American Journal of Computational Mathematics, 2022, 12, 297-305
https://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

Designing Physics Problems with *Mathematica* Example II

Haiduke Sarafian

The Pennsylvania State University, University College, York, USA
Email: has2@psu.edu

How to cite this paper: Sarafian, H. (2022) Designing Physics Problems with Mathematica Example II. *American Journal of Computational Mathematics*, **12**, 297-305.
https://doi.org/10.4236/ajcm.2022.123019

Received: May 12, 2022
Accepted: July 18, 2022
Published: July 21, 2022

Abstract

Customarily in the physics of sound, static-acoustic-related topics are addressed. For instance, the change in the sound level vs discrete change in the distance. In dynamic cases, e.g. the Doppler shit although the relative motion of the components, *i.e.* the source and the sensor are essential, the movements are limited to uniform motions. In this investigating report, scenarios departed from these limitations are considered. For the former case, time dependent sound level and for the latter case, nonuniform motions are analyzed. Aside from light long-hand mathematical formulations, the majority of the analysis is carried out utilizing a Computer Algebra System (CAS) specifically *Mathematica*. The analysis and format of the development are crafted flexibly conducive opportunities for furthering quests for the "what if" scenarios.

Keywords

Physics of Sound, Time-Dependent Sound Level, Computer Algebra System, *Mathematica*

1. Introduction

Acoustic sound level β measured in dB is given by,

$$\beta = 10\log\left(\frac{I}{I_0}\right), \tag{1}$$

where I_0 and I are the reference and sample intensities of the sound, respectively, and both are measured in W/m^2. The value of the former being the minimum audible intensity for the human ear is 1×10^{-12} W/m^2. In practice, the sample intensity, I, is related to the power of the source, P in watts, via

$$P = IA, \tag{2}$$

where A is the surface area through which the sound waves go through. For simplicity assuming the sound waves originate from a point-source spreading evenly throughout the space, *i.e.* space is homogeneous and isotropic area A is considered a sphere, with a surface area $A = 4\pi R^2$, with R being its radius.

With these assumptions (1) reads,

$$\beta = 10\log\left(\frac{p}{4\pi R^2 I_0}\right), \tag{3}$$

Customary, in practice for a chosen power, P and distance R from the source one measures the sound level β [1] [2] [3]. As pointed out in the abstract, this yields the discrete values of the sound level with practical applications. From an academic point of view, the continuous variation of the β as a function of distance could be a quantity of interest. Since the distance implicitly is a kinematic related entity, this can be related to the character of the movement of the source and ultimately to the run-time. In short, the sound level can be expressed as a function of continuous time-varying quantity.

With this insight to reach our honed objective, we consider two sound sources and their relative contributing sound levels. This eliminates the explicit need for utilizing I_0 and provides a forum to compare the impact of the different source powers.

With these objectives, we craft our report which is composed of three sections. In addition to the Introduction, in Section 2 and its subsections, we develop the needed formulation embodying various scenarios concerning the character of the motions of the sources. Applying a CAS, specifically *Mathematica* [4], we obtain symbolic and then numeric values for the relevant quantities. For better comprehension, the numerics are backed-up with appropriate graphs. The last section, the Conclusions, is the summary of the learned topics with suggestions for augmenting the scope of the investigation. *Mathematica* codes are embodied in the report, the interested reader may reproduce the results and extend based on personal interest.

2. Procedure

Here is the posed problem. Two loudspeakers, with output powers P_1 and P_2 are a distance d away from a sound sensor. Simultaneously, they put out identical sound notes and begin moving in the same direction with a zero initial speed toward the sensor. In scenario one, source 1 moves at a constant velocity v_1, and source 2 at a constant acceleration a_2. In scenario two, sources begin accelerating at constant rates a_1 and a_2, and one is modulated with oscillations. Question: When does the sound-level difference at the sensor reach the maximum?

The second somewhat-related major question concerns the classic Doppler shift, it stems from a classic problem [2] [3]. It poses: how the depth of a water well is measured using a watch only? Its solution hinges on knowing the sound speed in air and the measured run-time between dropping a stone in the well and the splash heard when it hits the water.

Literature search reveals this problem and its solution has never been modified. We altered the posed question by asking: Is it possible to measure the depth of a water well using only a tuning fork? *I.e.* no timer or no rope! Our solution is insightful.

