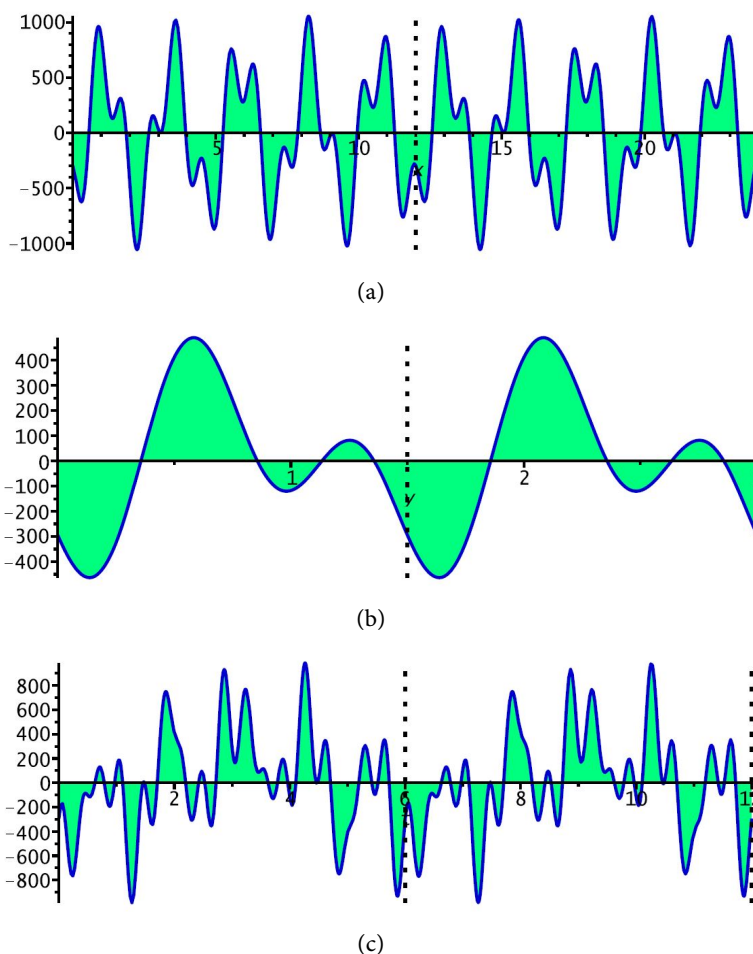


Figure 9. The diagonal wave oscillon $K_{w,b,2,b,3}$: (a)— $K_{w,b,2,b,3}(x)$ (240), (b)— $K_{w,b,2,b,3}(y)$ (243), (c)— $K_{w,b,2,b,3}(t)$ (245) for independent parameters (95).

Along the y -axis, the diagonal wave oscillons $K_{w,b,m,b,n}$ are displayed by 2-f neutral oscillons in y with periods $L_{y,m,n} = [9/2, 9/2, 3/2]$, $[2, 4, 2]$ local y -maximums, and $[2, 4, 2]$ local y -minimums, the numbers of which do not exceed or equal to $k_{y,1,m,n} = [2, 5, 2]$ due to $[0, 1, 0]$ merged y -maximums and y -minimums.

There are three sets of frequencies with $[m, n] = [1, 2], [1, 3], [2, 3]$ for [7-f, 7-f, 6-f] diagonal wave oscillons, since frequencies $\pi, 2\pi/3, [5\pi/3, 13\pi/3]$ are repeated in the first, second, and third sets, respectively. Due to smallness of amplitudes $N_{m,n}$ compared with $K_{m,n}, \Lambda_{m,n}, M_{m,n}$,

$$\begin{aligned} K_{1,2} &= +3.2955, & K_{1,3} &= +7.5680, & K_{2,3} &= +19.974, \\ \Lambda_{1,2} &= +2.9299, & \Lambda_{1,3} &= +8.9388, & \Lambda_{2,3} &= +26.554, \\ M_{1,2} &= +6.2198, & M_{1,3} &= +16.341, & M_{2,3} &= +46.293, \\ N_{1,2} &= -0.0056, & N_{1,3} &= -0.1658, & N_{2,3} &= -0.2350, \end{aligned} \tag{248}$$

the diagonal wave oscillons $K_{w,b,m,b,n}$ are visualized with a graph accuracy by [6-f, 5-f, 5-f] neutral oscillons in t with periods $T_{d,m,n} = [6, 3, 6]$, $[9, 7, 15]$ local t -maximums, and $[9, 7, 15]$ local t -minimums, the numbers of which do not exceed or equal to $k_{d,5,m,n} = [11, 8, 17]$ due to $[2, 1, 2]$ merged t -maximums and t -minimums.

4.5. The External Wave Oscillons

Substituting (197) in (64), we recast definitions (42) in the inhomogeneous Fourier expansion of the external wave oscillons along the x -axis

$$\begin{aligned} K_{w,a,m,b,n} &= 2A_{w,m,n} \left[-Q_{y,m} Q_{y,n} (\kappa_m \kappa_n - \mu_m \mu_n) f_{x,1,m,n} + R_{y,m} R_{y,n} \lambda_m \lambda_n f_{x,13,m,n} \right], \\ K_{w,c,m,d,n} &= 2A_{w,m,n} \left[+Q_{y,m} Q_{y,n} \lambda_m \lambda_n f_{x,1,m,n} - R_{y,m} R_{y,n} (\kappa_m \kappa_n - \mu_m \mu_n) f_{x,13,m,n} \right], \\ K_{w,a,m,c,n} &= A_{w,m,n} \left[+K_{m,n} (Q_{y,m} R_{y,n} f_{x,8,m,n} + R_{y,m} Q_{y,n} f_{x,12,m,n}) \right. \\ &\quad \left. + N_{m,n} (Q_{y,m} R_{y,n} f_{x,6,m,n} + R_{y,m} Q_{y,n} f_{x,10,m,n}) \right], \\ K_{w,b,m,d,n} &= A_{w,m,n} \left[+K_{m,n} (Q_{y,m} R_{y,n} f_{x,8,m,n} + R_{y,m} Q_{y,n} f_{x,12,m,n}) \right. \\ &\quad \left. - N_{m,n} (Q_{y,m} R_{y,n} f_{x,6,m,n} + R_{y,m} Q_{y,n} f_{x,10,m,n}) \right], \\ K_{w,a,m,d,n} &= A_{w,m,n} \left[+M_{m,n} (Q_{y,m} R_{y,n} f_{x,7,m,n} - R_{y,m} Q_{y,n} f_{x,11,m,n}) \right. \\ &\quad \left. - N_{m,n} (Q_{y,m} R_{y,n} f_{x,5,m,n} + R_{y,m} Q_{y,n} f_{x,9,m,n}) \right], \\ K_{w,b,m,c,n} &= A_{w,m,n} \left[-M_{m,n} (Q_{y,m} R_{y,n} f_{x,7,m,n} - R_{y,m} Q_{y,n} f_{x,11,m,n}) \right. \\ &\quad \left. - N_{m,n} (Q_{y,m} R_{y,n} f_{x,5,m,n} + R_{y,m} Q_{y,n} f_{x,9,m,n}) \right]. \end{aligned} \tag{249}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the first pair $K_{w,a,m,b,n}, K_{w,c,m,d,n}$ of external wave oscillons is reduced to 1-f neutral oscillons in x , which are formed by a superposition of sine waves $f_{x,1,m,n}, f_{x,13,m,n}$ with wavenumber $\kappa_{1,m,n}$. The second pair $K_{w,a,m,c,n}, K_{w,b,m,d,n}$ is transformed into at most 2-f neutral oscillons in x , which are generated by superpositions of two pairs of cosine waves $f_{x,6,m,n}, f_{x,8,m,n}$ and $f_{x,10,m,n}, f_{x,12,m,n}$ with two wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, re-

spectively. The third pair $K_{w,a,m,d,n}, K_{w,b,m,c,n}$ is converted into at most 2-f neutral oscillons in x , which are produced by superpositions of two pairs of sine waves $f_{x,5,m,n}, f_{x,7,m,n}$ and $f_{x,9,m,n}, f_{x,11,m,n}$ with two wavenumbers $\kappa_{1,m,n}, \kappa_{2,m,n}$, correspondingly.

For the first pair, amplitudes depend on $\kappa_m, \kappa_n, \lambda_m, \lambda_n, \mu_m, \mu_n$, and (180). For the second and third pairs, amplitudes are proportional to $K_{m,n}, M_{m,n}, N_{m,n}$, and (180). For all pairs, x -shifts depend on (181). The x -period of the first wave pair is $L_{x,1,m,n}$ and the x -period of the second and third wave pairs is $L_{x,m,n}$ that are defined by (182)-(183). If $n = m$, the external wave oscillons (249) with the help of (108)-(109) and

$$K_{m,m} = 2\kappa_m^2, \Lambda_{m,m} = 2\lambda_m^2, M_{m,m} = 2\mu_m^2, N_{m,m} = 0 \tag{250}$$

are reduced to the doubled internal wave oscillons (234).

Like the external elementary oscillons (198), the external wave oscillons are neutral since on average they do not transfer any kinetic energy along the x -axis as

$$\int_0^{L_{x,1,m,n}} K_{w,i,m,j,n} dx = 0, [i, j] = [a, b], [c, d], \tag{251}$$

$$\int_0^{L_{x,m,n}} K_{w,i,m,j,n} dx = 0, [i, j] = [a, c], [b, d], [a, d], [b, c].$$

Analogously, writing definitions (42) in y -eigenfunctions of external interaction $f_{y,l,m,n}$ with the help of (199) and (64) gives

$$K_{w,a,m,b,n} = A_{w,m,n} \left[+\Lambda_{m,n} \left(Q_{x,m} R_{x,n} f_{y,8,m,n} + R_{x,m} Q_{x,n} f_{y,12,m,n} \right) + N_{m,n} \left(Q_{x,m} R_{x,n} f_{y,6,m,n} + R_{x,m} Q_{x,n} f_{y,10,m,n} \right) \right],$$

$$K_{w,c,m,d,n} = A_{w,m,n} \left[+\Lambda_{m,n} \left(Q_{x,m} R_{x,n} f_{y,8,m,n} + R_{x,m} Q_{x,n} f_{y,12,m,n} \right) - N_{m,n} \left(Q_{x,m} R_{x,n} f_{y,6,m,n} + R_{x,m} Q_{x,n} f_{y,10,m,n} \right) \right],$$

$$K_{w,a,m,c,n} = 2A_{w,m,n} \left[-Q_{x,m} Q_{x,n} \left(\lambda_m \lambda_n - \mu_m \mu_n \right) f_{y,1,m,n} + R_{x,m} R_{x,n} \kappa_m \kappa_n f_{y,13,m,n} \right],$$

$$K_{w,b,m,d,n} = 2A_{w,m,n} \left[+Q_{x,m} Q_{x,n} \kappa_m \kappa_n f_{y,1,m,n} - R_{x,m} R_{x,n} \left(\lambda_m \lambda_n - \mu_m \mu_n \right) f_{y,13,m,n} \right],$$

$$K_{w,a,m,d,n} = A_{w,m,n} \left[+M_{m,n} \left(Q_{x,m} R_{x,n} f_{y,7,m,n} - R_{x,m} Q_{x,n} f_{y,11,m,n} \right) - N_{m,n} \left(Q_{x,m} R_{x,n} f_{y,5,m,n} + R_{x,m} Q_{x,n} f_{y,9,m,n} \right) \right], \tag{252}$$

$$K_{w,b,m,c,n} = A_{w,m,n} \left[-M_{m,n} \left(Q_{x,m} R_{x,n} f_{y,7,m,n} - R_{x,m} Q_{x,n} f_{y,11,m,n} \right) - N_{m,n} \left(Q_{x,m} R_{x,n} f_{y,5,m,n} + R_{x,m} Q_{x,n} f_{y,9,m,n} \right) \right].$$

For any frozen $x = x_0, z = z_0, t = t_0$, the first pair $K_{w,a,m,b,n}, K_{w,c,m,d,n}$ of external wave oscillons is transformed into at most 2-f neutral oscillons in y , which are produced by superpositions of two pairs of cosine waves $f_{y,6,m,n}, f_{y,8,m,n}$ and $f_{y,10,m,n}, f_{y,12,m,n}$ with wavenumbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, respectively. The second pair $K_{w,a,m,c,n}, K_{w,b,m,d,n}$ is reduced to 1-f neutral oscillons in y , which are formed by a superposition of sine waves $f_{y,1,m,n}, f_{y,13,m,n}$ with wavenumber $\lambda_{1,m,n}$. The third

pair $K_{w,a,m,d,n}, K_{w,b,m,c,n}$ is converted into at most 2-f neutral oscillons in y , which are generated by superpositions of two pairs of sine waves $f_{y,5,m,n}, f_{y,7,m,n}$ and $f_{y,9,m,n}, f_{y,11,m,n}$ with wavenumbers $\lambda_{1,m,n}, \lambda_{2,m,n}$, correspondingly.

For the first and third pairs, amplitudes are proportional to $\Lambda_{m,n}, M_{m,n}, N_{m,n}$, and (186). For the second pair, amplitudes depend on $\kappa_m, \kappa_n, \lambda_m, \lambda_n, \mu_m, \mu_n$, and (186). For all pairs, y -shifts depend on (187). The y -period of the first and third wave pairs is $L_{y,m,n}$ and the y -period of the second wave pair is $L_{y,1,m,n}$ that are specified by (188)-(189). If $n = m$, the external wave oscillons (252) are converted into the doubled internal wave oscillons (236).

