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Block Extension of a Single-Step Hybrid Multistep Method for Directly Solving Fourth-Order Initial Value Problems

Monday Kolawole Duromola[1], Adelegan Lukuman Momoh[1], Olusegun Mayowa Akinmoladun[2]

[1]Department of Mathematical Sciences, Federal University of Technology, Akure, Nigeria
[2]Department of Mathematical Sciences, Lagos State University of Sciences of Technology, Ikoroodu, Nigeria
Email: mkduromola@futa.edu.ng, almomoh@futa.edu.ng, olusegun_m@yahoo.com

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Abstract

This article aims to derive, analyse, and implement an efficient one-step implicit hybrid method with block extension comprised of seven off-step points to directly solve Initial Value Problems (IVPs) of general four-order ordinary differential equations. For the resolution of the fourth-order IVPs, the exact was approximated by a polynomial termed basis function. The partial sum of the basis function and its fourth derivative were interpolated and collocated at some selected grid and off-grid points for the unknown parameters to be determined. The derived method, when tested, is found to be consistent, convergent, and zero-stable. The method's accuracy and usability were experimented with using specific sample problems, and the findings revealed that it surpassed some cited methods in terms of accuracy.

Keywords

Implicit, Absolute Stability, Grid Points, Off-Grid Points, Convergence, Interpolation, and Collocation

1. Introduction

This paper proposed a one-step hybrid method for directly solving fourth-order initial value problem of ordinary differential equations of the form:

$$y^{(4)} = f\left(x, y, y', y'', y'''\right), y\left(x_0\right) = a_0, y'\left(x_0\right) = a_1, y''\left(x_0\right) = a_2, y'''\left(x_0\right) = a_3 \quad (1)$$

where, f is a given continuous real value function. High-order linear and nonlinear IVPs have been used to represent engineering and other areas' problems. The static deflection of a uniform beam or a cantilever beam (with the left end embedded and the right end free giving birth to fourth-order IVPs) is one of the applications of fourth-order problems (see [1]).

Traditionally, the reduction approach is adopted for numerically solving equation (1), as reported by [2] [3] [4] [5] and many others. Although this strategy has had much success, it does have certain disadvantages. For example, computer programs related to the method's implementation are frequently complex, particularly the subroutines that supply the starting values of the methods, resulting in longer computer time and computational work. However, [6] found that these approaches do not consider extra information connected with a particular ordinary differential equation, such as the solution's oscillatory nature.

A direct approach is introduced as an alternative method to overcome the setback inherent in the reduction approach. Several direct numerical methods exist in the literature, but a few are specially designed to solve fourth-order ordinary differential equations. For instance, [7] developed a Block Hybrid Collocation Method (BHCM) and applied it to solve fourth-order IVPs. Three off-grid points are utilized with the collocation. [8] developed a four-step implicit block method with three generalized off-step points and applied it to solve fourth-order IVPs directly. In a work by [9], an algorithmic collocation approach was presented for obtaining the approximation of fourth-order IVPs. [1] suggested Runge-Kutta type method for directly solving this kind of problem.

According to [10] [11], single-step methods are efficient in terms of accuracy due to the hybrid points incorporated into the method. The proposed method's efficiency is measured by the number of hybrid points included, either as solutions or function values. The success of [11] motivates this work where the approximation of (1) is sorted in the interval $\left[x_n, x_{n+1}\right]$ with seven (7) intermediate points.

2. Mathematical Formulation

Let's start by allowing the exact $y(x)$ of the fourth-order IVP of ordinary differential Equation (1) to be approximated by a partial sum of the polynomial $p(x)$ of the form:

$$y(x) \cong p(x) = \sum_{r=0}^{8k+5} a_r x^r. \tag{2}$$

Equation (2) is differentiated four times to obtain its fourth derivative given as:

$$y^{(4)}(x) \cong p^{(4)}(x) = \sum_{r=4}^{8k+5} r(r-1)(r-2)(r-3) a_r x^{r-4}. \tag{3}$$

Equating (3) and (1) yields the differential system:

$$f\left(x, y(x), y'(x), y''(x), y'''(x)\right) = \sum_{r=4}^{8k+5} r(r-1)(r-2)(r-3) a_r x^{r-4}. \tag{4}$$

Note that x is continuously differentiable, parameters a_r's in (2), (3), and (4) are linear terms to be determined. By applying, $x = x_{n+r}, r = \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}$ to (2) and $x = x_{n+r}, r = 0\left(\frac{1}{8}\right)1$ to (4) yields the following system of algebraic equations:

$$y_{n+r} = \sum_{r=0}^{13} a_r x^r, \quad r = \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \tag{5}$$

$$f_{n+r} = \sum_{r=4}^{13} r(r-1)(r-2)(r-3) a_r x^{r-4}, \quad r = 0\left(\frac{1}{8}\right)1. \tag{6}$$

By allowing $x_{n+\frac{r}{8}} = x_n + \frac{r}{8}h$ and $x_n = 0$, (5) and (6) are written as matrix equation and solved using CAS in Mathematica to obtain the parameters $a_{\frac{r}{8}}$'s for $r = 0,1,2,\cdots,1$ which were substituted into (2) and yields the following continuous scheme after some simplifications:

$$y(t) = \sum_{r=4}^{7} \alpha_{\frac{r}{8}}(t) y_{n+\frac{r}{8}}(t) + h^4 \sum_{r=0}^{8} \beta_{\frac{r}{8}}(t) f_{n+\frac{r}{8}}(t) \tag{7}$$

where, $x = x_{n+t} = x_n + th$, $\alpha_{\frac{r}{8}}(t)$'s and $\beta_{\frac{r}{8}}(t)$'s are the coefficients that defined the scheme and are given as: 8

$$\alpha_{\frac{1}{2}}(t) = \frac{1}{3}\left(105 - 428t + 576t^2 - 256t^3\right), \quad \alpha_{\frac{5}{8}}(t) = 4\left(-21 + 94t - 136t^2 + 64t^3\right),$$

$$\alpha_{\frac{3}{4}}(t) = -2\left(-35 + 166t - 256t^2 + 128t^3\right), \quad \alpha_{\frac{7}{8}}(t) = \frac{4}{3}\left(-15 + 74t - 120t^2 + 64t^3\right),$$

$$\begin{aligned}\beta_0(t) = \frac{h^4}{490497638400}\big(&-125875 - 7186478t + 232909496t^2 - 2931938944t^3 \\ &+ 20437401600t^4 - 88873500672t^5 + 255465357312t^6 \\ &- 498906169344t^7 + 665831079935t^8 - 597939978240t^9 \\ &+ 345467431872t^{10} - 115964116992t^{11} + 17179869184t^{12}\big),\end{aligned}$$