The forthcoming sections address these issues. Sections 2.1 and 2.2 are concerned with the sound level and 2.3 is the water well problem.

2.1. Case 1. One of the Sources Is Moving at a Constant Speed, the Other One Is at a Constant Acceleration

Figure 1 shows the situation at hand.

Sources put out power P_1 and P_2. Applying (3) the difference of the sound level with appropriately changed notations, yields,

$$\Delta\beta = \beta_1 - \beta_2 = 10\left[\log\left(\frac{P_1}{P_2}\right) + 20\log\left(\frac{R_{12}}{R_{11}}\right)\right], \tag{4}$$

According to Figure 1, the sources at $t = 0$ are at the origin, d-distance from the sensor. At a later time, t they are R distance away from the sensor and traveled a certain distance depending on their respective kinematics. Utilizing Figure 1 the scenario on hand yields,

$$\begin{cases} R_{11} = d - v_1 t \\ R_{12} = d - \frac{1}{2}a_2 t^2 \end{cases}, \tag{5}$$

Substituting (5) in (4) and assuming Doppler shit does not affect the intensity yields the explicit time-dependent sound-level difference,

$$\Delta\beta 1(t) = 10\left[\log\left(\frac{P_1}{P_2}\right) + 20\log\left(\frac{d - \frac{1}{2}a_2 t^2}{d - v_1 t}\right)\right], \tag{6}$$

One of our objectives is to find the time, t to maximize the sound-level difference. We set the slope of (6) zero and search for its root(s). These are,

Δ*β*1 [t_] = 10(Log [10, P$_1$/P$_2$] + 20 Log [10,(d − 1/2 a$_2$ t^2)/(d − v$_1$t)]);
slopeΔ*β*1 = D [Δ*β*1 [t], {t, 1}]//Simplify;
solt1 = Solve [slopeΔ*β*1 = = 0, t]

$$\left\{\left\{t \to \frac{da_2 - \sqrt{d^2 a_2^2 - 2da_2 v_1^2}}{a_2 v_1}\right\}, \left\{t \to \frac{da_2 + \sqrt{d^2 a_2^2 - 2da_2 v_1^2}}{a_2 v_1}\right\}\right\}$$

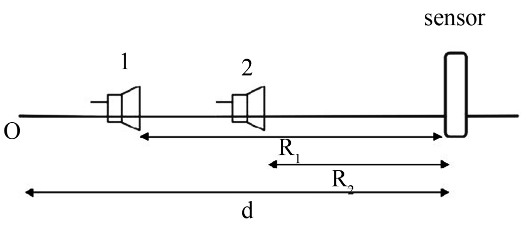


Figure 1. At time t loudspeakers are R distances away from the sensor. Both are approaching the sensor with respective kinematics.

To obtain a realistic meaningful output after trial and error we choose a set of practical parameters storing them in **values 1**. Units are MKS and symbols correspond one-to-one to the passage in the text.

values 1 = {v_1 → 2, a_2 → 5, d → 10, P_1 → 200, P_2 → 30};

solt1/.values1//N

$$\{\{t \to 0.4174\}, \{t \to 9.582\}\}$$

At these time instances, the sources are at the same distances from the sensor yielding the maximum intensity sound-level differences. The distance corresponding to the second time instance is ignored; the distance exceeds *d*. The acceptable $[\Delta\beta(t)]_{\max}$ is,

Δ*β*1 [t]/.solt1 [[1]]/.values1//N

11.9426 sec

It is insightful to display $\Delta\beta(t)$ side-by-side with the "distance to the sensor" vs *t* see Figure 2. A useful table is included as well.

plot11 = Plot [Δ*β*1 [t]/.values1, {t, 0, 1}, AxesLabel → {"t(s)", "Δ*β*1(dB)"}, PlotStyle → Black, GridLines → Automatic];

plot12 = Plot [Evaluate [{d − v_1 t, d − 1/2 a_2 t^2}/.values1], {t, 0, 1}, PlotStyle → {Blue, Red}, AxesLabel → {"t(s", "distance to sensor(m)"}, GridLines → Automatic];

table11 = NumberForm [TableForm [Table [{v_1t − 1/2 a_2 t^2, t}/.values1, {t, 0.3, 0.6, 0.05}], TableHeadings → {None, {"[Δd = v_1t − 1/2$a_2$$t^2$](m)", "t(s)"}}], {3,3}];