Similar to the external elementary oscillons (200), the external wave oscillons do not transfer any kinetic energy along the y -axis because

$$\int_0^{L_{y,m,n}} K_{w,i,m,j,n} dy = 0, [i, j] = [a, b], [c, d], [a, d], [b, c],$$

$$\int_0^{L_{y,1,m,n}} K_{w,i,m,j,n} dy = 0, [i, j] = [a, c], [b, d].$$
(253)

Using (64) and (201), we compute the external wave oscillons in the inhomogeneous Fourier expansion via the t -eigenfunctions of external interaction (117) as follows:

$$K_{w,a,m,b,n} = \frac{\rho_c e z_m e z_n}{4} \left[+\Lambda_{m,n} (Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n}) + N_{m,n} (Q_m Q_n f_{1,m,n} + R_m R_n f_{13,m,n}) \right],$$

$$K_{w,c,m,d,n} = \frac{\rho_c e z_m e z_n}{4} \left[+\Lambda_{m,n} (Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n}) - N_{m,n} (Q_m Q_n f_{1,m,n} + R_m R_n f_{13,m,n}) \right],$$

$$K_{w,a,m,c,n} = \frac{\rho_c e z_m e z_n}{4} \left[-K_{m,n} (Q_m R_n f_{7,m,n} - R_m Q_n f_{11,m,n}) + N_{m,n} (Q_m Q_n f_{1,m,n} - R_m R_n f_{13,m,n}) \right],$$

$$K_{w,b,m,d,n} = \frac{\rho_c e z_m e z_n}{4} \left[-K_{m,n} (Q_m R_n f_{7,m,n} - R_m Q_n f_{11,m,n}) - N_{m,n} (Q_m Q_n f_{1,m,n} - R_m R_n f_{13,m,n}) \right],$$

$$K_{w,a,m,d,n} = \frac{\rho_c e z_m e z_n}{4} \left[-M_{m,n} (Q_m Q_n f_{4,m,n} - R_m R_n f_{16,m,n}) - N_{m,n} (Q_m Q_n f_{2,m,n} - R_m R_n f_{14,m,n}) \right],$$

$$K_{w,b,m,c,n} = \frac{\rho_c e z_m e z_n}{4} \left[+M_{m,n} (Q_m Q_n f_{4,m,n} - R_m R_n f_{16,m,n}) - N_{m,n} (Q_m Q_n f_{2,m,n} - R_m R_n f_{14,m,n}) \right].$$
(254)

For any frozen $x = x_0, y = y_0, z = z_0$, the first pair $K_{w,a,m,b,n}, K_{w,c,m,d,n}$ of external wave oscillons is reduced to at most 4-f neutral oscillons in t , which are formed by superpositions of four sine waves $f_{1,m,n}, f_{5,m,n}, f_{9,m,n}, f_{13,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{3,m,n}, \omega_{5,m,n}, \omega_{7,m,n}$ correspondingly. The second pair $K_{w,a,m,c,n}, K_{w,b,m,d,n}$ is transformed into at most 4-f neutral oscillons in t , which

are generated by superpositions of four sine waves $f_{1,m,n}, f_{7,m,n}, f_{11,m,n}, f_{13,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{4,m,n}, \omega_{6,m,n}, \omega_{7,m,n}$. The third pair $K_{w,a,m,d,n}, K_{w,b,m,c,n}$ is converted into at most 4-f neutral oscillons in t , which are produced by superpositions of four cosine waves $f_{2,m,n}, f_{4,m,n}, f_{14,m,n}, f_{16,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{2,m,n}, \omega_{7,m,n}, \omega_{8,m,n}$, correspondingly.

For all pairs, amplitudes are proportional to $K_{m,n}, \Lambda_{m,n}, M_{m,n}, N_{m,n}$, and (192) and t -shifts depend on (193). Temporal periods of all pairs are given by (202)-(204), respectively. If $n = m$, the external wave oscillons (254) are transformed by (250), (127), and (171) into the doubled internal wave oscillons (238).

Because of (205) and (64),

$$\begin{aligned} \int_0^{T_{o,1,m,n}} K_{w,i,m,j,n} dt &= 0, \quad [i, j] = [a, b], [c, d], \\ \int_0^{T_{o,2,m,n}} K_{w,i,m,j,n} dt &= 0, \quad [i, j] = [a, c], [b, d], \\ \int_0^{T_{o,3,m,n}} K_{w,i,m,j,n} dt &= 0, \quad [i, j] = [a, d], [b, c], \end{aligned} \tag{255}$$

the external wave oscillons are also neutral in time since on average they do not transfer any kinetic energy.

The external wave oscillon $K_{w,a,2,c,3}$ with $L_{x,2,3} = 12, L_{y,1,2,3} = 3/4, T_{o,2,2,3} = 6$ is shown in **Figure 10** for independent parameters (95). Due to smallness of amplitudes $N_{m,n}$ compared with $K_{m,n}$ (248), the second pair $K_{w,a,m,c,n}, K_{w,b,m,d,n}$ coincide with a graph accuracy with 1-f neutral oscillons in x , which are formed by cosine waves $f_{x,8,m,n}, f_{x,12,m,n}$ with wavenumber $\kappa_{2,m,n}$, periods $L_{x,m,n} = [24, 8, 12]$, $[5, 5, 5]$ local x -maximums, and $[5, 5, 5]$ local x -minimums, the numbers of which are specified by the magnitude of $k_{x,2,m,n} = [-5, -5, -5]$.

Along the y -axis, the second pair is reduced to 1-f neutral oscillons in y with periods $L_{y,1,m,n} = [9/4, 9/10, 3/4]$. The second pair is displayed with a graph accuracy by 2-f neutral oscillons in t with frequencies $\omega_{4,m,n}, \omega_{6,m,n}$, periods $T_{o,2,m,n} = [2, 3, 6]$, $[3, 7, 13]$ local t -maximums, and $[3, 7, 13]$ local t -minimums, the numbers of which are specified by the magnitude of $k_{o,2,6,m,n} = [-3, -7, -13]$.

5. Group Oscillons and Pulsons

5.1. The Group Oscillon of Propagation

In agreement with (53) and (206), the group oscillon of propagation of the velocity potential

$$K_{g,i,m} = \sum_{m=1}^M e z_m Q_{w,y,m} \sin(\kappa_m(x + s_{w,\alpha,y,m})). \tag{256}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the group oscillon of propagation is represented by a M -f neutral oscillon in x that is formed by M wave oscillons of propagation, *i.e.* the superposition of sine waves with wavenumbers

$\kappa_1, \dots, \kappa_m, \dots, \kappa_M$ and parameters of (206) for $m = 1, 2, \dots, M$.

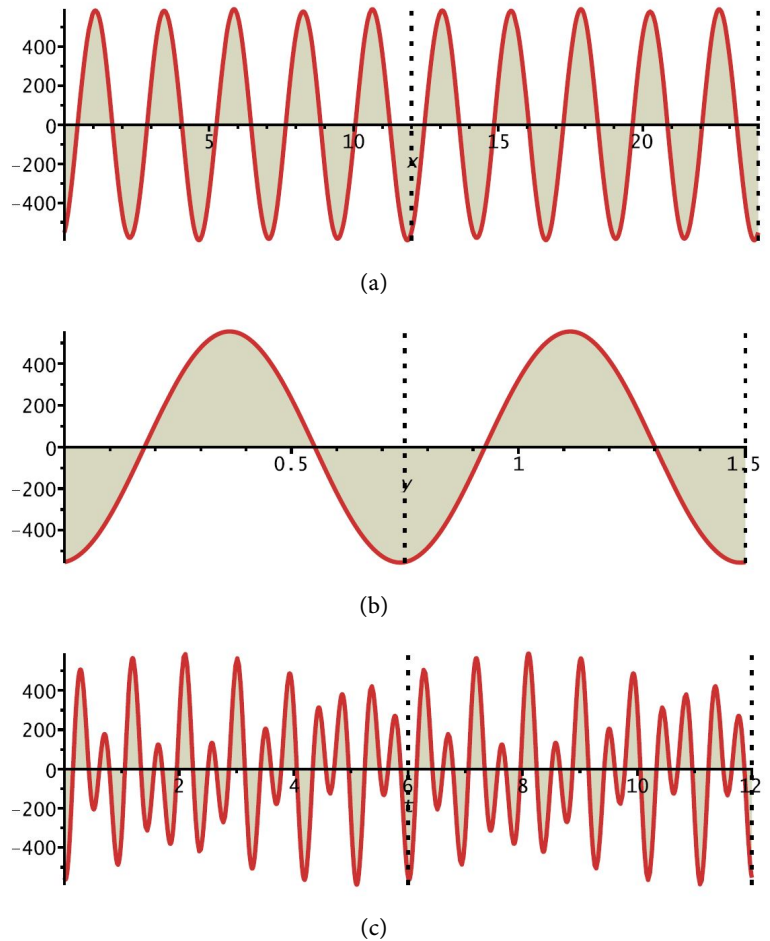


Figure 10. The external wave oscillon $K_{w,a,2,c,3}$: (a)— $K_{w,a,2,c,3}(x)$ (249), (b)— $K_{w,a,2,c,3}(y)$ (252), (c)— $K_{w,a,2,c,3}(t)$ (254) for independent parameters (95).

An x -period of the group oscillon of propagation

$$L_{g,x} = LCM(L_{x,1}, \dots, L_{x,m}, \dots, L_{x,M}) = k_{g,x,1}L_{x,1} \dots = k_{g,x,m}L_{x,m} \dots = k_{g,x,M}L_{x,M}, \quad (257)$$

where $k_{g,x,m}$ is an integer and $L_{x,m}$ is given by (74) for all m .

Due to (53), (211), (257), the group oscillon of propagation is neutral in the x -direction since

$$\int_0^{L_{g,x}} K_{g,i,m} dx = 0. \quad (258)$$

Similarly, we use (53) and (212) to find the group oscillon of propagation

$$K_{g,i,m} = \sum_{m=1}^M e z_m Q_{w,x,m} \sin(\lambda_m(y + s_{w,\alpha,x,m})). \quad (259)$$

For any frozen $x = x_0, z = z_0, t = t_0$, the group oscillon of propagation is displayed by a M -f neutral oscillon in y that is generated by M wave oscillons of propagation, *i.e.* the superposition of sine waves with wavenumbers

$\lambda_1, \dots, \lambda_m, \dots, \lambda_M$ and parameters of (212) for $m = 1, 2, \dots, M$.

A y -period of the group oscillon of propagation

$$L_{g,y} = LCM(L_{y,1}, \dots, L_{y,m}, \dots, L_{y,M}) = k_{g,y,1} L_{y,1} \dots = k_{g,y,m} L_{y,m} \dots = k_{g,y,M} L_{y,M}, \quad (260)$$

where $k_{g,y,m}$ is an integer and $L_{y,m}$ is provided by (82) for all m .

Due to (53), (217), (260), the group oscillon of propagation is also neutral in the y -direction because

$$\int_0^{L_{g,y}} K_{g,i,m} dy = 0. \quad (261)$$

Usage (53) and (218) yields

$$K_{g,i,m} = \frac{1}{2} \sum_{m=1}^M e z_m (-Q_{w,m} f_{w,1,m} + R_{w,m} f_{w,3,m}). \quad (262)$$

For any frozen $x = x_0, y = y_0, z = z_0$, the group oscillon of propagation is presented by a $2M$ -f neutral oscillon in t that is produced by M wave oscillons of propagation, *i.e.* the superposition of sine waves $f_{w,1,m}, f_{w,3,m}$ with frequencies $\omega_{x,m} + \omega_{y,m}, \omega_{x,m} - \omega_{y,m}$, and parameters of (218) for $m = 1, 2, \dots, M$.

A t -period of the group oscillon of propagation

$$T_g = LCM(T_{o,1}, \dots, T_{o,m}, \dots, T_{o,M}) = k_{g,1} T_{o,1} \dots = k_{g,m} T_{o,m} \dots = k_{g,M} T_{o,M}, \quad (263)$$

where $k_{g,m}$ is an integer and $T_{o,m}$ is specified by (93) for all m .

Due to (53), (223), (263),

$$\int_0^{T_g} K_{g,i,m} dt = 0. \quad (264)$$

The group oscillon of propagation of the velocity potential is neutral in time, as well.

The group oscillon of propagation $K_{g,i,m}$ with $L_{g,x} = 24, L_{g,y} = 9, T_g = 6$ is shown in **Figure 11** for independent parameters (95). The group oscillon of propagation is visualized as a 3-f neutral oscillon in x with period $L_{g,x}$, 14 local x -maximums, and 14 local x -minimums, the numbers of which do not exceed $k_{g,x,3} = 18$ due to four merged x -maximums and x -minimums.

The group oscillon of propagation is depicted by a 3-f neutral oscillon in y with period $L_{g,y}$, seven local y -maximums, and seven local y -minimums, the numbers of which do not exceed $k_{g,y,3} = 9$ due to two merged y -maximums and y -minimums. The group oscillon of propagation is displayed in terms of a 6-f neutral oscillon in t with period T_g , 10 local t -maximums, and 10 local t -minimums, the numbers of which do not exceed $k_{g,3} = 15$ due to five merged t -maximums and t -minimums.