$$\begin{aligned}\beta_{\frac{1}{8}}(t) = -\frac{h^4}{61312204800}\big(&-2558325 + 58784710t - 528368824t^2 + 2030364160t^3 \\ &- 32699842560t^5 + 149796421632t^6 - 362293493760t^7 \\ &+ 546450702336t^8 - 530579456000t^9 + 323330506752t^{10} \\ &- 112742891520t^{11} + 17179689184t^{12}\big),\end{aligned}$$

$$\begin{aligned}\beta_{\frac{1}{4}}(t) = \frac{h^4}{122624409600}\big(&39090975 - 476020370t + 1697474216t^2 - 86879232t^3 \\ &- 114449448960t^5 + 67688740992t^6 - 1905207017472t^7 \\ &+ 3174272335872t^8 - 3303433830400t^9 + 2117150441472t^{10} \\ &- 766651662336t^{11} + 120256084288t^{12}\big),\end{aligned}$$

$$\beta_{\frac{3}{8}}(t) = -\frac{h^4}{61312204800}\left(-68220075 + 543730330t - 1591532296t^2 + 2867507456t^3 - 76299632640t^5 + 485168775168t^6 - 1489244258304t^7 + 2670798569472t^8 - 2948176281600t^9 + 1979845705728t^{10} - 744103084032t^{11} + 120259084288t^{12}\right),$$

$$\beta_{\frac{1}{2}}(t) = \frac{h^4}{49049763840}\left(127767255 - 764794898t + 1418313224t^2 + 146591104t^3 - 57224724480t^5 + 376593186816t^6 - 1209997983744t^7 + 2282345201664t^8 - 26427470643t^9 + 1851399340032t^{10} - 721554505728t^{11} + 120259084288t^{12}\right),$$

$$\beta_{\frac{3}{4}}(t) = \frac{h^4}{122624409600}\left(103817175 - 509826050t + 7510153512t^2 - 323121920t^3 - 38149816320t^5 + 259539861504t^6 - 873552936960t^7 + 1745860165632t^8 - 2159227699200t^9 + 1621081718784t^{10} - 676457349120t^{11} + 120256084288t^{12}\right),$$

$$\beta_{\frac{7}{8}}(t) = -\frac{h^4}{61312204800}\left(768075 - 3551450t - 3513592t^2 + 897994424t^3 - 4671406080t^5 + 32076988416t^6 - 109414711296t^7 + 222566547456t^8 - 281437798400t^9 + 217030066176t^{10} - 93415538688t^{11} + 17179689184t^{12}\right),$$

$$\beta_{1}(t) = \frac{h^4}{490497638400}\left(594825 - 2755790t - 3575368t^2 + 78427008t^3 - 4087480320t^5 + 28262006784t^6 - 97372864512t^7 + 200766652416t^8 - 258369126400t^9 + 203742511104t^{10} - 90194313216t^{11} + 17179869184t^{12}\right).$$

Evaluating (7) at $t = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, 1$ to obtain the discrete one-step formula,

$$y_n - 35y_{n+\frac{1}{2}} + 84y_{n+\frac{5}{8}} - 70y_{n+\frac{3}{4}} + 20y_{n+\frac{7}{8}} = \frac{-h^4}{424673280}\left(109f_n - 17720f_{n+\frac{1}{8}} - 135380f_{n+\frac{1}{4}} - 472520f_{n+\frac{3}{8}} - 1106210f_{n+\frac{1}{2}} - 1542344f_{n+\frac{5}{8}} - 359540f_{n+\frac{3}{4}} + 5320f_{n+\frac{7}{8}} - 515f_{n+1}\right) \quad (8a)$$

$$y_{n+\frac{1}{8}} - 20y_{n+\frac{1}{2}} + 45y_{n+\frac{5}{8}} - 36y_{n+\frac{3}{4}} + 10y_{n+\frac{7}{8}} = \frac{-h^4}{990904320}\left(125f_n - 1144f_{n+\frac{1}{8}} - 34660f_{n+\frac{1}{4}} - 332440f_{n+\frac{3}{8}} - 1087810f_{n+\frac{1}{2}} - 1758760f_{n+\frac{5}{8}} - 4197f_{n+\frac{3}{4}} + 6200f_{n+\frac{7}{8}} - 595f_{n+1}\right) \quad (8b)$$

$$y_{n+\frac{1}{4}}-10y_{n+\frac{1}{2}}+20y_{n+\frac{5}{8}}-15y_{n+\frac{3}{4}}+4y_{n+\frac{7}{8}}$$
$$=\frac{-h^4}{2972712960}\Big(205f_n-1880f_{n+\frac{1}{8}}-7228f_{n+\frac{1}{4}}-130280f_{n+\frac{3}{8}} \tag{8c}$$
$$-973010f_{n+\frac{1}{2}}-2032040f_{n+\frac{5}{8}}-506180f_{n+\frac{3}{4}}+7912f_{n+\frac{7}{8}}-755f_{n+1}\Big)$$

$$y_{n+\frac{3}{8}}-4y_{n+\frac{1}{2}}+6y_{n+\frac{5}{8}}-4y_{n+\frac{3}{4}}+y_{n+\frac{7}{8}}$$
$$=\frac{-h^4}{2972712960}\Big(41f_n-328f_{n+\frac{1}{8}}+908f_{n+\frac{1}{4}}+152f_{n+\frac{3}{8}}-125722f_{n+\frac{1}{2}} \tag{8d}$$
$$-475288f_{n+\frac{5}{8}}-127444f_{n+\frac{3}{4}}+2120f_{n+\frac{7}{8}}-199f_{n+1}\Big)$$

$$y_{n+1}-4y_{n+\frac{7}{8}}+6y_{n+\frac{3}{4}}-4y_{n+\frac{5}{8}}+y_{n+\frac{1}{2}}$$
$$=\frac{h^4}{2972712960}\Big(199f_n-1832f_{n+\frac{1}{8}}+7492f_{n+\frac{1}{4}}-17624f_{n+\frac{3}{8}}+24922f_{n+\frac{1}{2}} \tag{8e}$$
$$+100648f_{n+\frac{5}{8}}+492004f_{n+\frac{3}{4}}+120280f_{n+\frac{7}{8}}-329f_{n+1}\Big)$$

Derivation of the Block Method

In the spirit of [12], the normalized form of the general block method is given by

$$AY_i=Ey_n+h^{\mu-\rho}df\left(y_n\right)+h^{\mu-\rho}BF\left(y_n\right) \tag{9}$$