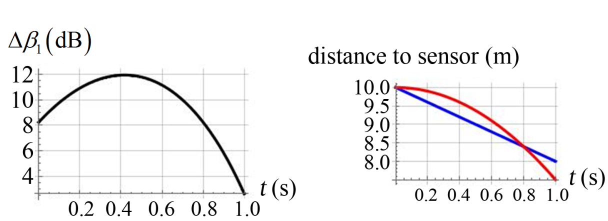
The intersecting point of these two curves shown in the middle panel of Figure 2 hints that at $t = 0.8$ s both sources are at the same distance from the sensor. Therefore, the sound level difference at this point is the same as when they were at the origin. Numerically, $\Delta\beta(t)$ at $t = 0$, and 0.8 s are,

{N [Δ*β*1 [t]/.values1/.t → 0], Δ*β*1 [t]/.values1/.t → 0.8}

{8.23909, 8.23909}

For identical sources, *i.e.* for $P_1 = P_2$, this leads to $\Delta\beta = 0$.

One of the objectives of this report is to demonstrate how to design a physics problem. Throughout crafting this report we realized to maximize the sound-level



$\left[\Delta d = v_1 t - \frac{1}{2} a_2 t^2\right](m)$	t (s)
0.375	0.300
0.394	0.350
0.400	0.400
0.394	0.450
0.375	0.500
0.344	0.550
0.300	0.600

Figure 2. The sound-level difference $\Delta\beta(t)$ vs t is shown on the left panel. The middle panel is the display of the kinematics of the sources; the slanted blue line is the uniform motion of the 1st source. The red curve is the accelerated kinematics of the 2nd source. The right-most table is the numeric values of the ordinate differences between the blue and red curves associated with the corresponding *t*.

difference the two sound sources ought to move with different kinematics. In this example, we selected one to move with a constant speed and the other with a constant acceleration. Their character differences are shown on the middle panel of Figure 2.

Intrigued by the learned lesson we extend the design of the physics problem by considering a modified version of the aforementioned case. The details are discussed in Subsection 2.2.

2.2. Case 2. Both Sources Are Moving at the Same Constant Acceleration, One of Them Is Modulated with an Oscillation

Here both sources are moving at the same constant acceleration, one of them has modified modulated oscillations. Their kinematics is given by,

$$\begin{cases} R_{21} = \dfrac{1}{2} a_{22} t^2 + A\sin\left[2\pi f t\right] \\ R_{22} = \dfrac{1}{2} a_{22} t^2 \end{cases}, \tag{7}$$

Inserting (7) in (6) gives,

$$\Delta\beta 2\left[t\_\right] = 10\left(\log\left[10, \frac{P_1}{P_2}\right] + 20\log\left[10, \frac{d - \frac{1}{2} a_{22} t^2}{d - \frac{1}{2} a_{22} t^2 - A\sin\left[2\pi f t\right]}\right]\right), \tag{8}$$

Similar to the previous objective we search for the instance that maximizes the sound-level difference. In fact, because of oscillations, there are multiple such instances. We set the slope of (8) zero and search for its root(s). Theoretically, this sounds, but in practice, it is challenging that even *Mathematica* is unable to resolve symbolically. Alternatively, for a reasonable set of parameters that are stored in **values 2**, we solve the problem numerically.

values 2 = {A → 1., f → 1., a_{22} → 5., P_1 → 200., P_2 → 30., d → 10.};

The amplitude of the oscillation and its frequency are set at, $f = 1$ Hz and $A = 1$, respectively. These quantities are chosen such that they are compatible with the acceleration $a_{22} = 5$ m/s^2.