5.2. The Group Pulson of Propagation

Because of (56) and (132), the group pulson of propagation of the kinetic energy

$$K_{g,i,m,i,m} = \rho_c \mu_m^2 e z_m^2 (Q_{y,m}^2 + R_{y,m}^2) = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \quad (265)$$

For any frozen $y = y_0, z = z_0, t = t_0$, the group pulson of propagation is reduced to a 0-f supercritical pulson in x with an energy shift proportional to $\rho_c \mu_m^2 e z_m^2(z_0), Q_{y,m}^2(y_0, t_0) + R_{y,m}^2(y_0, t_0)$.

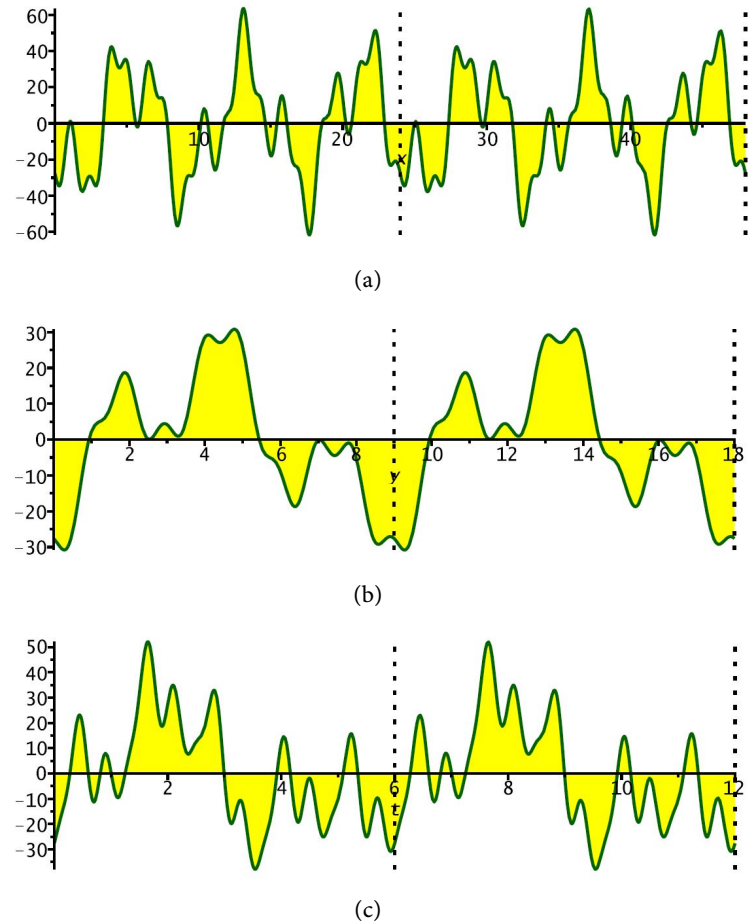


Figure 11. The group oscillon of propagation $K_{g,i,m}$: (a)— $K_{g,i,m}(x)$ (256), (b)— $K_{g,i,m}(y)$ (259), (c)— $K_{g,i,m}(t)$ (262) for independent parameters (95).

In agreement with (56) and (135),

$$\frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{g,i,m,i,m} dx = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \quad (266)$$

The group pulson of propagation transfers a positive amount of the kinetic energy along the x -axis.

Using (56) and (138), we compute the group pulson of propagation as follows

$$K_{g,i,m,i,m} = \rho_c \mu_m^2 e z_m^2 (Q_{x,m}^2 + R_{x,m}^2) = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \quad (267)$$

For any frozen $x = x_0, z = z_0, t = t_0$, the group pulson of propagation is transformed into the 0-f supercritical pulson in y with the energy shift proportional to $\rho_c \mu_m^2, e z_m^2(z_0), Q_{x,m}^2(x_0, t_0) + R_{x,m}^2(x_0, t_0)$.

Because of (56) and (141), the group pulson of propagation also transfers the positive amount of the kinetic energy along the y -axis since

$$\frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{g,i,m,i,m} dy = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \quad (268)$$

Usage of (56) and (143)-(144) gives the group pulson of propagation in the following form:

$$K_{g,i,m,i,m} = \frac{1}{2} \rho_c \mu_m^2 e z_m^2 (Q_m^2 + R_m^2) = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \quad (269)$$

For any frozen $x = x_0, y = y_0, z = z_0$, the group pulson of propagation is converted into the 0-f supercritical pulson in t with the energy shift proportional to μ_m^2 and (146).

Due to (56) and (149),

$$\frac{1}{T_{w,m,m}} \int_0^{T_{w,m,m}} K_{g,i,m,i,m} dt = \rho_c \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2), \quad (270)$$

i.e. the group pulson of propagation transfers the positive amount of the kinetic energy in time, as well.

5.3. The Internal Group Oscillon

Combining (59) and (152) returns the inhomogeneous Fourier expansion of the internal group oscillon along the x -axis

$$K_{g,i,m,j,m} = \rho_c e z_m^2 \left[2 \kappa_m^2 Q_{y,m} R_{y,m} h_{y,2,m} + \lambda_m^2 (Q_{y,m}^2 g_{x,1,m} + R_{y,m}^2 g_{x,5,m}) \right]. \quad (271)$$

For any frozen $y = y_0, z = z_0, t = t_0$, the internal group oscillon is converted into a 1-f nonneutral oscillon in x , which is formed by a superposition of sine waves $g_{x,1,m}, g_{x,5,m}$ with wavenumber $2 \kappa_m$, amplitudes proportional to λ_m^2 and (133), x -shifts (72), and an energy shift proportional to κ_m^2 and (153) except $h_{y,1,m}(y_0, t_0)$. The x -period of the internal group oscillon is specified by (134).

For the same reason as for the internal elementary oscillons,

$$\frac{1}{L_{x,m,m}} \int_0^{L_{x,m,m}} K_{g,i,m,j,m} dx = 2 \rho_c \kappa_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,2,m}. \quad (272)$$

Integral (272) proves that the internal group oscillon on average is nonneutral as it transfers a kinetic energy along the x -axis.

Similarly, we use (59) and (165) to find the inhomogeneous Fourier expansion of the internal group oscillon along the y -axis

$$K_{g,i,m,j,m} = \rho_c e z_m^2 \left[2 \lambda_m^2 Q_{x,m} R_{x,m} h_{x,2,m} + \kappa_m^2 (Q_{x,m}^2 g_{y,1,m} + R_{x,m}^2 g_{y,5,m}) \right]. \quad (273)$$

For any frozen $x = x_0, z = z_0, t = t_0$, the internal group oscillon is reduced to a 1-f nonneutral oscillon in y , which is produced by a superposition of sine waves $g_{y,1,m}, g_{y,5,m}$ with wavenumber $2 \lambda_m$, amplitudes proportional to κ_m^2 and (139), y -shifts (81), and an energy shift proportional to λ_m^2 and (166) except $h_{x,1,m}(y_0, t_0)$. The y -period of the internal group oscillon is given by (140).

Analogously to the internal elementary oscillons, the internal group oscillon on average is nonneutral as it also transfers a kinetic energy along the y -axis because

$$\frac{1}{L_{y,m,m}} \int_0^{L_{y,m,m}} K_{g,i,m,j,m} dy = 2 \rho_c \lambda_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m}. \quad (274)$$

Usage of (59) and (170) yields the inhomogeneous Fourier expansion of the internal group oscillon in time

$$K_{g,i,m,j,m} = \rho_c e z_m^2 Q_m R_m \left(\lambda_m^2 g_{3,m} - \kappa_m^2 g_{5,m} \right). \tag{275}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the internal group oscillon is transformed into a 2-f neutral oscillon in t , which is generated by a superposition of sine waves $g_{3,m}, g_{5,m}$ with frequencies $2\omega_{x,m}, 2\omega_{y,m}$, respectively, amplitudes proportional to κ_m^2, λ_m^2 , and (145), and t -shifts depending on (91). The temporal period of the group oscillon $T_{w,m,m}$ is provided by (229).

The internal group oscillon on average is neutral as does not transfer in time any kinetic energy since

$$\int_0^{T_{w,m,m}} K_{g,i,m,j,m} dt = 0. \tag{276}$$

The internal group oscillon $K_{g,i,3,j,3}$ with $L_{x,3,3} = 2/3, L_{y,3,3} = 1/2, T_{w,3,3} = 1$ is shown in **Figure 12** for independent parameters (95). The internal group oscillons $K_{g,i,m,j,m}$ are visualized by 1-f nonneutral oscillons in x with periods $L_{x,m,m} = [4, 3/2, 2/3]$, in terms of 1-f nonneutral oscillons in y with periods $L_{y,m,m} = [9/2, 3/2, 1/2]$, and via 2-f neutral oscillons in t with periods $T_{w,m,m} = [3, 3/2, 1], [3, 3, 3]$ local t -maximums and $[3, 3, 3]$ local t -minimums, the numbers of which are specified by $k_{w,2,m,m} = [3, 3, 3]$

5.4. The Diagonal Group Oscillon

Combining (62) and (179) yields the inhomogeneous Fourier expansion of the diagonal group oscillon along the x -axis

$$K_{g,i,m,i,n} = \rho_c e z_m e z_n M_{m,n} \left(Q_{y,m} Q_{y,n} f_{x,4,m,n} + R_{y,m} R_{y,n} f_{x,16,m,n} \right). \tag{277}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal group oscillon is transformed into a 1-f neutral oscillon in x , which is generated by a superposition of two cosine waves $f_{x,4,m,n}, f_{x,16,m,n}$ with wavenumber $\kappa_{2,m,n}$, amplitudes proportional to $M_{m,n}$ and (180), and x -shifts depending on (181). The x -wavelength of the diagonal group oscillon is provided by $L_{x,2,m,n}$ (183). If $n = m$, the diagonal group oscillon (277) by (108)-(109) is transformed into the doubled group pulson of propagation (265).

The diagonal group oscillon on average also is neutral along the x -axis as

$$\int_0^{L_{x,2,m,m}} K_{g,i,m,i,n} dx = 0. \tag{278}$$

Analogously, we combine (62) and (185) to find the inhomogeneous Fourier expansion of the diagonal group oscillon along the y -axis

$$K_{g,i,m,i,n} = \rho_c e z_m e z_n M_{m,n} \left(Q_{x,m} Q_{x,n} f_{y,4,m,n} + R_{x,m} R_{x,n} f_{y,16,m,n} \right). \tag{279}$$

For any frozen $x = x_0, z = z_0, t = t_0$, the diagonal group oscillon is reduced to a 1-f neutral oscillon in y , which is formed by a superposition of two cosine waves $f_{y,4,m,n}, f_{y,16,m,n}$ with wavenumber $\lambda_{2,m,n}$, amplitudes proportional to $M_{m,n}$ and (186), and y -shifts depending on (187). The y -wavelength of the diagonal group oscillon is specified by $L_{y,2,m,n}$ (189). If $n = m$, the diagonal

group oscillon (279) is reduced to the doubled group pulson of propagation (267).

The diagonal group oscillon is also neutral along the y -axis since it does not transfer on average any kinetic energy because

$$\int_0^{L_{y,2,m,m}} K_{g,i,m,i,n} dy = 0. \tag{280}$$

Application of (62) and (191) gives the inhomogeneous Fourier expansion of the diagonal group oscillon in time

$$K_{g,i,m,i,n} = \frac{1}{2} \rho_c e z_m e z_n M_{m,n} (Q_m Q_n f_{4,m,n} + R_m R_n f_{16,m,n}). \tag{281}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the diagonal group oscillon is converted into a 2-f neutral oscillon in t , which is produced by a superposition of cosine waves $f_{4,m,n}, f_{16,m,n}$ with frequencies $\omega_{2,m,n}, \omega_{8,m,n}$ correspondingly, amplitudes proportional to $M_{m,n}$ and (192), and t -shifts depending on (193). If $n = m$, the diagonal group oscillon (281) is transformed into the doubled group pulson of propagation (269).

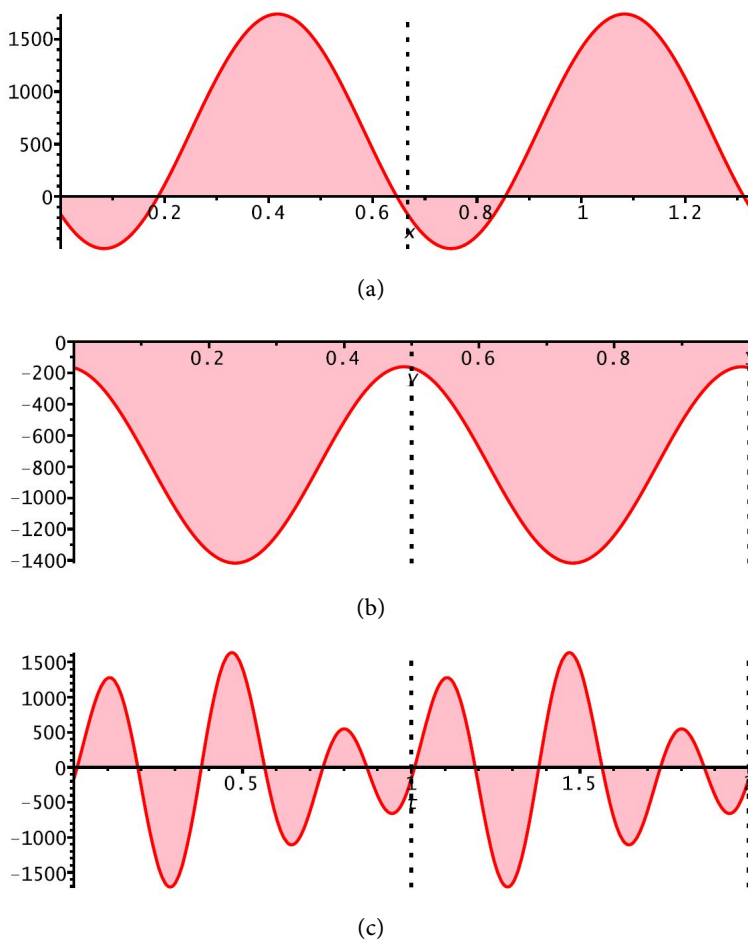


Figure 12. The internal group oscillon $K_{g,i,3,j,3}$: (a)— $K_{g,i,3,j,3}(x)$ (271), (b)— $K_{g,i,3,j,3}(y)$ (273), (c)— $K_{g,i,3,j,3}(t)$ (275) for independent parameters (95).