By combining the formulas in (8) and the additional first, second, and third derivatives formulas obtained from (7) and writing in block form, using the definition of the implicit block method in (9) to get the block formula described as follows:

$$h^a\sum_{r=0}^{q}\phi_{m,r}y_{n+r}^{\rho}=h^{\rho}\sum_{r=0}^{q}\nabla_{m,r}y_n^{\rho}+h^{p-\rho}\left(\sum_{r=0}^{q}\Delta_{m,r}f_n+\sum_{r=0}^{q}\delta_{m,r}f_{n+r}\right) \tag{10}$$

where σ is the power of the derivative of the continuous method and p is the order of the problem to solve. Equation (10) is solved for $r=0,\frac{1}{8},\frac{2}{8},\cdots,1$ to obtain the following proposed Single-step Hybrid Multistep Method (SHMM):

$$y_{n+\frac{1}{8}}=y_n+\frac{1}{8}hy_n'+\frac{1}{128}h^2y_n''+\frac{1}{3072}h^3y'''+\frac{h^4}{3923981107200}\Big(24396497f_n$$
$$+36501816f_{n+\frac{1}{8}}-52883276f_{n+\frac{1}{4}}+67126376f_{n+\frac{3}{8}}-61500210f_{n+\frac{1}{2}}$$
$$+38838088f_{n+\frac{5}{8}}-16041916f_{n+\frac{3}{4}}+3904536f_{n+\frac{7}{8}}-425111f_{n+1}\Big),$$

$$y_{n+\frac{1}{4}}=y_n+\frac{1}{4}hy_n'+\frac{1}{32}h^2y_n''+\frac{1}{384}h^3y'''+\frac{h^4}{1532805100}\Big(1035731f_n$$
$$+2719504f_{n+\frac{1}{8}}-3139836f_{n+\frac{1}{4}}+3933392f_{n+\frac{3}{8}}-3577790f_{n+\frac{1}{2}}$$
$$+2249904f_{n+\frac{5}{8}}-926764f_{n+\frac{3}{4}}+225136f_{n+\frac{7}{8}}-24477f_{n+1}\Big),$$

$$
\begin{aligned}
y_{n+\frac{3}{8}} &= y_n + \frac{3}{8}hy_n' + \frac{9}{128}h^2 y_n'' + \frac{9}{1024}h^3 y''' + \frac{h^4}{16148070400}\Big(4104531 f_n \\
&\quad + 13764168 f_{n+\frac{1}{8}} - 12476916 f_{n+\frac{1}{4}} + 16614360 f_{n+\frac{3}{8}} - 15168870 f_{n+\frac{1}{2}} \\
&\quad + 9553464 f_{n+\frac{5}{8}} - 3938148 f_{n+\frac{3}{4}} + 957096 f_{n+\frac{7}{8}} - 104085 f_{n+1}\Big), \\
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy_n' + \frac{1}{8}h^2 y_n'' + \frac{1}{48}h^3 y''' + \frac{h^4}{59875200}\Big(38084 f_n \\
&\quad + 145056 f_{n+\frac{1}{8}} - 104780 f_{n+\frac{1}{4}} + 160352 f_{n+\frac{3}{8}} - 144375 f_{n+\frac{1}{2}} \\
&\quad + 90976 f_{n+\frac{5}{8}} - 37516 f_{n+\frac{3}{4}} + 9120 f_{n+\frac{7}{8}} - 992 f_{n+1}\Big), \\
y_{n+\frac{5}{8}} &= y_n + \frac{5}{8}hy_n' + \frac{25}{128}h^2 y_n'' + \frac{125}{3072}h^3 y''' + \frac{h^4}{156959244288}\Big(201421625 f_n \\
&\quad + 828115000 f_{n+\frac{1}{8}} - 490807500 f_{n+\frac{1}{4}} + 895985000 f_{n+\frac{3}{8}} - 766681250 f_{n+\frac{1}{2}} \\
&\quad + 487389000 f_{n+\frac{5}{8}} - 201077500 f_{n+\frac{3}{4}} + 48895000 f_{n+\frac{7}{8}} - 5319375 f_{n+1}\Big), \\
y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy_n' + \frac{9}{32}h^2 y_n'' + \frac{9}{128}h^3 y''' + \frac{h^4}{63078400}\Big(142929 f_n \\
&\quad + 618192 f_{n+\frac{1}{8}} - 308772 f_{n+\frac{1}{4}} + 672912 f_{n+\frac{3}{8}} - 532170 f_{n+\frac{1}{2}} \\
&\quad + 351216 f_{n+\frac{5}{8}} - 143892 f_{n+\frac{3}{4}} + 34992 f_{n+\frac{7}{8}} - 3807 f_{n+1}\Big), \\
y_{n+\frac{7}{8}} &= y_n + \frac{7}{8}hy_n' + \frac{49}{128}h^2 y_n'' + \frac{343}{3072}h^3 y''' + \frac{h^4}{560568729600}\Big(2048300303 f_n \\
&\quad + 9184285992 f_{n+\frac{1}{8}} - 3949712228 f_{n+\frac{1}{4}} + 10148873336 f_{n+\frac{3}{8}} \\
&\quad - 7344827070 f_{n+\frac{1}{2}} + 5205732952 f_{n+\frac{5}{8}} - 2050386772 f_{n+\frac{3}{4}} \\
&\quad + 504037128 f_{n+\frac{7}{8}} - 54841241 f_{n+1}\Big), \\
y_{n+1} &= y_n + hy_n' + \frac{1}{2}h^2 y_n'' + \frac{1}{6}h^3 y''' + \frac{h^4}{3742200}\Big(20648 f_n + 95104 f_{n+\frac{1}{8}} \\
&\quad - 35808 f_{n+\frac{1}{4}} + 108880 f_{n+\frac{3}{8}} - 70760 f_{n+\frac{1}{2}} + 54912 f_{n+\frac{5}{8}} \\
&\quad - 19744 f_{n+\frac{3}{4}} + 6248 f_{n+\frac{7}{8}} - 555 f_{n+1}\Big), \\
y'_{n+\frac{1}{8}} &= y_n' + \frac{1}{8}hy_n'' + \frac{1}{128}h^2 y''' + \frac{h^3}{20437401600}\Big(3619903 f_n + 6779886 f_{n+\frac{1}{8}} \\
&\quad - 9359135 f_{n+\frac{1}{4}} + 11774146 f_{n+\frac{3}{8}} - 10745445 f_{n+\frac{1}{2}} + 6771082 f_{n+\frac{5}{8}} \\
&\quad - 2792861 f_{n+\frac{3}{4}} + 679110 f_{n+\frac{7}{8}} - 73886 f_{n+1}\Big),
\end{aligned} \tag{11}
$$