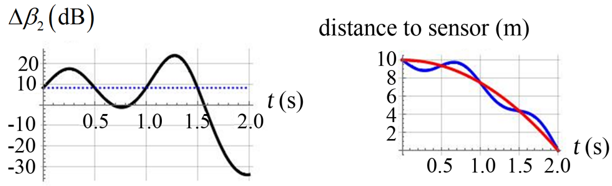
Δ*β*2 [t_] = 10(Log [10, P_1/P_2] + 20 Log [10, (d − 1/2 a_{22} t^2)/(d − 1/2 a_{22} t^2 − A Sin [2π f t])])/.values2;

plot21 = Show [{Plot [Δ*β*2 [t], {t, 0, 2}, AxesLabel → {"t(s)", "Δ*β*2(dB)"}, PlotStyle → Black, GridLines → Automatic],

Graphics [{Blue, Dashing [0.01], Line [{{0,8.23}, {2.,8.23}}]}]}];

plot22 = Show [{Plot [Evaluate [{d − 1/2 a_{22} t^2 − A Sin [2π f t], d − 1/2 a_{22} t^2}/.values2], {t, 0, 2}, PlotStyle → {Blue, Red}]}, AxesLabel → {"t(s)", "distance to sensor(m)"}, GridLines → Automatic];

Figure 3 is insightful and the panels are complimentary. The middle plate is included because the source oscillates and there are multiple instances where its distance from the sensor is the same as of the second source; noted by their intersections. Consequently, this is conducive to the instances where the sound-level



t (s)	$\Delta\beta_2$ (dB)
0.253	17.550
0.761	-1.344
1.277	24.050
1.981	-34.190

Figure 3. The left graph is the sound-level difference, $\Delta\beta(t)$ vs time. The middle graph is the display of "distance to sensor"; the red and the blue curves are the associated kinematics described in the text. The table is the maximum $\Delta\beta$ occurring at the instances shown on the left panel.

differences are maxima. These are shown by the four intersects of the horizontal dashed line in black shown on the left panel. The ordinate differences between the red and the blue curves are extreme. These are tabulated in the right-most table. Because the frequency of the oscillations is $f = 1$ Hz this corresponds to the even time-intervals of 0.5 s shown on the left panel.

2.3. Depth of a Water Well Using Only a Tuning Fork

In the introduction paragraph of Section 2, Procedure, we referred to the classic water well problem. Briefly, this is about measuring the depth of a water well without lowering a rope into the well. Assuming the only available tool is a stop-watch the run-time for a freely dropped stone from the instance a stone is dropped in the well to the time the splash is heard is conducive to the measured depth.

Aiming at the same objective, here we offer an alternate approach that solution doesn't require a watch; all that is needed is a tuning fork! Our solution employs 1) the principle of the Doppler shift and 2) uses the modified version of the latter. Noting, the standard classic Doppler shift utilizes the change of frequency due to relative motion at a constant speed, a falling tuning fork is falling at a constant acceleration!

The scenario on hand is depicted in **Figure 4**.

Here is the outline of our solution. Take a tunning fork of frequency f Hz. Drop it in a well of unknown depth h. While it is accelerating and keeps vibrating it emits sound waves that reach back at the edge of the well at frequency f_1. The waves also bounce off from the bottom of the well at frequency f_2. Both frequencies f_1 and f_2 are time-dependent and are continuously changing; the Doppler shift. Noting, that the farther the fork drops the faster it moves and at the splash, it moves the fastest corresponding to the highest frequency, (Subscript [f, 2])$_{\mathrm{Max}}$. The reflected frequency at the top of the well produces a sound-beat, $B = (f_2)_{\max} - f_1$. Knowing the B we formulate a strategy conducive to measuring the depth, h.

2.3.1. Symbolic Analysis

The frequency of the sound at the top and the bottom of the well while the fork falls are, respectfully,

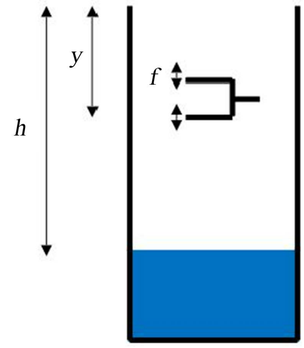


Figure 4. A falling tuning fork of frequency f at its instantaneous depth from the top of the well, y.

$$f_1 = f\left(\frac{1}{1+\frac{v_0}{v}}\right), \tag{9}$$

$$f_2 = f\left(\frac{1}{1-\frac{v_0}{v}}\right), \tag{10}$$

where the instantaneous speed of the fork is, $v_0 = \sqrt{2gy}$, the y is the fallen depth, and v is the sound-speed. Utilizing (9) and (10) the number of the beats is,

$$B = \left(f_2\right)_{\max} - f_1, \tag{11}$$

Rearranging (11) for the depth, h yields,

$$h = \frac{1}{2g}v^2\left[1-2\frac{f}{B}\left[\sqrt{\left(\frac{f}{B}\right)^2+1}-\frac{f}{B}\right]\right], \tag{12}$$