A temporal period of the diagonal group oscillon

$$T_{g,d,m,n} = LCM(T_{2,m,n}, T_{8,m,n}) = k_{g,d,2,m,n} T_{1,m,n} = k_{g,d,8,m,n} T_{8,m,n}, \quad (282)$$

where $k_{g,d,2,m,n}, k_{g,d,8,m,n}$ are integers and $T_{2,m,n}, T_{8,m,n}$ are specified by (194).

Integral over the t -period of the diagonal group oscillon

$$\int_0^{T_{g,d,m,n}} K_{g,i,m,i,n} dt = 0. \quad (283)$$

The diagonal group oscillon on average is neutral in time, as well, because it does not transfer any kinetic energy.

The diagonal group oscillon $K_{g,i,2,i,3}$ with $L_{x,2,2,3} = 12/5, L_{y,2,2,3} = 3/2, T_{g,d,2,3} = 6$ is shown in **Figure 13** for independent parameters (95). The diagonal group oscillons $K_{g,i,m,i,n}$ are displayed in terms of 1-f neutral oscillons in x with periods specified by the magnitude of $L_{x,2,m,n} = [-24/5, -8/4, -12/5]$, via 1-f neutral oscillons in y with periods indicated by the magnitude of $L_{y,2,m,n} = [-9/2, -9/8, -2]$, and by 2-f neutral oscillons in t with periods $T_{g,d,m,n} = [6, 3, 6], [5, 5, 5]$ local t -maximums and $[5, 5, 5]$ local t -minimums, the numbers of which are specified by the magnitude of $k_{g,d,2,m,n} = [-5, -5, -5]$.

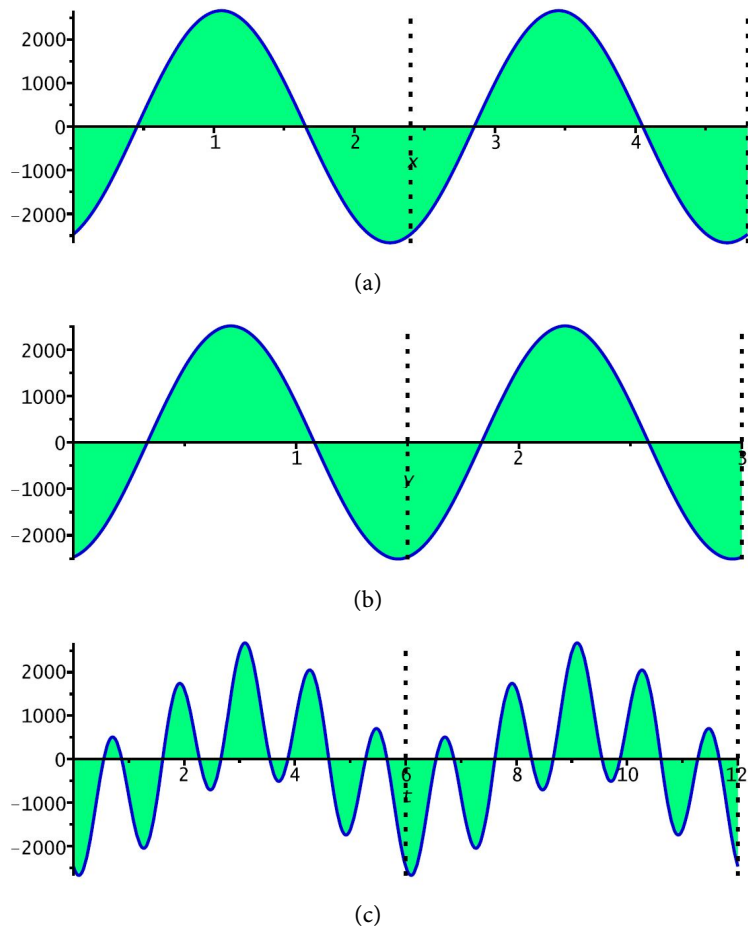


Figure 13. The diagonal group oscillon $K_{g,i,2,i,3}$: (a)— $K_{g,i,2,i,3}(x)$ (277), (b)— $K_{g,i,2,i,3}(y)$ (279), (c)— $K_{g,i,2,i,3}(t)$ (281) for independent parameters (95).

5.5. The External Group Oscillon

Substituting (197) in (65), we recast definition (43) in the inhomogeneous Fourier expansion of the external group oscillon along the x -axis

$$K_{g,i,m,j,n} = \rho_c e z_m e z_n \left[K_{m,n} \left(Q_{y,m} R_{y,n} f_{x,8,m,n} + R_{y,m} Q_{y,n} f_{x,12,m,n} \right) + \Lambda_{m,n} \left(Q_{y,m} Q_{y,n} f_{x,1,m,n} + R_{y,m} R_{y,n} f_{x,13,m,n} \right) - N_{m,n} \left(Q_{y,m} R_{y,n} f_{x,5,m,n} + R_{y,m} Q_{y,n} f_{x,9,m,n} \right) \right]. \quad (284)$$

For any frozen $y = y_0, z = z_0, t = t_0$, the external group oscillon is reduced to a 2-f neutral oscillon in x , which is generated by a superposition of four sine waves $f_{x,1,m,n}, f_{x,5,m,n}, f_{x,9,m,n}, f_{x,13,m,n}$ with wavenumber $\kappa_{1,m,n}$ and two cosine waves $f_{x,8,m,n}, f_{x,12,m,n}$ with wavenumber $\kappa_{2,m,n}$, amplitudes proportional to $K_{m,n}, \Lambda_{m,n}, N_{m,n}$, and (180), and x -shifts depending on (181). The x -wavelength of the external group oscillon is specified by $L_{x,m,n}$ (182)-(183). If $n = m$, the external group oscillon (284) is transformed by (108), (109), (250) into the doubled internal group oscillon (271).

Like the external elementary oscillons, the external group oscillon on average does not transfer any kinetic energy along the x -axis as

$$\int_0^{L_{x,m,n}} K_{g,i,m,j,n} dx = 0. \quad (285)$$

Similarly, writing definition (43) in y -eigenfunctions of external interaction $f_{y,l,m,n}$ with the help of (199) and (65) yields

$$K_{g,i,m,j,n} = \rho_c e z_m e z_n \left[\Lambda_{m,n} \left(Q_{x,m} R_{x,n} f_{y,8,m,n} + R_{x,m} Q_{x,n} f_{y,12,m,n} \right) + K_{m,n} \left(Q_{x,m} Q_{x,n} f_{y,1,m,n} + R_{x,m} R_{x,n} f_{y,13,m,n} \right) - N_{m,n} \left(Q_{x,m} R_{x,n} f_{y,5,m,n} + R_{x,m} Q_{x,n} f_{y,9,m,n} \right) \right]. \quad (286)$$

For any frozen $x = x_0, z = z_0, t = t_0$, the external group oscillon is converted into a 2-f neutral oscillon in y , which is produced by a superposition of four sine waves $f_{y,1,m,n}, f_{y,5,m,n}, f_{y,9,m,n}, f_{y,13,m,n}$ with wavenumber $\lambda_{1,m,n}$ and two cosine waves $f_{y,8,m,n}, f_{y,12,m,n}$ with wavenumber $\lambda_{2,m,n}$, amplitudes proportional to $K_{m,n}, \Lambda_{m,n}, N_{m,n}$, and (186), and y -shifts depending on (187). The y -wavelength of the external group oscillon $L_{y,m,n}$ is defined by (188)-(189). If $n = m$, the external group oscillon (286) is reduced to the doubled internal group oscillon (273).

Because

$$\int_0^{L_{y,m,n}} K_{g,i,m,j,n} dy = 0, \quad (287)$$

the external group oscillon on average does not transfer any kinetic energy along the y -axis, as well.

Substitution of (201) in (65) and collection of like terms produce the inhomogeneous Fourier expansion of the external group oscillon in time

$$\begin{aligned}
 K_{g,i,m,j,n} = \frac{\rho_c}{2} e z_m e z_n & \left[\Lambda_{m,n} \left(+Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n} \right) \right. \\
 & + K_{m,n} \left(-Q_m R_n f_{7,m,n} + R_m Q_n f_{11,m,n} \right) \\
 & \left. + N_{m,n} \left(-Q_m Q_n f_{2,m,n} + R_m R_n f_{14,m,n} \right) \right].
 \end{aligned}
 \tag{288}$$

For any frozen $x = x_0, y = y_0, z = z_0$, the external group oscillon is converted into at most sextet-frequency (6-f) neutral oscillon in t , which are formed by a superposition of four sine waves $f_{5,m,n}, f_{7,m,n}, f_{9,m,n}, f_{11,m,n}$ with frequencies $\omega_{3,m,n}, \omega_{4,m,n}, \omega_{5,m,n}, \omega_{6,m,n}$ and two cosine waves $f_{2,m,n}, f_{14,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{7,m,n}$, respectively, amplitudes proportional to $K_{m,n}, \Lambda_{m,n}, N_{m,n}$, and (192), and t -shifts depending on (193). If $n = m$, the external group oscillon (288) is transformed by (127) and (250) into the doubled internal group oscillon (275).

A temporal period of the external group oscillon

$$\begin{aligned}
 T_{g,e,m,n} &= LCM(T_{1,m,n}, \dots, T_{l,m,n}, \dots, T_{7,m,n}) \\
 &= k_{g,e,l,m,n} T_{1,m,n} \dots = k_{g,e,l,m,n} T_{l,m,n} \dots = k_{g,e,7,m,n} T_{7,m,n},
 \end{aligned}
 \tag{289}$$

where $k_{g,e,l,m,n}$ is an integer and $T_{l,m,n}$ is specified by (194) for $l = 1, 3, 4, 5, 6, 7$.

The external group oscillon on average does not transfer in time any kinetic energy since

$$\int_0^{T_{g,e,m,n}} K_{g,i,m,j,n} dt = 0.
 \tag{290}$$

The external group oscillon $K_{g,i,2,j,3}$ with $L_{x,2,3} = 12, L_{y,2,3} = 3/2, T_{g,e,2,3} = 6$ is shown in **Figure 14** for independent parameters (95). The external group oscillons $K_{g,i,m,j,n}$ are visualized by 2-f neutral oscillons in x with periods $L_{x,m,n} = [24, 8, 12]$, $[11, 7, 13]$ local x -maximums, and $[11, 7, 13]$ local x -minimums, the numbers of which are specified by $k_{x,1,m,n} = [11, 7, 13]$.

The external group oscillons are depicted by 2-f neutral oscillons in y with periods $L_{y,m,n} = [9/2, 9/2, 3/2]$, $[1, 4, 2]$ local y -maximums, and $[1, 4, 2]$ local y -minimums, the numbers of which do not exceed or equal to $k_{y,1,m,n} = [2, 5, 2]$ due to $[1, 1, 0]$ merged y -maximums and y -minimums.

Because of smallness of amplitudes $N_{m,n}$ compared with $K_{m,n}, \Lambda_{m,n}$ (248) and repeated frequencies, the external group oscillons are displayed with a graph accuracy as 4-f neutral oscillons in t with periods $T_{g,e,m,n} = [6, 3, 6]$, $[9, 7, 17]$ local t -maximums and $[9, 7, 17]$ local t -minimums, the numbers of which do not exceed or equal to $k_{g,e,5,m,n} = [11, 8, 17]$ due to $[2, 1, 0]$ merged t -maximums and t -minimums.