$$y'_{n+\frac{1}{4}} = y'_n + \frac{1}{4}hy''_n + \frac{1}{32}h^2 y''' + \frac{h^3}{319334400}\Big(286967 f_n + 911204 f_{n+\frac{1}{8}} - 926646 f_{n+\frac{1}{4}} + 1173140 f_{n+\frac{3}{8}} - 1067950 f_{n+\frac{1}{2}} + 67162 f_{n+\frac{5}{8}} - 276634 f_{n+\frac{3}{4}} + 67196 f_{n+\frac{7}{8}} - 7305 f_{n+1}\Big),$$

$$y'_{n+\frac{3}{8}} = y'_n + \frac{3}{8}hy''_n + \frac{9}{128}h^2 y''' + \frac{h^3}{252313600}\Big(550152 f_n + 2135754 f_{n+\frac{1}{8}} - 1563651 f_{n+\frac{1}{4}} + 2298870 f_{n+\frac{3}{8}} - 2099655 f_{n+\frac{1}{2}} + 1323918 f_{n+\frac{5}{8}} - 546129 f_{n+\frac{3}{4}} + 132786 f_{n+\frac{7}{8}} - 14445 f_{n+1}\Big),$$

$$y'_{n+\frac{1}{2}} = y'_n + \frac{1}{2}hy''_n + \frac{1}{8}h^2 y''' + \frac{h^3}{19958400}\Big(80293 f_n + 342816 f_{n+\frac{1}{8}} - 188120 f_{n+\frac{1}{4}} + 358816 f_{n+\frac{3}{8}} - 310800 f_{n+\frac{1}{2}} + 196192 f_{n+\frac{5}{8}} - 80936 f_{n+\frac{3}{4}} + 19680 f_{n+\frac{7}{8}} - 2141 f_{n+1}\Big),$$

$$y'_{n+\frac{5}{8}} = y'_n + \frac{5}{8}hy''_n + \frac{25}{128}h^2 y''' + \frac{h^3}{817496064}\Big(5253125 f_n + 23702750 f_{n+\frac{1}{8}} - 10296375 f_{n+\frac{1}{4}} + 25537250 f_{n+\frac{3}{8}} - 19680625 f_{n+\frac{1}{2}} + 129202250 f_{n+\frac{5}{8}} - 5327125 f_{n+\frac{3}{4}} + 1295750 f_{n+\frac{7}{8}} - 141000 f_{n+1}\Big),$$

$$y'_{n+\frac{3}{4}} = y'_n + \frac{3}{4}hy''_n + \frac{9}{32}h^2 y''' + \frac{h^3}{3942400}\Big(37017 f_n + 173124 f_{n+\frac{1}{8}} - 61830 f_{n+\frac{1}{4}} + 192564 f_{n+\frac{3}{8}} - 129330 f_{n+\frac{1}{2}} + 95148 f_{n+\frac{5}{8}} - 37674 f_{n+\frac{3}{4}} + 9180 f_{n+\frac{7}{8}} - 999 f_{n+1}\Big),$$

$$y'_{n+\frac{7}{8}} = y'_n + \frac{7}{8}hy''_n + \frac{49}{128}h^2 y''' + \frac{h^3}{2919628800}\Big(37701874 f_n + 180838518 f_{n+\frac{1}{8}} - 54639557 f_{n+\frac{1}{4}} + 206894170 f_{n+\frac{3}{8}} - 122270925 f_{n+\frac{1}{2}} + 104842066 f_{n+\frac{5}{8}} - 35782103 f_{n+\frac{3}{4}} + 9423582 f_{n+\frac{7}{8}} - 1020425 f_{n+1}\Big),$$

$$y'_{n+1} = y'_n + hy''_n + \frac{1}{2}h^2 y''' + \frac{h^3}{1247400}\Big(21203 f_n + 103616 f_{n+\frac{1}{8}} - 27024 f_{n+\frac{1}{4}} + 121280 f_{n+\frac{3}{8}} - 63820 f_{n+\frac{1}{2}} + 63552 f_{n+\frac{5}{8}} - 16816 f_{n+\frac{3}{4}} + 6464 f_{n+\frac{7}{8}} - 555 f_{n+1}\Big), \quad (12)$$

$$y''_{n+\frac{1}{8}} = y''_n + \frac{1}{8}hy''' + \frac{h^2}{464486400}\Big(1624505 f_n + 4124232 f_{n+\frac{1}{8}} - 5225624 f_{n+\frac{1}{4}} + 6488192 f_{n+\frac{3}{8}} - 5888310 f_{n+\frac{1}{2}} + 3698920 f_{n+\frac{5}{8}} - 1522672 f_{n+\frac{3}{4}} + 369744 f_{n+\frac{7}{8}} - 40187 f_{n+1}\Big),$$

$$y''_{n+\frac{1}{4}} = y''_n + \frac{1}{4}hy''' + \frac{h^2}{7257600}\Big(58193 f_n + 235072 f_{n+\frac{1}{8}} - 183708 f_{n+\frac{1}{4}} + 247328 f_{n+\frac{3}{8}} - 227030 f_{n+\frac{1}{2}} + 143232 f_{n+\frac{5}{8}} - 59092 f_{n+\frac{3}{4}} + 14368 f_{n+\frac{7}{8}} - 1563 f_{n+1}\Big),$$

$$y''_{n+\frac{3}{8}} = y''_n + \frac{3}{8}hy''' + \frac{h^2}{5734400}\Big(71661 f_n + 328608 f_{n+\frac{1}{8}} - 150624 f_{n+\frac{1}{4}} + 315000 f_{n+\frac{3}{8}} - 281430 f_{n+\frac{1}{2}} + 177264 f_{n+\frac{5}{8}} - 73128 f_{n+\frac{3}{4}} + 17784 f_{n+\frac{7}{8}} - 1935 f_{n+1}\Big),$$

$$y''_{n+\frac{1}{2}} = y''_n + \frac{1}{2}hy''' + \frac{h^2}{453600}\Big(7703 f_n + 37248 f_{n+\frac{1}{8}} - 11600 f_{n+\frac{1}{4}} + 40064 f_{n+\frac{3}{8}} - 29610 f_{n+\frac{1}{2}} + 19072 f_{n+\frac{5}{8}} - 7888 f_{n+\frac{3}{4}} + 1920 f_{n+\frac{7}{8}} - 209 f_{n+1}\Big),$$