2.3.2. Graphic and Numeric Results

To demonstrate the usefulness of (12) we use the physiological data applied to a normal human ear [2] [3]. Accordingly, the sensitivity frequency range is: 20 Hz ≤ f ≤ 20 kHz and the capability of counting the sound-beat is: 6 Hz ≤ B ≤ 12 Hz. In addition we use: the gravity constant g = 9.8 m/s^2 and sound-speed v = 340.0 m/s.

The h in (12) is a two-variable function: $h(f, B)$. For a tunning fork with frequency and beats within the range of 50 Hz ≤ f ≤ 70 Hz and 6 ≤ B ≤ 12 Hz its contour plot is shown on the left panel of **Figure 5** this corresponds to a well 19 m deep. It shows when a 60 Hz vibrating fork is dropped 7 beats are heard at a 19 m depth. Or, for the same fork, an 8-beat yields a 26 m depth. The information in the contour plot usefully is converted to the shown graph on the right panel. For instance, the eight-beat count of a vibrating 50 Hz fork (the blue curve) yields a 37 m depth. Or a 10-beat count of a 70 Hz fork (the mustard curve) yields to a 30 m depth.

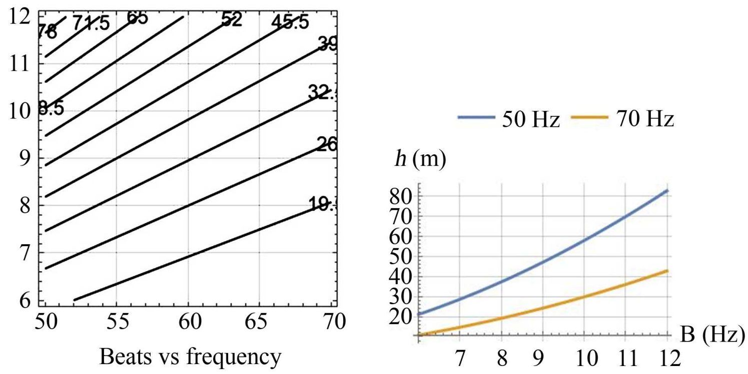


Figure 5. The left graph is the contour plot of the Beats vs the frequency, both in Hz. The right graph is the display of the Beats(Hz) vs the depth(m) of the water well.

3. Conclusions

The sound-level changes as a discrete function of distance from the source. This report extends its variation by replacing the discrete distances with distances that are due to the continuous movement of the source. Two such sources with two different kinematics in two different scenarios are considered. The interplay of their respective kinematics utilizing a CAS specifically *Mathematica* reveals features not reported in the literature. Based on the progress made in this report, similar scenarios considering various kinematics of interest may be analyzed.

This report also revisits the classic "water well" problem. The classic objective of the problem is intact but we offer a different solution providing only a tuning fork instead of a watch.

The interested reader may find a motivational source related to the current study [5], and resourceful for coding and plotting the graphs [6] [7]. *Mathematica* codes are in boldface.

Acknowledgements

The author acknowledges the John T. and Page S. Smith Professorship funds for completing and publishing this work.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Olson, H.F. (1952) Music, Physics and Engineering. Dover Publications, Mineola.

[2] Halliday, D., Resnick, R. and Walker, J. (2021) Fundamentals of Physics. 11th Edition, John Wiley & Sons, Inc., New Jersey.

[3] Tipler, P. and Mosca, G. (2022) Physics for Scientists and Engineers. 6th Edition, McGraw Hills Companies, Inc., New York.

[4] Mathematica V13. (2202). http://Wolfram.com

[5] Sarafian, H. (2022) Designing Physics Problems with *Mathematica*. *American*

Journal of Computational Mathematics, **12**, 191-196.
https://doi.org/10.4236/ajcm.2022.122010 .

[6] Wolfram, S. (2003) Mathematica Book. 5th Edition, Cambridge University Press, New York.

[7] Sarafian, H. (2019) Mathematica Graphics Examples. 2nd Edition, Scientific Research Publishing, Wuhan.