6. Energy Oscillons and Pulsons

6.1. The Energy Pulson of Propagation

Due to (30) and (265), the cumulative energy of all group pulsons of propagation along the x -axis is given by the energy pulson of propagation

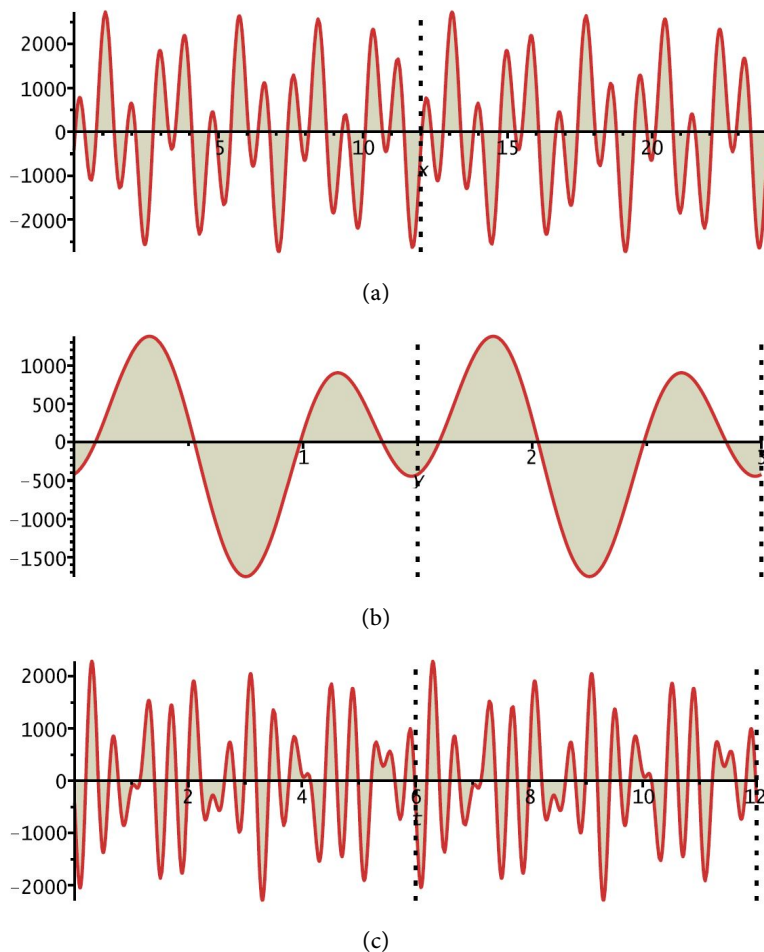


Figure 14. The external group oscillon $K_{g,i2,j3}$: (a)— $K_{g,i2,j3}(x)$ (284), (b)— $K_{g,i2,j3}(y)$ (286), (c)— $K_{g,i2,j3}(t)$ (288) for independent parameters (95).

$$K_{e,i,m,i,m} = \rho_c \sum_{m=1}^M \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \tag{291}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the energy pulson of propagation is formed by the superposition of M 0-f supercritical pulsions in x with energy shifts of (265).

A global x -period of internal interaction

$$\begin{aligned} L_{e,x,i} &= LCM(L_{x,1,1}, \dots, L_{x,m,m}, \dots, L_{x,M,M}) \\ &= k_{e,x,1,1} L_{x,1,1} \dots = k_{e,x,m,m} L_{x,m,m} \dots = k_{e,x,M,M} L_{x,M,M}, \end{aligned} \tag{292}$$

where $k_{e,x,m,m}$ is an integer and $L_{x,m,m}$ is specified by (134) for all m .

The energy pulson of propagation transfers a positive amount of the kinetic energy along the x -axis as

$$\frac{1}{L_{e,x,i}} \int_0^{L_{e,x,i}} K_{e,i,m,i,m} dx = \rho_c \sum_{m=1}^M \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \tag{293}$$

Because of (30) and (267), the cumulative energy of all group pulsions of propagation along the y -axis is also given by (291). For any frozen

$x = x_0, z = z_0, t = t_0$, the energy pulson of propagation is produced by the superposition of M 0-f supercritical pulsions in y with energy shifts of (267).

A global y -period of internal interaction

$$\begin{aligned} L_{e,y,i} &= LCM(L_{y,1,1}, \dots, L_{y,m,m}, \dots, L_{y,M,M}) \\ &= k_{e,y,1,1} L_{y,1,1} \dots = k_{e,y,m,m} L_{y,m,m} \dots = k_{e,y,M,M} L_{y,M,M}, \end{aligned} \tag{294}$$

where $k_{e,y,m,m}$ is an integer and $L_{y,m,m}$ is defined by (140) for all m .

The energy pulson of propagation also transfers a positive amount of the kinetic energy along the y -axis since

$$\frac{1}{L_{e,y,i}} \int_0^{L_{e,y,i}} K_{e,i,m,i,m} dy = \rho_c \sum_{m=1}^M \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2). \tag{295}$$

In the view of (30) and (269), the energy pulson of propagation in time is once more described by (291). For any frozen $x = x_0, y = y_0, z = z_0$, the energy pulson of propagation is generated by the superposition of M 0-f supercritical pulsions in t with energy shifts of (269).

A global t -period of internal interaction

$$\begin{aligned} T_{e,i} &= LCM(T_{w,1,1}, \dots, T_{w,m,m}, \dots, T_{w,M,M}) \\ &= k_{e,i,1,1} T_{w,1,1} \dots = k_{e,i,m,m} T_{w,m,m} \dots = k_{e,i,M,M} T_{w,M,M}, \end{aligned} \tag{296}$$

where $k_{e,i,m,m}$ is an integer and $T_{w,m,m}$ is given by (229) for all m .

Because

$$\frac{1}{T_{e,i}} \int_0^{T_{e,i}} K_{e,i,m,i,m} dt = \rho_c \sum_{m=1}^M \mu_m^2 e z_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2), \tag{297}$$

the energy pulson of propagation transfers a positive amount of the kinetic energy in time, as well.

6.2. The Internal Energy Oscillon

In agreement with (40) and (271), the cumulative energy of all internal group oscillons along the x -axis is described the internal energy oscillon

$$K_{e,i,m,j,m} = \rho_c \sum_{m=1}^m e z_m^2 \left[2 \kappa_m^2 Q_{y,m} R_{y,m} h_{y,2,m} + \lambda_m^2 (Q_{y,m}^2 g_{x,1,m} + R_{y,m}^2 g_{x,5,m}) \right]. \tag{298}$$

For any frozen $y = y_0, z = z_0, t = t_0$, the internal energy oscillon is represented by an M -f nonneutral oscillon in x that is generated by M internal group oscillons, *i.e.* the superposition of sine waves $g_{x,1,m}, g_{x,5,m}$ for $m = 1, 2, \dots, M$ with wavenumbers $2 \kappa_1, \dots, 2 \kappa_m, \dots, 2 \kappa_M$, parameters of (271), and x -period (292).

Average of (298) over the global x -period of internal interaction

$$\frac{1}{L_{e,x,i}} \int_0^{L_{e,x,i}} K_{e,i,m,j,m} dx = 2 \rho_c \sum_{m=1}^M \kappa_m^2 e z_m^2 Q_{y,m} R_{y,m} h_{y,2,m} \tag{299}$$

due to (292), (272), and the additive interval property. The internal energy oscillon on average is nonneutral since it transfers the kinetic energy along the x -axis.

In the view of (40) and (273), the cumulative energy of all internal group oscillons along the y -axis is given the internal energy oscillon

$$K_{e,i,m,j,m} = \rho_c \sum_{m=1}^M e z_m^2 \left[2\lambda_m^2 Q_{x,m} R_{x,m} h_{x,2,m} + \kappa_m^2 \left(Q_{x,m}^2 g_{y,1,m} + R_{x,m}^2 g_{y,5,m} \right) \right]. \quad (300)$$

For any frozen $x = x_0, z = z_0, t = t_0$, the internal energy oscillon is displayed by an M -f nonneutral oscillon in y , which is produced by M internal group oscillons, namely, the superposition of sine waves $g_{y,1,m}, g_{y,5,m}$ for $m = 1, 2, \dots, M$ with wavenumbers $2\lambda_1, \dots, 2\lambda_m, \dots, 2\lambda_M$, parameters of (273), and y -period (294).

The internal energy oscillon on average is also nonneutral along the y -axis because it transfers the kinetic energy as average of (300) over the global y -period (294)

$$\frac{1}{L_{e,y,i}} \int_0^{L_{e,y,i}} K_{e,i,m,j,m} dy = 2\rho_c \sum_{m=1}^M \lambda_m^2 e z_m^2 Q_{x,m} R_{x,m} h_{x,2,m}. \quad (301)$$

Because of (40) and (275), the internal energy oscillon takes the following t -form:

$$K_{e,i,m,j,m} = \rho_c \sum_{m=1}^M e z_m^2 Q_m R_m \left(\lambda_m^2 g_{3,m} - \kappa_m^2 g_{5,m} \right). \quad (302)$$

For any frozen $x = x_0, y = y_0, z = z_0$, the internal energy oscillon is at most $2M$ -f neutral oscillon in t , which is formed by M internal group oscillons, *viz.* the superposition of sine waves $g_{3,m}$ and $g_{5,m}$ with frequencies $2\omega_{x,1}, \dots, 2\omega_{x,m}, \dots, 2\omega_{x,M}$ and $2\omega_{y,1}, \dots, 2\omega_{y,m}, \dots, 2\omega_{y,M}$, respectively, parameters of (275), and t -period (296).

Integral of (302) over the global t -period of internal interaction

$$\int_0^{T_{e,i}} K_{e,i,m,j,m} dt = 0 \quad (303)$$

shows that $K_{e,i,m,j,m}$ on average does not transfer in time any kinetic energy.

The internal energy oscillon $K_{e,i,m,j,m}$ with $L_{e,x,i} = 12, L_{e,y,i} = 9/2, T_{e,i} = 3$ is shown in **Figure 15** for independent parameters (95). $K_{e,i,m,j,m}$ is displayed by a 3-f nonneutral oscillon in x with 18 local x -maximums and 18 local x -minimums, the numbers of which are specified by $k_{e,x,3,3} = 18$. $K_{e,i,m,j,m}$ is depicted as a 3-f nonneutral oscillon in y with nine local y -maximums and nine local y -minimums, the numbers of which are given by $k_{e,y,3,3} = 9$. Due to the repeated frequency 4π , $K_{e,i,m,j,m}$ is visualized by a 5-f neutral oscillon in t with nine local t -maximums, and nine local t -minimums, the number of which are specified by $m_{e,i,2,3,3} = 9$, where $m_{e,i,k,m,m}$ are multipliers of $T_{k,m,m}$ (147) in $T_{e,i}$ for $k = 1, 2, 3, 4$.

6.3. The Diagonal Energy Oscillon

Due to (35) and (277), a cumulative energy of all diagonal group oscillons along the x -axis is expressed via the diagonal energy oscillon

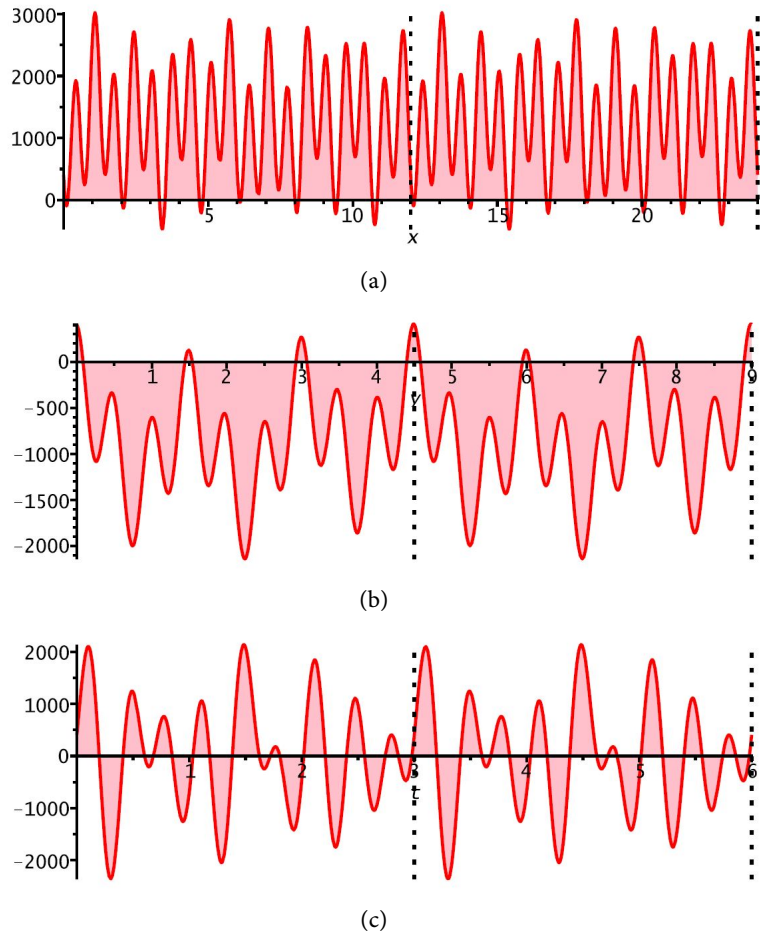


Figure 15. The internal energy oscillon $K_{e,i,m,j,m}$: (a)— $K_{e,i,m,j,m}(x)$ (298), (b)— $K_{e,i,m,j,m}(y)$ (300), (c)— $K_{e,i,m,j,m}(t)$ (302) for independent parameters (95).

$$K_{e,i,m,i,n} = \rho_c \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n M_{m,n} (Q_{y,m} Q_{y,n} f_{x,4,m,n} + R_{y,m} R_{y,n} f_{x,16,m,n}) \quad (304)$$

with a global x -period of diagonal interaction

$$\begin{aligned} L_{e,x,d} &= LCM(L_{x,2,1,2}, \dots, L_{x,2,m,n}, \dots, L_{x,2,M-1,M}) \\ &= k_{e,x,2,1,2} L_{x,2,1,2} \dots = k_{e,x,2,m,n} L_{x,2,m,n} \dots = k_{e,x,2,M-1,M} L_{x,2,M-1,M}, \end{aligned} \quad (305)$$

where $k_{e,x,2,m,n}$ is an integer and $L_{x,2,m,n}$ is specified by (183) for all m, n .