$$y''_{n+\frac{5}{8}} = y''_n + \frac{5}{8}hy''' + \frac{h^2}{18579456}\Big(398825 f_n + 1987000 f_{n+\frac{1}{8}} - 465000 f_{n+\frac{1}{4}} + 2294000 f_{n+\frac{3}{8}} - 128350 f_{n+\frac{1}{2}} + 1020600 f_{n+\frac{5}{8}} - 412000 f_{n+\frac{3}{4}} + 100000 f_{n+\frac{7}{8}} - 10875 f_{n+1}\Big),$$

$$y''_{n+\frac{3}{4}} = y''_n + \frac{3}{4}hy''' + \frac{h^2}{89600}\Big(2325 f_n + 11808 f_{n+\frac{1}{8}} - 2196 f_{n+\frac{1}{4}} + 14208 f_{n+\frac{3}{8}} - 6390 f_{n+\frac{1}{2}} + 7200 f_{n+\frac{5}{8}} - 2268 f_{n+\frac{3}{4}} + 576 f_{n+\frac{7}{8}} - 63 f_{n+1}\Big),$$

$$y''_{n+\frac{7}{8}} = y''_n + \frac{7}{8}hy''' + \frac{h^2}{66355200}\Big(2019731 f_n + 10388784 f_{n+\frac{1}{8}} - 1575056 f_{n+\frac{1}{4}} + 12811736 f_{n+\frac{3}{8}} - 4826010 f_{n+\frac{1}{2}} + 7068544 f_{n+\frac{5}{8}} - 1018024 f_{n+\frac{3}{4}} + 589176 f_{n+\frac{7}{8}} - 57281 f_{n+1}\Big),$$

$$y''_{n+1} = y''_n + hy''' + \frac{h^2}{28350}\Big(989 f_n + 5152 f_{n+\frac{1}{8}} - 696 f_{n+\frac{1}{4}} + 6560 f_{n+\frac{3}{8}} - 2270 f_{n+\frac{1}{2}} + 3936 f_{n+\frac{5}{8}} - 232 f_{n+\frac{3}{4}} + 736 f_{n+\frac{7}{8}}\Big), \tag{13}$$

$$y'''_{n+\frac{1}{8}} = y''' + \frac{h}{29030400}\Big(1070017 f_n + 4467094 f_{n+\frac{1}{8}} - 4604594 f_{n+\frac{1}{4}} + 5595358 f_{n+\frac{3}{8}} - 5033120 f_{n+\frac{1}{2}} + 31463388 f_{n+\frac{5}{8}} - 1291214 f_{n+\frac{3}{4}} + 312874 f_{n+\frac{7}{8}} - 33953 f_{n+1}\Big),$$

$$y'''_{n+\frac{1}{4}} = y''' + \frac{h}{907200}\Big(32377 f_n + 182584 f_{n+\frac{1}{8}} - 42494 f_{n+\frac{1}{4}} + 120088 f_{n+\frac{3}{8}} - 116120 f_{n+\frac{1}{2}} + 74728 f_{n+\frac{5}{8}} - 31154 f_{n+\frac{3}{4}} + 7624 f_{n+\frac{7}{8}} - 833 f_{n+1}\Big).$$

$$y'''_{n+\frac{3}{8}} = y''' + \frac{h}{358400}\Big(12881 f_n + 70902 f_{n+\frac{1}{8}} + 3438 f_{n+\frac{1}{4}} + 79934 f_{n+\frac{3}{8}} - 56160 f_{n+\frac{1}{2}} + 34434 f_{n+\frac{5}{8}} - 14062 f_{n+\frac{3}{4}} + 3402 f_{n+\frac{7}{8}} - 369 f_{n+1}\Big),$$

$$y'''_{n+\frac{1}{2}} = y''' + \frac{h}{113400}\Big(4063 f_n + 22576 f_{n+\frac{1}{8}} + 244 f_{n+\frac{1}{4}} + 32752 f_{n+\frac{3}{8}} - 9080 f_{n+\frac{1}{2}} + 9232 f_{n+\frac{5}{8}} - 3956 f_{n+\frac{3}{4}} + 976 f_{n+\frac{7}{8}} - 107 f_{n+1}\Big),$$

$$y'''_{n+\frac{5}{8}} = y''' + \frac{h}{1161216}\Big(41705 f_n + 230150 f_{n+\frac{1}{8}} + 7550 f_{n+\frac{1}{4}} + 318350 f_{n+\frac{3}{8}} - 4000 f_{n+\frac{1}{2}} + 170930 f_{n+\frac{5}{8}} - 49150 f_{n+\frac{3}{4}} + 11450 f_{n+\frac{7}{8}} - 1225 f_{n+1}\Big),$$

$$y'''_{n+\frac{3}{4}} = y''' + \frac{h}{11200}\Big(401 f_n + 2232 f_{n+\frac{1}{8}} + 18 f_{n+\frac{1}{4}} + 3224 f_{n+\frac{3}{8}} - 360 f_{n+\frac{1}{2}} + 2664 f_{n+\frac{5}{8}} + 158 f_{n+\frac{3}{4}} + 72 f_{n+\frac{7}{8}} - 9 f_{n+1}\Big),$$

$$y'''_{n+\frac{7}{8}} = y''' + \frac{h}{4147200}\Big(149527 f_n + 816634 f_{n+\frac{1}{8}} + 48706 f_{n+\frac{1}{4}} + 1085938 f_{n+\frac{3}{8}} + 54880 f_{n+\frac{1}{2}} + 736078 f_{n+\frac{5}{8}} + 522046 f_{n+\frac{3}{4}} + 223174 f_{n+\frac{7}{8}} - 8183 f_{n+1}\Big),$$

$$y'''_{n+1} = y''' + \frac{h}{28350}\Big(989 f_n + 5888 f_{n+\frac{1}{8}} - 928 f_{n+\frac{1}{4}} + 10496 f_{n+\frac{3}{8}} - 4540 f_{n+\frac{1}{2}} + 10496 f_{n+\frac{5}{8}} - 928 f_{n+\frac{3}{4}} + 5888 f_{n+\frac{7}{8}} + 989 f_{n+1}\Big). \tag{14}$$

3. Analysis of the Properties of the Method

This section presented the analysis of the basic properties of the proposed single-step hybrid multistep method.