For any frozen $y = y_0, z = z_0, t = t_0$, the diagonal energy oscillon is transformed into at most $M(M-1)/2$ -f neutral oscillon in x that is formed by $M(M-1)/2$ diagonal group oscillons, *i.e.* the superposition of cosine waves $f_{x,4,m,n}, f_{x,16,m,n}$ with wavenumber $\kappa_{2,m,n}$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, and parameters of (277).

Because integral of (304) over the global x -period of diagonal interaction

$$\int_0^{L_{e,x,d}} K_{e,i,m,i,n} dx = 0, \quad (306)$$

the diagonal energy oscillon on average does not transfer any kinetic energy

along the x -axis.

In agreement with (35) and (279), a cumulative energy of all diagonal group oscillons along the y -axis is described by the diagonal energy oscillon

$$K_{e,i,m,i,n} = \rho_c \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left(Q_{x,m} Q_{x,n} f_{y,4,m,n} + R_{x,m} R_{x,n} f_{y,16,m,n} \right) \quad (307)$$

with a global y -period of diagonal interaction

$$\begin{aligned} L_{e,y,d} &= LCM \left(L_{y,2,1,2}, \dots, L_{y,2,m,n}, \dots, L_{y,2,M-1,M} \right) \\ &= k_{e,y,2,1,2} L_{y,2,1,2} \dots = k_{e,y,2,m,n} L_{y,2,m,n} \dots = k_{e,y,2,M-1,M} L_{y,2,M-1,M}, \end{aligned} \quad (308)$$

where $k_{e,y,2,m,n}$ is an integer and $L_{y,2,m,n}$ is defined by (189) for all m, n .

For any frozen $x = x_0, z = z_0, t = t_0$, the diagonal energy oscillon is converted into at most $M(M-1)/2$ -f neutral oscillon in y that is generated by $M(M-1)/2$ diagonal group oscillons, viz. the superposition of cosine waves $f_{y,4,m,n}, f_{y,16,m,n}$ with wavenumber $\lambda_{2,m,n}$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, and parameters of (279).

The diagonal energy oscillon on average also does not transfer any kinetic energy along the y -axis since

$$\int_0^{L_{e,y,d}} K_{e,i,m,i,n} dy = 0. \quad (309)$$

Because of (35) and (281), the diagonal energy oscillon takes the following t -form:

$$K_{e,i,m,i,n} = \frac{\rho_c}{2} \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n M_{m,n} \left(Q_m Q_n f_{4,m,n} + R_m R_n f_{16,m,n} \right) \quad (310)$$

with a global t -period of diagonal interaction

$$\begin{aligned} T_{e,d} &= LCM \left(T_{g,d,1,2}, \dots, T_{g,d,m,n}, \dots, T_{g,d,M-1,M} \right) \\ &= k_{e,d,1,2} T_{g,d,1,2} \dots = k_{e,d,m,n} T_{g,d,m,n} \dots = k_{e,d,M-1,M} T_{g,d,M-1,M}, \end{aligned} \quad (311)$$

where $k_{e,d,m,n}$ is an integer and $T_{g,d,m,n}$ is given by (282) for all m, n .

For any frozen $x = x_0, y = y_0, z = z_0$, the diagonal energy oscillon is reduced to at most $M(M-1)$ -f neutral oscillon in t , which is produced by $M(M-1)/2$ diagonal group oscillons, specifically, the superposition of cosine waves $f_{4,m,n}, f_{16,m,n}$ with frequencies $\omega_{2,m,n}, \omega_{8,m,n}$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, respectively, and parameters of (281).

Since

$$\int_0^{T_{e,d}} K_{e,i,m,i,n} dt = 0, \quad (312)$$

the diagonal energy oscillon on average does not transfer any kinetic energy in time, as well.

The diagonal energy oscillon $K_{e,i,m,i,n}$ with $L_{e,x,d} = 24/5, L_{e,y,d} = 9/2, T_{e,d} = 6$ is shown in **Figure 16** for independent parameters (95). $K_{e,i,m,i,n}$ is represented by a 3-f neutral oscillon in x with three local x -maximums and three local x -minimums, the numbers of which are specified by the magnitude of

$k_{e,x,2,1,3} = -3$. $K_{e,i,m,i,n}$ is displayed as a 3-f neutral oscillon in y with three local y -maximums and three local y -minimums, the numbers of which do not exceed the magnitude of $k_{e,y,2,1,3} = -4$ due to a single merged y -maximum and y -minimum. As the frequencies $-5\pi/3, -\pi/3$ are repeated, $K_{e,i,m,i,n}$ is visualized via a 4-f neutral oscillon with nine local t -maximums and nine local t -minimums, the numbers of which do not exceed the magnitude of $m_{e,d,2,1,3} = -10$ due to a single merged t -maximum and t -minimum, where $m_{e,d,l,m,n}$ are multipliers of $T_{l,m,n}$ (194) in $T_{e,d}$ for $l = 2, 8$.

6.4. The External Energy Oscillon

Combining (45) and (284), we find the cumulative energy of all external group oscillons along the x -axis that is given by the external energy oscillon

$$K_{e,i,m,j,n} = \rho_c \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[K_{m,n} (Q_{y,m} R_{y,n} f_{x,8,m,n} + R_{y,m} Q_{y,n} f_{x,12,m,n}) \right. \\ \left. + \Lambda_{m,n} (Q_{y,m} Q_{y,n} f_{x,1,m,n} + R_{y,m} R_{y,n} f_{x,13,m,n}) \right. \\ \left. - N_{m,n} (Q_{y,m} R_{y,n} f_{x,5,m,n} + R_{y,m} Q_{y,n} f_{x,9,m,n}) \right] \quad (313)$$

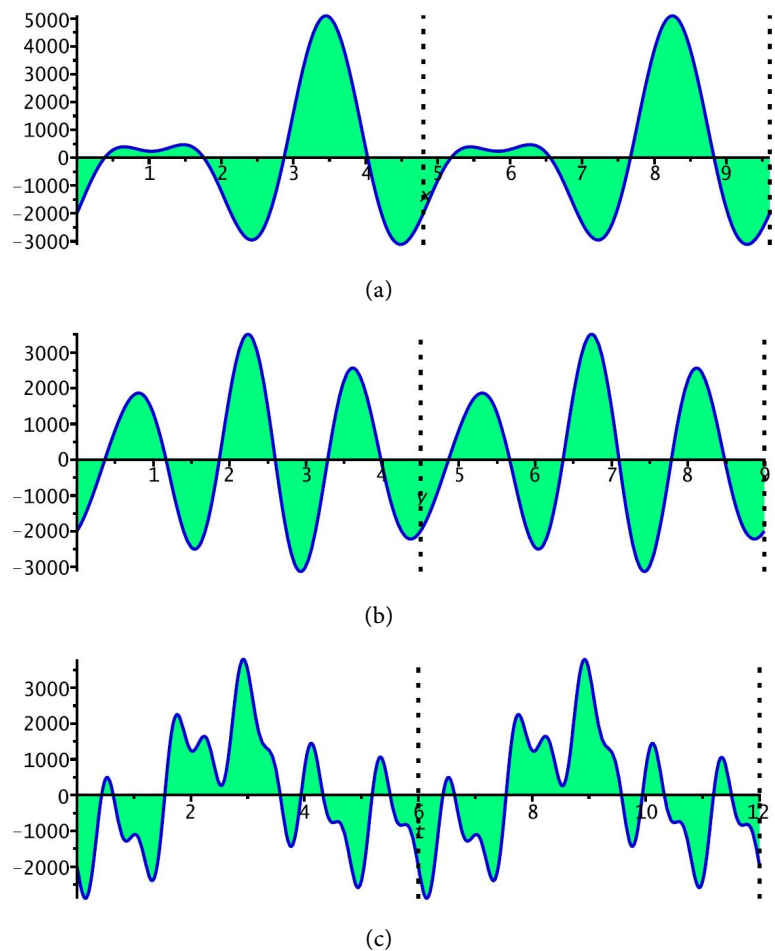


Figure 16. The diagonal energy oscillon $K_{e,i,m,i,n}$: (a)— $K_{e,i,m,i,n}(x)$ (304), (b)— $K_{e,i,m,i,n}(y)$ (307), (c)— $K_{e,i,m,i,n}(t)$ (310) for independent parameters (95).

with a global x -period of external interaction

$$L_{e,x,e} = LCM(L_{x,1,2}, \dots, L_{x,m,n}, \dots, L_{x,M-1,M}) \tag{314}$$

$$= k_{e,x,1,2} L_{x,1,2} \dots = k_{e,x,m,n} L_{x,m,n} \dots = k_{e,x,M-1,M} L_{x,M-1,M},$$

where $k_{e,x,m,n}$ is an integer and $L_{x,m,n}$ is provided by (182)-(183) for all m, n .

For any frozen $y = y_0, z = z_0, t = t_0$, the external energy oscillon is transformed into at most $M(M-1)$ -f neutral oscillon in x that is formed by $M(M-1)/2$ external group oscillons, *i.e.* the superposition of sine waves $f_{x,1,m,n}, f_{x,5,m,n}, f_{x,9,m,n}, f_{x,13,m,n}$ with wavenumber $\kappa_{1,m,n}$ and cosine waves $f_{x,8,m,n}, f_{x,12,m,n}$ with wavenumber $\kappa_{2,m,n}$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, and parameters of (284).

Because

$$\int_0^{L_{e,x,e}} K_{e,i,m,j,n} dx = 0, \tag{315}$$

the external energy oscillon on average does not transfer any kinetic energy along the x -axis.

In the view of (45) and (286), the external energy oscillon along the y -axis takes the following y -form:

$$K_{e,i,m,j,n} = \rho_c \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[\Lambda_{m,n} (Q_{x,m} R_{x,n} f_{y,8,m,n} + R_{x,m} Q_{x,n} f_{y,12,m,n}) \right. \\ \left. + K_{m,n} (Q_{x,m} Q_{x,n} f_{y,1,m,n} + R_{x,m} R_{x,n} f_{y,13,m,n}) \right. \\ \left. - N_{m,n} (Q_{x,m} R_{x,n} f_{y,5,m,n} + R_{x,m} Q_{x,n} f_{y,9,m,n}) \right] \tag{316}$$

with a global y -period of external interaction

$$L_{e,y,e} = LCM(L_{y,1,2}, \dots, L_{y,m,n}, \dots, L_{y,M-1,M}) \tag{317}$$

$$= k_{e,y,1,2} L_{y,1,2} \dots = k_{e,y,m,n} L_{y,m,n} \dots = k_{e,y,M-1,M} L_{y,M-1,M},$$

where $k_{e,y,m,n}$ is an integer and $L_{y,m,n}$ is defined by (188)-(189) for all m, n .

For any frozen $x = x_0, z = z_0, t = t_0$, the external energy oscillon is converted into at most $M(M-1)$ -f neutral oscillon in y , which is produced by $M(M-1)/2$ external group oscillons, namely, the superposition of sine waves $f_{y,1,m,n}, f_{y,5,m,n}, f_{y,9,m,n}, f_{y,13,m,n}$ with wavenumber $\lambda_{1,m,n}$ and cosine waves $f_{y,8,m,n}, f_{y,12,m,n}$ with wavenumber $\lambda_{2,m,n}$ for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, and parameters of (286).

The external energy oscillon is also neutral along the y -axis since

$$\int_0^{L_{e,y,e}} K_{e,i,m,j,n} dy = 0. \tag{318}$$

We use (45) and (288) to compute the inhomogeneous Fourier expansion the external energy oscillon in time

$$K_{e,i,m,j,n} = \frac{\rho_c}{2} \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[\Lambda_{m,n} (+Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n}) \right. \\ \left. + K_{m,n} (-Q_m R_n f_{7,m,n} + R_m Q_n f_{11,m,n}) \right. \\ \left. + N_{m,n} (-Q_m Q_n f_{2,m,n} + R_m R_n f_{14,m,n}) \right] \tag{319}$$

with a global t -period of external interaction

$$T_{e,e} = LCM(T_{g,e,1,2}, \dots, T_{g,e,m,n}, \dots, T_{g,e,M-1,M}) \tag{320}$$

$$= k_{e,e,1,2} T_{g,e,1,2} \dots = k_{e,e,m,n} T_{g,e,m,n} \dots = k_{e,e,M-1,M} T_{g,e,M-1,M},$$

where $k_{e,e,m,n}$ is an integer and $T_{g,e,m,n}$ is specified by (289) for all m, n .

For any frozen $x = x_0, y = y_0, z = z_0$, the external energy oscillon is transformed into at most $3M(M-1)$ -f neutral oscillon in t , which is generated by $M(M-1)/2$ external group oscillon, viz. the superposition of sine waves $f_{5,m,n}, f_{7,m,n}, f_{9,m,n}, f_{11,m,n}$ with frequencies $\omega_{3,m,n}, \omega_{4,m,n}, \omega_{5,m,n}, \omega_{6,m,n}$ and cosine waves $f_{2,m,n}, f_{14,m,n}$ with frequencies $\omega_{1,m,n}, \omega_{7,m,n}$, for $n = m+1, m+2, \dots, M$, $m = 1, 2, \dots, M-1$, respectively, and parameters of (288).

For the reason that

$$\int_0^{T_{e,e}} K_{e,i,m,j,n} dt = 0, \tag{321}$$

the external energy oscillon on average does not transfer any kinetic energy in time, as well.

The external energy oscillon $K_{e,i,m,j,n}$ with $L_{e,x,e} = 24, L_{e,y,e} = 9/2, T_{e,e} = 6$ is shown in **Figure 17** for independent parameters (95). $K_{e,i,m,j,n}$ is displayed by a 6-f neutral oscillon in x with 26 local x -maximums and 26 local x -minimums, the numbers of which are specified by $m_{e,x,1,2,3} = 26$, where $m_{e,x,l,m,n}$ are multipliers of $L_{x,l,m,n}$ (183) in $L_{e,x,e}$ for $l=1,2$. $K_{e,i,m,j,n}$ is visualized by a 6-f neutral oscillon in y with six local y -maximums and six local y -minimums, the numbers of which are specified by $m_{e,y,1,2,3} = 6$, where $m_{e,y,l,m,n}$ are multipliers of $L_{y,l,m,n}$ (189) in $L_{e,y,e}$ for $l=1,2$.

Due to repeated frequencies $\pi, 7\pi/3, 13\pi/3$, $K_{e,i,m,j,n}$ is represented via a 15-f neutral oscillon with 16 local t -maximums and 16 local t -minimums, the numbers of which do not exceed $m_{e,e,5,2,3} = 17$ due to a single merged t -maximum and t -minimum, where $m_{e,e,l,m,n}$ are multipliers of $T_{l,m,n}$ (194) in $T_{e,e}$ for $l=1,3,4,5,6,7$.

6.5. The Cumulative Energy Pulson

We then use (66), (291), (298), (304), and (313), to find the inhomogeneous Fourier expansion of the cumulative pulson of the kinetic energy along the x -axis

$$K_{e,i,m,j,n} = \rho_c \left\{ \sum_{m=1}^m e z_m^2 \left[\mu_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) \right. \right. \\ \left. \left. + 2 \kappa_m^2 Q_{y,m} R_{y,m} h_{y,2,m} + \lambda_m^2 (Q_{y,m}^2 g_{x,1,m} + R_{y,m}^2 g_{x,5,m}) \right] \right. \\ \left. + \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[M_{m,n} (Q_{y,m} Q_{y,n} f_{x,4,m,n} + R_{y,m} R_{y,n} f_{x,16,m,n}) \right. \right. \\ \left. \left. + K_{m,n} (Q_{y,m} R_{y,n} f_{x,8,m,n} + R_{y,m} Q_{y,n} f_{x,12,m,n}) \right. \right. \\ \left. \left. + \Lambda_{m,n} (Q_{y,m} Q_{y,n} f_{x,1,m,n} + R_{y,m} R_{y,n} f_{x,13,m,n}) \right. \right. \\ \left. \left. - N_{m,n} (Q_{y,m} R_{y,n} f_{x,5,m,n} + R_{y,m} Q_{y,n} f_{x,9,m,n}) \right] \right\}, \tag{322}$$

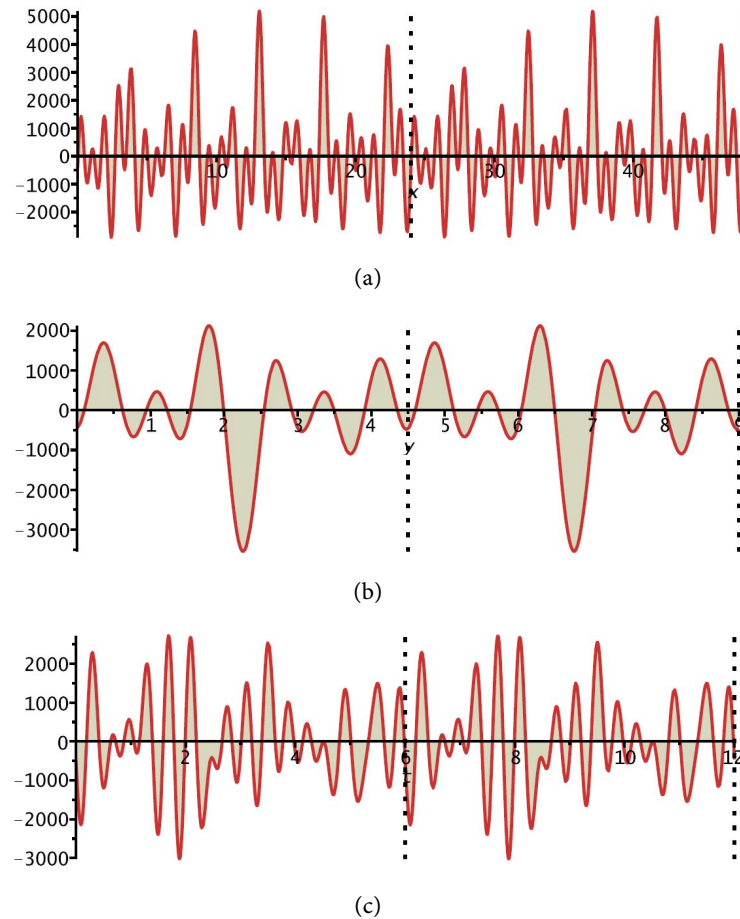


Figure 17. The external energy oscillon $K_{e,i,m,j,n}$: (a)— $K_{e,i,m,j,n}(x)$ (313), (b)— $K_{e,i,m,j,n}(y)$ (316), (c)— $K_{e,i,m,j,n}(t)$ (319) for independent parameters (95).

where ρ_c is the constant density, $[\kappa_m, \lambda_m, \mu_m]$ are the wave numbers in the (x, y, z) directions, $[Av_m, Bv_m, Cv_m, Dv_m]$ are functional amplitudes of the 3-d DSK functions (48), $[Q_{y,m}, Q_{y,n}, R_{y,m}, R_{y,n}]$ are trigonometric amplitudes (68), $h_{y,2,m}$ is energy shift (109), $[g_{x,1,m}, g_{x,5,m}]$ are sine waves (109) with wavenumber $2\kappa_m$, $[K_{m,n}, \Lambda_{m,n}, M_{m,n}, N_{m,n}]$ are nonlinear amplitudes (246), $[f_{x,4,m,n}, f_{x,8,m,n}, f_{x,12,m,n}, f_{x,16,m,n}]$ are cosine waves (101) with wavenumber $\kappa_m - \kappa_n$, and $[f_{x,1,m,n}, f_{x,5,m,n}, f_{x,9,m,n}, f_{x,13,m,n}]$ are sine waves (101) with wavenumber $\kappa_m + \kappa_n$.

For any frozen $y = y_0, z = z_0, t = t_0$, the cumulative energy pulson is converted to at most M^2 -f supercritical pulson in x , which is formed by the superposition of the energy pulson of propagation $K_{e,i,m,i,m}$, the internal energy oscillon $K_{e,i,m,j,m}$, the diagonal energy oscillon $K_{e,i,m,i,n}$, and the external energy oscillon $K_{e,i,m,j,n}$. The total number of various wavenumbers is at most M^2 since there are M wave numbers $2\kappa_m$ and $M(M-1)/2$ wavenumbers $[\kappa_m - \kappa_n, \kappa_m + \kappa_n]$.

A global x -wavelength of the cumulative energy pulson

$$L_{e,x} = LCM(L_{e,x,i}, L_{e,x,d}, L_{e,x,e}) = k_{e,x,i} L_{e,x,i} = k_{e,x,d} L_{e,x,d} = k_{e,x,e} L_{e,x,e}, \quad (323)$$

where $k_{e,x,i}, k_{e,x,d}, k_{e,x,e}$ are integers, $L_{e,x,i}$ is the global x -period of internal interaction (292), $L_{e,x,d}$ is the global x -period of diagonal interaction (305), and $L_{e,x,e}$ is the global x -period of external interaction (314).

Combining (323), (292), (305), and (314) yields the global x -wavelength of K_e

$$L_{e,x} = LCM(\dots, L_{x,m,m}, \dots, L_{x,1,m,n}, \dots, L_{x,2,m,n}, \dots) \tag{324}$$

$$\dots = l_{e,x,m,m} L_{x,m,m} \dots = l_{e,x,1,m,n} L_{x,1,m,n} \dots = l_{e,x,2,m,n} L_{x,2,m,n} \dots,$$

where $l_{e,x,m,m}, l_{e,x,1,m,n}, l_{e,x,2,m,n}$ are integers, $L_{x,m,m}$ (134) is the local x -period of internal interaction for $m = 1, 2, \dots, M$, $L_{x,1,m,n}$ and $L_{x,2,m,n}$ (183) are the local x -periods of external interaction for $n = m + 1, m + 2, \dots, M$, $m = 1, 2, \dots, M - 1$.

The cumulative energy pulson on average transfer a positive amount of the kinetic energy along the x -axis since

$$\frac{1}{L_{e,x}} \int_0^{L_{e,x}} K_e dx = \rho_c \sum_{m=1}^M e z_m^2 \left[\mu_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \kappa_m^2 Q_{y,m} R_{y,m} h_{y,2,m} \right] \tag{325}$$

in the view of (66), (293), (299), (306), (315), (323), and the additive interval property.

Combining (66), (291), (300), (307), and (316) yields the inhomogeneous Fourier expansion of the cumulative energy pulson along the y -axis

$$K_{e,i,m,j,n} = \rho_c \left\{ \sum_{m=1}^m e z_m^2 \left[\mu_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \lambda_m^2 Q_{x,m} R_{x,m} h_{x,2,m} + \kappa_m^2 (Q_{x,m}^2 g_{y,1,m} + R_{x,m}^2 g_{y,5,m}) \right] + \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[M_{m,n} (Q_{x,m} Q_{x,n} f_{y,4,m,n} + R_{x,m} R_{x,n} f_{y,16,m,n}) + \Lambda_{m,n} (Q_{x,m} R_{x,n} f_{y,8,m,n} + R_{x,m} Q_{x,n} f_{y,12,m,n}) + K_{m,n} (Q_{x,m} Q_{x,n} f_{y,1,m,n} + R_{x,m} R_{x,n} f_{y,13,m,n}) - N_{m,n} (Q_{x,m} R_{x,n} f_{y,5,m,n} + R_{x,m} Q_{x,n} f_{y,9,m,n}) \right] \right\}, \tag{326}$$

where $[Q_{x,m}, Q_{x,n}, R_{x,m}, R_{x,n}]$ are trigonometric amplitudes (77), $h_{x,2,m}$ is the energy shift, $[g_{y,1,m}, g_{y,5,m}]$ are the sine waves with wavenumber $2\lambda_m$, $[f_{y,4,m,n}, f_{y,8,m,n}, f_{y,12,m,n}, f_{y,16,m,n}]$ are the cosine waves with wavenumber $\lambda_m - \lambda_n$, and $[f_{y,1,m,n}, f_{y,5,m,n}, f_{y,9,m,n}, f_{y,13,m,n}]$ are the sine waves with wavenumber $\lambda_m + \lambda_n$.

For any frozen $x = x_0, z = z_0, t = t_0$, the cumulative energy pulson is transformed into at most M^2 -f supercritical pulson in y , which is generated by the superposition of $K_{e,i,m,i,m}, K_{e,i,m,j,m}, K_{e,i,m,i,n}, K_{e,i,m,j,n}$. The total number of various wavenumbers is at most M^2 since there are M wavenumbers $2\lambda_m$ and $M(M-1)/2$ wavenumbers $[\lambda_m - \lambda_n, \lambda_m + \lambda_n]$.

A global y -wavelength of the cumulative energy pulson

$$L_{e,y} = LCM(L_{e,y,i}, L_{e,y,d}, L_{e,y,e}) = k_{e,y,i} L_{e,y,i} = k_{e,y,d} L_{e,y,d} = k_{e,y,e} L_{e,y,e}, \tag{327}$$

where $k_{e,y,i}, k_{e,y,d}, k_{e,y,e}$ are integers, $L_{e,y,i}$ is the global y -period of internal

interaction (294), $L_{e,y,d}$ is the global y -period of diagonal interaction (308), and $L_{e,y,e}$ is the global y -period of external interaction (317).

We then use (327), (294), (308), and (317) to compute the global y -wavelength of K_e

$$L_{e,y} = LCM(\dots, L_{y,m,m}, \dots, L_{y,1,m,n}, \dots, L_{y,2,m,n}, \dots) \tag{328}$$

$$\dots = l_{e,y,m,m} L_{y,m,m} \dots = l_{e,y,1,m,n} L_{y,1,m,n} \dots = l_{e,y,2,m,n} L_{y,2,m,n} \dots,$$

where $l_{e,y,m,m}, l_{e,y,1,m,n}, l_{e,y,2,m,n}$ are integers, $L_{y,m,m}$ (140) is the local y -period of internal interaction for $m = 1, 2, \dots, M$, $L_{y,1,m,n}$ and $L_{y,2,m,n}$ (189) are the local y -periods of external interaction for $n = m + 1, m + 2, \dots, M$, $m = 1, 2, \dots, M - 1$.