3.1. Order of the Method

Since $y(x)$ is continuously differentiable, following [2], the linear difference operator associated with formulas in (11) is defined by:

$$\begin{aligned}&\Upsilon_{\frac{r}{8}}\left\{y(x):h\right\}\\&=y\left(x_n+\frac{r}{8}h\right)-\left\{\sum_{b=0}^{3}\alpha_b\left(\frac{r}{8}\right)y^{(b)}(x)h^b-h^4\sum_{b=0}^{3}\beta_b\left(\frac{r}{8}\right)y^{(4)}\left(x_n+\frac{r}{8}h\right),r=1,2,3\right\}.\end{aligned}\tag{16}$$

Taking $y(x)$ as the valid solution of (1), the Taylor series expansion about the point x after using (16) gives a formula for the local truncation error written as:

$$\begin{aligned}\Upsilon_{\frac{r}{8}}\left\{y(x):h\right\}&=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+\cdots+c_{p+3}h^{p+3}y^{(p+3)}(x)\\&\quad+c_{p+4}h^{p+4}y^{(p+4)}(x).\end{aligned}\tag{17}$$

The term c_{p+4} is called the error constant and implies that the local truncation error is given by:

$$\Upsilon_{\frac{r}{8}}\left\{y(x):h\right\}=c_{p+4}h^{p+4}y^{(p+4)}(x_n)+Oh^{p+5}\tag{18}$$

Since $c_0=c_1=\cdots=c_{p+3}=0,c_{p+4}\neq 0$, refer to [13]; then formulas in (11) have uniform order $p=9$ with error constant given by

$$\begin{aligned}c_{p+4}=\Bigg(&\frac{100009549}{143780317296063204556800},\frac{100009549}{143780317296063204556800},\\&\frac{2851897}{280820932218873446400},\frac{8093367}{197229516181156659200},\\&\frac{14479}{137119595809996800},\frac{1243375625}{575121269184252818227 2},\\&\frac{148383}{385213898791321600},\frac{1833526051}{293429218971557560320 0}\frac{8123}{8569974738124800}\Bigg)\end{aligned}$$

This procedure can be repeated for the formulas in (12)-(14) to obtain their respective error constants.

3.2. Zero Stability of the Block Method

According to [3] and [14], a block method is zero stable if the zeros of the characteristic polynomial satisfy $|\eta_i|\leq 1$ and the root $|\eta_i|=1$ has multiplicity not exceeding the order of the differential equation. Moreover, this kind of stability tells the behaviour of the numerical method as $h^\rho\to 0$. Setting h to zero in the formulas (11)-(14), its reduced to the form that satisfies $\Pi(\eta_i)=\left|\eta_i(\phi_{m,r})-\nabla_{m,r}\right|$, whose the characteristics equation is $\eta_i^{28}(\eta_i-1)^4=0$. It is evident that the SHMM is zero stable since the multiplicity of roots $|\eta_i|=1$ does not exceed the order of the differential equation understudy.

3.3. Convergence

According to [15], the necessary and sufficient condition for a numerical method to be convergent is to be consistent and Zero stable. The satisfactory condition for the proposed method SHMM to be consistent is that it must have order $p \geq 1$ (see [16]). Section 3.2 established zero stability of the proposed SHMM, and it is correct to conclude that SHMM is convergent.

3.4. Region of Absolute Stability of the Method SHMM

This section studies the region of absolute stability of the proposed Single-step Hybrid Multistep Method (SHMM). Substituting the test problems:

$$y' = -vy, \quad y'' = -v^2 y, \quad y''' = -v^3 y, \quad y^{(4)} = -v^4 y$$

into formulas in (11) and then combined as a block:

$$\widehat{U}_0 Y_{n+r} = \widehat{U}_1 Y_n + h\widehat{U}_2 Y_n' + h^2 \widehat{U}_3 Y_n'' + h^3 \widehat{U}_4 Y_n''' + h^4 \left(Б_0 F_n + Б_1 F_{n+1}\right), \tag{19}$$

where vectors,

$$Y_{n+r} = \left(y_{n+\frac{1}{8}}, y_{n+\frac{2}{8}}, y_{n+\frac{3}{8}}, y_{n+\frac{4}{8}}, y_{n+\frac{5}{8}}, y_{n+\frac{6}{8}}, y_{n+\frac{7}{8}}, y_{n+1}\right)^{\mathrm{T}},$$

$$Y_n = \left(y_{n-\frac{1}{8}}, y_{n-\frac{2}{8}}, y_{n-\frac{3}{8}}, y_{n-\frac{4}{8}}, y_{n-\frac{5}{8}}, y_{n-\frac{6}{8}}, y_{n-\frac{7}{8}}, y_n\right)^{\mathrm{T}},$$

$$Y_n' = \left(y'_{n-\frac{1}{8}}, y'_{n-\frac{2}{8}}, y'_{n-\frac{3}{8}}, y'_{n-\frac{4}{8}}, y'_{n-\frac{5}{8}}, y'_{n-\frac{6}{8}}, y'_{n-\frac{7}{8}}, y'_n\right)^{\mathrm{T}},$$

$$Y_n'' = \left(y''_{n-\frac{1}{8}}, y''_{n-\frac{2}{8}}, y''_{n-\frac{3}{8}}, y''_{n-\frac{4}{8}}, y''_{n-\frac{5}{8}}, y''_{n-\frac{6}{8}}, y''_{n-\frac{7}{8}}, y''_n\right)^{\mathrm{T}},$$

$$Y_n''' = \left(y'''_{n-\frac{1}{8}}, y'''_{n-\frac{2}{8}}, y'''_{n-\frac{3}{8}}, y'''_{n-\frac{4}{8}}, y'''_{n-\frac{5}{8}}, y'''_{n-\frac{6}{8}}, y'''_{n-\frac{7}{8}}, y'''_n\right)^{\mathrm{T}},$$

$$F_n = \left(f_{n-\frac{1}{8}}, f_{n-\frac{2}{8}}, f_{n-\frac{3}{8}}, f_{n-\frac{4}{8}}, f_{n-\frac{5}{8}}, f_{n-\frac{6}{8}}, f_{n-\frac{7}{8}}, f_n\right)^{\mathrm{T}},$$

$$F_{n+1} = \left(f_{n+\frac{1}{8}}, f_{n+\frac{2}{8}}, f_{n+\frac{3}{8}}, f_{n+\frac{4}{8}}, f_{n+\frac{5}{8}}, f_{n+\frac{6}{8}}, f_{n+\frac{7}{8}}, f_{n+1}\right)^{\mathrm{T}},$$

$\widehat{U}_0$, $\widehat{U}_1$, $\widehat{U}_2$, $\widehat{U}_3$, $\widehat{U}_4$, $Б_0$, and $Б_1$ are matrices of dimension eight (8) whose entries are the coefficients of formulas in (11). Regarding (19) the amplification matrix obtained is:

$$M(\zeta) = \Gamma^{-1} \boldsymbol{Z} \tag{20}$$

where $\Gamma = \left(\widehat{U}_0 - Б_1\right)$ and $\boldsymbol{Z} = \left(\widehat{U}_1 + \widehat{U}_2 + \widehat{U}_3 + \widehat{U}_4 + Б_0\right)$. Further analysis of the amplification matrix $M(\zeta)$ give the eigenvalues $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\} = \{0,0,0,0,0,0,0,\gamma_8\}$. The dominant eigenvalue γ_8

is a function of ζ where $\zeta = v^4h^4$. **Figure 1** shows the absolute stability region where the proposed method exhibited the behaviours of the true solution.