The cumulative energy pulson on average also transfer a positive amount of the kinetic energy along the y -axis because

$$\frac{1}{L_{e,y}} \int_0^{L_{e,y}} K_e dy = \rho_c \sum_{m=1}^M e z_m^2 \left[\mu_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + 2 \lambda_m^2 Q_{x,m} R_{x,m} h_{x,2,m} \right] \tag{329}$$

due to (66), (295), (301), (309), (318), (327), and the additive interval property.

Usage of (66), (291), (302), (310), and (319) gives the inhomogeneous Fourier expansion of the cumulative energy in time

$$K_{e,i,m,j,n} = \rho_c \left\{ \sum_{m=1}^M e z_m^2 \left[\mu_m^2 (A v_m^2 + B v_m^2 + C v_m^2 + D v_m^2) + Q_m R_m (\lambda_m^2 g_{3,m} - \kappa_m^2 g_{5,m}) \right] \right. \\ \left. + \frac{1}{2} \sum_{m=1}^{M-1} \sum_{n=m+1}^M e z_m e z_n \left[M_{m,n} (+Q_m Q_n f_{4,m,n} + R_m R_n f_{16,m,n}) \right. \right. \\ \left. \left. + \Lambda_{m,n} (+Q_m R_n f_{5,m,n} + R_m Q_n f_{9,m,n}) \right. \right. \\ \left. \left. + K_{m,n} (-Q_m R_n f_{7,m,n} + R_m Q_n f_{11,m,n}) \right. \right. \\ \left. \left. + N_{m,n} (-Q_m Q_n f_{2,m,n} + R_m R_n f_{14,m,n}) \right] \right\}, \tag{330}$$

where $[Q_m, Q_n, R_m, R_n]$ are trigonometric amplitudes (87), $[g_{3,m}, g_{5,m}]$ are sine waves (126) with frequencies $[2\omega_{x,m}, 2\omega_{y,m}]$, respectively, $[f_{2,m,n}, f_{4,m,n}, f_{14,m,n}, f_{16,m,n}]$ are cosine waves (117) with frequencies $[\omega_{1,m,n}, \omega_{2,m,n}, \omega_{7,m,n}, \omega_{8,m,n}]$, and $[f_{5,m,n}, f_{7,m,n}, f_{9,m,n}, f_{11,m,n}]$ are sine waves (117) with frequencies $[\omega_{3,m,n}, \omega_{4,m,n}, \omega_{5,m,n}, \omega_{6,m,n}]$, correspondingly.

For any frozen $x = x_0, y = y_0, z = z_0$, the cumulative energy pulson is reduced to at most $2M(2M - 1)$ -f supercritical pulson in t , which is produced by the superposition of $K_{e,i,m,i,m}, K_{e,i,m,j,m}, K_{e,i,m,i,n}, K_{e,i,m,j,n}$. The total number of various frequencies is at most $2M(2M - 1)$ because there are M frequencies $[2\omega_{x,m}, 2\omega_{y,m}]$ and $M(M - 1)/2$ frequencies $\omega_{l,m,n}$ for $l = 1, 2, \dots, 8$.

A global t -wavelength of the cumulative energy pulson

$$T_e = LCM(T_{e,i}, T_{e,d}, T_{e,e}) = k_{e,i} T_{e,i} = k_{e,d} T_{e,d} = k_{e,e} T_{e,e}, \tag{331}$$

where $k_{e,i}, k_{e,d}, k_{e,e}$ are integers, $T_{e,i}$ is the global t -period of internal interaction (296), $T_{e,d}$ is the global t -period of diagonal interaction (311), and $T_{e,e}$ is the global t -period of external interaction (320).

Using (331), (296), (311), and (320) returns the global t -wavelength of K_e

$$T_e = LCM(\dots, T_{k,m,m}, \dots, T_{l,m,n}, \dots) \dots = l_{e,k,m,m} T_{k,m,m} \dots = l_{e,l,m,n} T_{l,m,n} \dots, \quad (332)$$

where $l_{e,k,m,m}, l_{e,l,m,n}$ are integers, $T_{k,m,m}$ (147) are the local t -periods of internal interaction for $k = 2, 3, m = 1, 2, \dots, M$, $T_{l,m,n}$ (194) are the local t -periods of external interaction for $l = 1, 2, \dots, 8, n = m + 1, m + 2, \dots, M, m = 1, 2, \dots, M - 1$.

In agreement with (66), (297), (303), (312), (321), (331), and the additive interval property,

$$\frac{1}{T_e} \int_0^{T_e} K_e dt = \rho_c \sum_{m=1}^M \mu_m^2 (Av_m^2 + Bv_m^2 + Cv_m^2 + Dv_m^2) ez_m^2. \quad (333)$$

Indeed, the cumulative energy pulson remains positive for all (x, y, z, t) despite numerous oscillons of internal and external interaction, which locally may transfer negative amounts of the kinetic energy.

The cumulative energy pulson K_e with $L_{e,x} = 24, L_{e,y} = 9/2, T_{e,e} = 6$ is shown in **Figure 18** for independent parameters (95). K_e is visualized by a 9-f supercritical pulson in x with 23 local x -maximums and 23 local x -minimums, the numbers of which do not exceed $l_{e,x,3,3} = 36$ due to 13 merged x -maximums and x -minimums. By the reason of repeated frequencies $4\pi/3, 4\pi/9$, K_e is depicted by a 7-f supercritical pulson in y with six local y -maximums and six local y -minimums, the numbers of which do not exceed $l_{e,y,3,3} = 9$ due to three merged y -maximums and y -minimums.

Because of smallness of amplitudes $N_{m,n}$ compared with $K_{m,n}, \Lambda_{m,n}, M_{m,n}$ (248) and repeated frequencies $\pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3, 7\pi/3, 4\pi, 13\pi/3$, K_e is displayed with a graph accuracy as a 19-f supercritical pulson with 15 local t -maximums and 15 local t -minimums, the numbers of which do not exceed $l_{e,2,3,3} = 18$ due to three merged t -maximums and t -minimums.

7. Discussion

The elementary pulsions of propagation, the internal elementary oscillons, the diagonal elementary oscillons, and the external elementary oscillons describe various scalar interactions between the scalar fields $a_m, b_m, c_m, d_m, a_n, b_n, c_n, d_n$ of the velocity potential of the elementary oscillons of propagation. The wave pulsions of propagation, the internal wave oscillons, the diagonal wave oscillons, and the external wave oscillons express different vector interactions between the velocity fields of wave groups $a_m, b_m, c_m, d_m, a_n, b_n, c_n, d_n$. Superpositions of the various vector interactions with frozen m and n are represented by the group pulsions of propagation, the internal group oscillons, the diagonal group oscillons, and the external group oscillons. Summation of the group pulsions of propagation, the internal group oscillons, the diagonal group oscillons, and the external group oscillons with respect to all m and n results in the energy pulson of propagation, the internal energy oscillon, the diagonal energy oscillon, and the external energy oscillon. The superposition of the energy pulson of propagation, the internal energy oscillon, the diagonal energy oscillon, and the external energy oscillon generates the cumulative energy pulson.

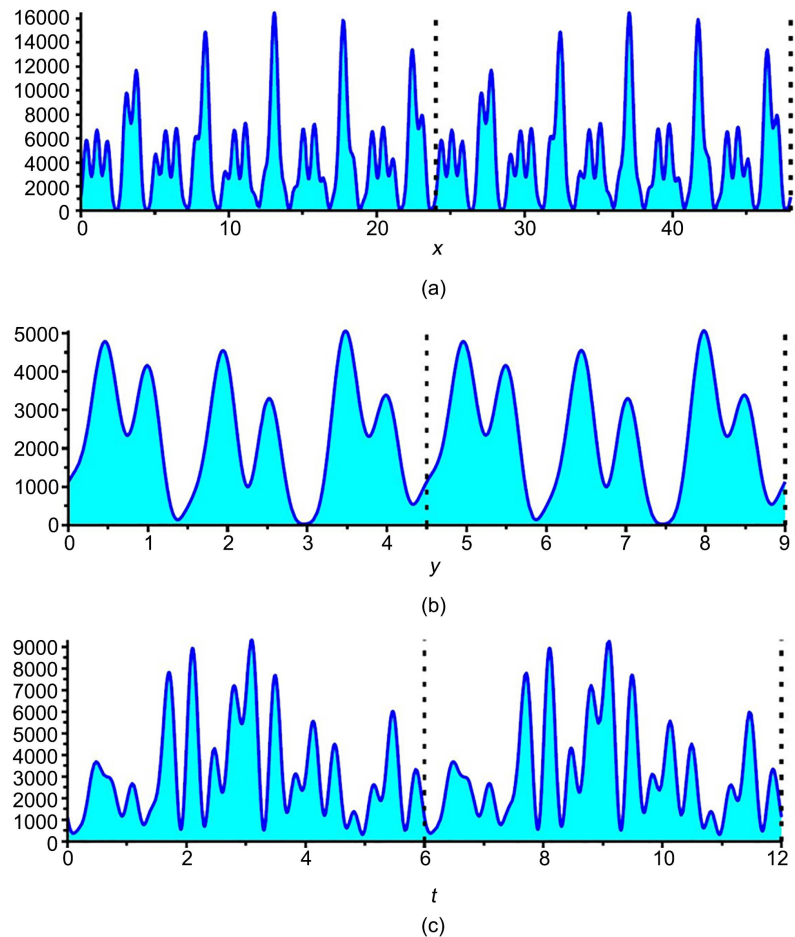


Figure 18. The cumulative energy pulson K_c : (a)— $K_c(x)$ (322), (b)— $K_c(y)$ (326), (c)— $K_c(t)$ (330) for independent parameters (95).

Classification of various types of pulsons and oscillons, which are decomposed in the inhomogeneous Fourier expansion, is considered in **Section 3.5**. The nested structure of the deterministic quantization of the kinetic energy of deterministic chaos of the elementary pulsons of propagation, the internal, diagonal, and external elementary oscillons, the wave pulsons of propagation, the internal, diagonal, and external wave oscillons, the group pulsons of propagation, the internal, diagonal, and external group oscillons, the energy pulsons of propagation, the internal, diagonal, and external energy oscillons, and the cumulative energy pulson is considered in **Section 2.7**. The nested structure of the deterministic quantization resembles the nested structure of objects of quantum mechanics: elementary particles, atoms, molecules, etc. This resemblance correlates with the quantum-like properties of the scalar and vector kinematic structures that are considered in [3]: the scalar-vector duality, the quadrality of the theoretical DSK and DVK structures, and the equiprobability the experimental DSK and DVK structures.

Amplitudes of the wave pulsons and oscillons are larger than amplitudes of the elementary pulsons and oscillons, amplitudes of the group pulsons and os-

cillons exceed amplitudes of the wave pulsons and oscillons, amplitudes of the energy pulsons and oscillons surpass amplitudes of the group pulsons and oscillons, and amplitudes of the cumulative energy pulson go above amplitudes of the energy pulsons and oscillons. Consequently, the elementary, wave, group, and energy pulsons and oscillons constitute also nested structures in amplitudes.

The x -, y -, t -periods of the wave pulsons and oscillons are larger or equal to the x -, y -, t -periods of the elementary pulsons and oscillons. Similarly, the x -, y -, t -periods of the group pulsons and oscillons exceed or equal to the x -, y -, t -periods of the wave pulsons and oscillons. Analogously, the x -, y -, t -periods of the energy pulsons and oscillons surpass or equal to the x -, y -, t -periods of the group pulsons and oscillons. Eventually, the x -, y -, t -periods of the cumulative energy pulson go beyond or equal to the x -, y -, t -periods of the energy pulsons and oscillons. Therefore, the elementary, wave, group, and energy pulsons and oscillons constitute nested structures with respect to the x -, y -, t -periods, as well.

Topology of the elementary pulson of propagation, the wave pulson of propagation, and the cumulative energy pulson in **Figure 2**, **Figure 7**, and **Figure 18**, respectively, is the same as of the solitons on shallow water, the solitary waves on shallow water with uniform [4] and linear [5] vorticity, the solitary waves generated by crossed electric and magnetic fields [6], and the pulsatory waves of the Korteweg-de Vries equation [7]. Topology of the diagonal elementary oscillon, the external elementary oscillon, the diagonal wave oscillon, the external wave oscillon, the diagonal group oscillon, the external group oscillon, the diagonal energy oscillon, and the external energy oscillon in **Figure 4**, **Figure 5**, **Figure 9**, **Figure 10**, **Figure 13**, **Figure 14**, **Figure 16**, and **Figure 17**, correspondingly, resembles the topology of nonlinear waves and solitons on deep water [8].

Pulsatory and oscillatory structure of the wave pulsons and oscillons is less complicated than the structure of the elementary pulsons and oscillons. Pulsatory and oscillatory structure of the group pulsons and oscillons is also less sophisticated than the structure of the wave pulsons and oscillons. This simplification is produced by compensation of undulations of complementary structures caused by the Pythagorean resonance, *i.e.* the Pythagorean identity. Oscillatory structure of the energy oscillons is more complex than the structure of the group oscillons due to a substantial growth in number of independent wavenumbers and frequencies.

Since chaos of physical systems could be explained by superposing a large number of deterministic solutions [9], it looks interesting to explore the effect of independent parameters on the rate of chaotization of the exact solution for deterministic chaos [3]. It also looks appealing to consider the Lagrangian and the Eulerian properties of the elementary, wave, group, energy pulsons and oscillons in the view of various types of wave lattices [2].

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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