4. Numerical Experiments

Four sample problems are considered as numerical examples to test the usability of the proposed SHMM. The accuracy of the method was evaluated by calculating the absolute error generated when applied to the sample problems.

4.1. Test Problem 1

The general fourth-order IVP of ordinary differential equation

$$y^{(4)} = y''' + y'' + y' + 2y,\ \ y(0)=1,\ \ y'(0)=0,\ \ y''(0)=0,\ \ y'''(0)=30$$

whose exact solution is reported as $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$ is considered as the first test problem. The solutions to problem 1 were obtained within [0, 1] over 20 iterations and are compared with the exact solution, as presented in **Figure 2(a)**. It is clear from **Table 1** and **Figure 2(b)** that SHMM shows good performance over the methods in [7] and [8] up to at least seven decimal digits.

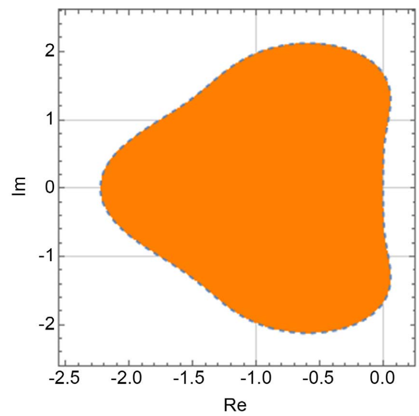

Figure 1. Stability plot of SHMM.

Table 1. Solution of problem 2 obtained using the proposed method.

x	y-computed	y-exact	Error SHMM	Error in [8]	Error in [7]
0.2	8.229478917040623	8.229478917040623	1.5152 E-20	3.5129E-13	2.319E-13
0.4	7.0610999883939325	7.061099988393933	1.4159E-19	4.1833E-12	2.2603E-12
0.6	6.778516610116943	6.778516610116943	5.6530E-19	1.4302E-11	1.9651E-11
0.8	7.786624979322129	7.786624979322128	1.6039E-18	3.5924E-11	9.9145E-11
1.	10.665177458051863	10.665177458051861	3.7919E-18	7.2762E-11	3.3114E-10
1.2	16.251045310569197	16.251045310569197	8.0188E-18	1.3360E-10	9.0000E-10
1.4	25.763132220706712	25.76313222070671	1.5750E-17	2.2345E-10	2.1176E-09
1.6	40.99078198991456	40.99078198991457	2.9382E-17	3.5796E-10	4.5506E-09
1.8	64.57672836648089	64.57672836648088	5.2804E-17	5.4334E-10	9.1180E-09
2.	100.44085913139898	100.44085913139897	9.2291E-17	8.0796E-10	1.7409E-08

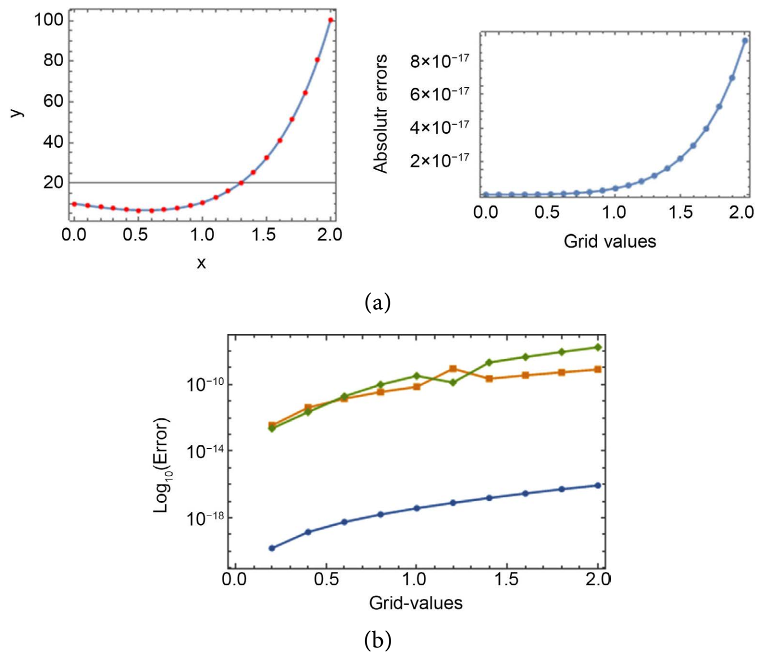

Figure 2. (a) Curves of solutions and behaviour of absolute errors on test problem 21 using the SHMM; (b) Efficiency curves of the maximum absolute errors against grid-values for problem 1.

4.2. Test Problem 2

The second example considered as a test problem is another fourth-order IVP of ordinary differential equation

$$y^{(4)} = y'', y(0) = 0, y'(0) = \frac{-1.1}{72-50\pi}, y''(0) = \frac{1}{144-100\pi}, y'''(0) = \frac{1.2}{144-100\pi},$$

whose exact solution is given as $y(x) = 1 - x - \cos x - y''(0) = \frac{1.2\sin x}{144-100\pi}$. Problem 2 was iterated within $[0,1]$ for 320 steps. The results are as presented in Table 2 and Figure 3(a) and Figure 3(b). Table 2 and Figure 3(b) make it evident that SHMM outperforms the approach in [17] up to at least seven decimal digits.

4.3. Test Problem 3: Application: Ship Dynamic Problem (See [8])

We applied the proposed method to solve a physical problem that occurs in ship dynamics. In particular, this problem has been studied and solved numerically by [18], and [19], which describes how the sinusoidal wave of frequency ∇ passes along a ship or offshore structure to lead to a fourth-order differential equation relates to the action of the fluids with time x as below

$$y^{(4)} = -3y'' - y(2 + \phi\cos(\nabla x)), y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0,$$

whose exact solution is $y(x) = 2\cos x - \cos(x\sqrt{2})$ for when $\phi = 0$. This problem was solved within [0, 15] over 150 iterations. The results are reported in **Table 3** and **Figure 4**. The results of the problem for $h = 0.25$ and 0.1 are compared with those of cited [7] [8] and [18], the suggested method compares well with the cited works (see **Table 4**).

4.4. Test Problem 4

The following nonlinear fourth-order IVP of ordinary differential equation

$$y^{(4)} = (y')^2 - yy'' - 4x^2 + e^x (1 + x^2 - 4x),$$
$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, [0,1]$$

with the exact solution $y(x) = x^2 + e^x$ is also considered as a test problem. The test problem was approximated using SHMM within [0, 1] over 5 and 10 iterations, respectively. See **Table 5** for the results as compared with some cited works in the literatures.

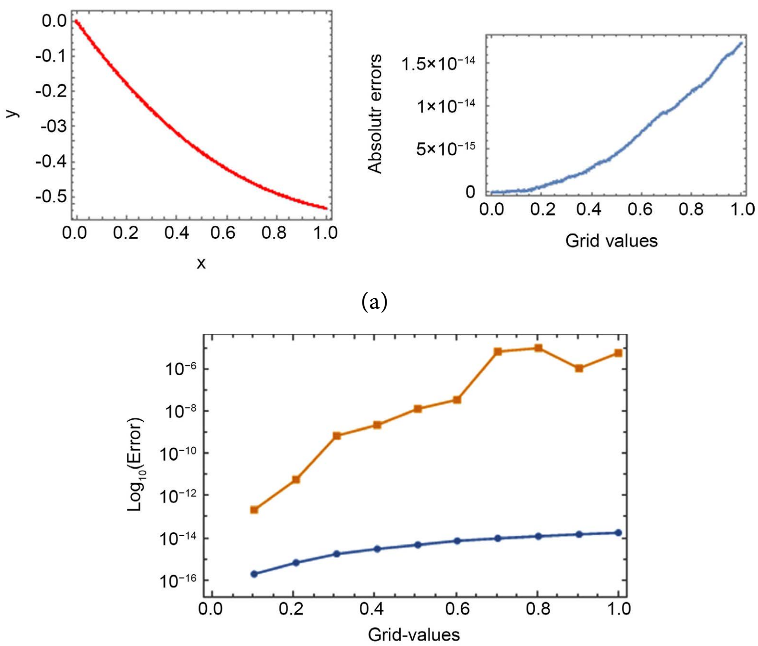

Figure 3. (a) Curves of solutions and behaviour of absolute errors on test problem 2 using the SHMM; (b) Efficiency curves of the maximum absolute errors against grid-values for Example 2.

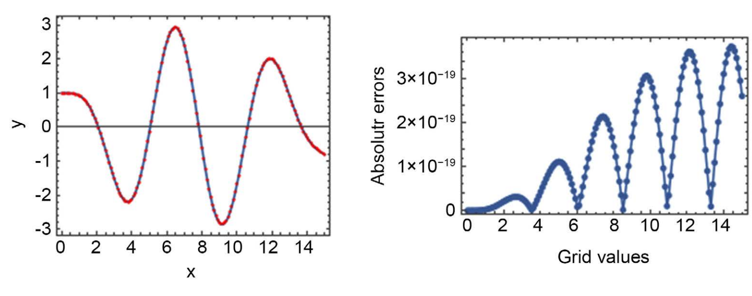

Figure 4. Curves of solutions and behaviours of absolute errors on test problem 3 using the SHMM.

Table 2. Solution of test problem 2 obtained by the proposed method.

x	y-computed	y-exact	Error in SHMM	Error in [16]
0.103125	−0.09708635645894467	−0.09708635645894448	1.9429E-16	2.116E-13
0.206250	−0.18361153147140596	−0.1836115314714053	6.6613E-16	5.6987E-12
0.306250	−0.2575947017186572	−0.2575947017186555	1.7208E-15	6.8031E-10
0.406250	−0.3220723501790842	−0.3220723501790812	2.9976E-15	2.2072E-9
0.506250	−0.3773994044471421	−0.37739940444713743	4.6629E-15	1.2741E-8
0.603125	−0.4226919296391457	−0.4226919296391385	7.2165E-15	3.4561E-8
0.703125	−0.46139027951012934	−0.4613902795101199	9.4369E-15	6.5534E-6
0.803125	−0.492512293357425	−0.4925122933574131	1.1935E-14	9.5865E-6
0.903125	−0.5167461772506352	−0.5167461772506207	1.4544E-14	1.0493E-6
1.	−0.5343680701999484	−0.5343680701999309	1.7431E-14	5.6962E-6

Table 3. Solution of the applied example obtained using SHMM.

x	**y-computed**	**y-exact**	**Error**
1.	0.92466091697090490	0.9246609169709051	2.3651E-21
2.	0.11906945503156266	0.11906945503156263	2.1163E-20
3.	−1.5273231359085384	−1.5273231359085389	2.6887E-20
4.	−2.1174708448420190	−2.1174708448420194	4.3095E-20
5.	−0.13802353538198978	−0.13802353538198964	1.1155E-19
6.	2.510535059206008	2.5105350592060085	8.3824E-21
7.	2.3972266325194904	2.3972266325194904	1.8763E-19
8.	−0.603795009129371	−0.6037950091293718	1.4646E-19
9.	−2.809239445368881	−2.8092394453688807	1.6758E-19
10.	−1.6731743960203111	−1.6731743960203105	2.9449E-19
11.	0.9973799806376236	0.9973799806376238	3.4453E-20
12.	1.9910488550789984	1.9910488550789966	3.5679E-19
13.	0.920973191409115	0.9209731914091138	1.3780E-19
14.	−0.30866899231111894	−0.30866899231111894	3.1111E-19
15.	−0.8070186485444422	−0.8070186485444416	2.6094E-19

Table 4. Comparison of the absolute error on test problem 3.

h	Method	Error at $x_N = 15$
0.25	SHMM	2.48E-15
	[8]	8.15E-08
	[7]	5.40E-07
	[19]	1.90E-04
0.1	SHMM	2.61E-19
	[8]	3.60E-11
	[7]	2.80E-10
	[18]	3.7E-05

Table 5. Comparison of the absolute error on test problem 4.

h	Method	Error at $x_N = 15$
0.2	SHMM	9.45E-23
	[20]	1.10E-16
	BHCM4 [20]	5.20E-12
	Adams [20]	5.01E-07
	[9]	5.84E-04
0.1	SHMM	9.71E-20
	[20]	1.77E-11
	BHCM4 [20]	1.95E-14
	Adams [20]	2.44E-06
	[9]	9.26E-05

5. Conclusion

This work proposes a single-step, linear multistep formula (LMF) with block extension for the direct solution of fourth-order ordinary differential equations. The ship dynamics problem and three other standard fourth-order ordinary differential equations are considered test problems to establish the methods' usefulness. The analysis and numerical experiments revealed that the proposed method is efficient with a better degree of accuracy for handling the direct solution of fourth-order ordinary differential equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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