American Journal of Computational Mathematics, 2022, 12, 1-6
https://www.scirp.org/journal/ajcm
ISSN Online: 2161-1211
ISSN Print: 2161-1203

Coulomb, Yukawa, and Hooke's Oscillations

Haiduke Sarafian

The Pennsylvania State University, University College, York, USA
Email: has2@psu.edu

How to cite this paper: Sarafian, H. (2022) Coulomb, Yukawa, and Hooke's Oscillations. *American Journal of Computational Mathematics*, **12**, 1-6.
https://doi.org/10.4236/ajcm.2022.121001

Received: January 1, 2022
Accepted: February 18, 2022
Published: February 21, 2022

Abstract

Oscillations due to three different forces in three areas of physics: electrostatic, nuclear, and mechanics, are analyzed. The electrostatic long-range Coulomb force has a different character than the nucleonic short-range Yukawa force. Both are different from the linear Hooke's force. The equation of motion of each case is solved applying a Computer Algebra System (CAS). It is shown that these oscillations have similarities and differences. Phase diagrams of all three cases are compared.

Keywords

Nonlinear Oscillations, Yukawa Force, Coulomb Force, Computer Algebra System (CAS), *Mathematica*

1. Introduction

We consider three distinct areas of physics: mechanical, electrical, and nuclear. We search a common scenario overlapping these areas, *i.e.*, oscillations. In the area of mechanics, we consider a linear retractable force, namely a spring that is subject to Hooke's law [1]. The impact of this force on a massive object is linear oscillations. Compatible with forthcoming cases, a setup composed of a spring and an inclined is considered. For the electrostatic theme, we consider a force that is in proportion to the inverse squared distance between two point-like charges *i.e.*, the Coulomb force [2]. A setup like the previous case, a paired of charges and an inclined are considered. This is conducive to non-linear oscillations. And for the nuclear theme, we consider a pair of interacting nucleons. The nucleonic interaction, the Yukawa force [3] is the strengthened version of the Coulomb force.

With these envisioned common settings, a forum is set to compare the characteristics of the induced oscillations.

This report is composed of three sections. In addition to Section 1, in Section

2, by applying the fundamental laws of mechanics, we craft the equations of motions for each of the three cases. Except for the first case where the equation of motion is a trivial analytic solvable ODE, the other two cases are nonlinear ODE. Applying a Computer Algebra System (CAS) specifically Mathematica [4], these are solved numerically. Plots of these solutions are compared characterizing the oscillations [5]. Additional auxiliary information is obtained by plotting their associated phase diagrams. Section 3 is the conclusions highlighting the lessons learned.

2. Calculation

We begin with the Coulomb force. Then we move to Yukawa force and then for a reference point we include the Hooke's force.

A) Coulomb force [2]. To weaken the gravity force we place a charged marble q of mass m at the top of an incline. A second identical charged marble is fastened at the bottom of the ramp. By adjusting the inclination angle, we adjust the gravity force. The effective weight of the marble along the incline is mgsin(θ), g is the gravity acceleration. **Figure 1** depicts the setup. The length of the incline, *i.e.*, the initial distance between the masses is d.

Equations of motion for all three cases are formed applying Newton's law, $\boldsymbol{F}_{net} = m\boldsymbol{a}$. The $\boldsymbol{F}_{net}$ includes two terms: 1) the above-mentioned effective weight and 2) one of the case-related forces. For the Coulomb interaction, this is,

$$F_C = kq^2 \frac{1}{(d-x)^2} \tag{1}$$

where k is the electrostatic coupling constants, $k = \frac{1}{4\pi\epsilon_0} = 8.95 \times 10^9 \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}$. For identical charges F_c is repulsive.

B) Yukawa force [3]. The force between two nucleons is attractive and short-range. It is formulated as,

$$F_Y = g^2 \left[1 + \frac{1}{\xi(d-x)}\right] Y_0\left[\xi(d-x)\right] \tag{2}$$

In (2), g^2 is the nucleonic coupling constant, ξ is the scale parameter that controls the interaction range, and $Y_0(\xi x) = \frac{1}{\xi x} \mathrm{e}^{-\xi x}$, respectively. We make two notes concerning F_Y within the content of our aimed objectives. Acknowledging that 1) Yukawa force, when applied to nucleons within a nucleus, is attractive. And 2) its effective range is being controlled by the inverse mass of the virtual pion, ξ as such the effective range of (2) is within femtometers. For the objective of our investigation a) by inserting a minus sign the force becomes repulsive and, b) we assign values to ξ such that (2) becomes a long-range compatible with our objectives.

It is insightful comparing these two forces, *i.e.* F_C and F_Y vs. their effective range. For the sake of comparison in **Figure 2** we set, $kq^2 = g^2$.

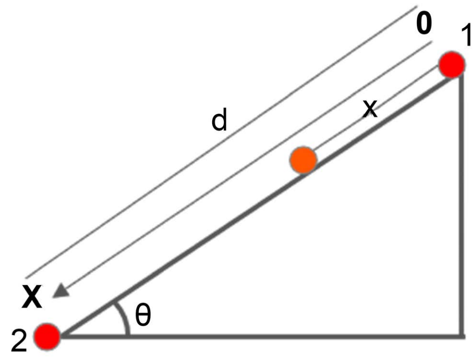

Figure 1. Two point-like massive, charged marbles are placed on the ramp. Their initial separation distance is *d* and their instantaneous distance is *d-x*. The top marble is loose the bottom one is fastened to the base.

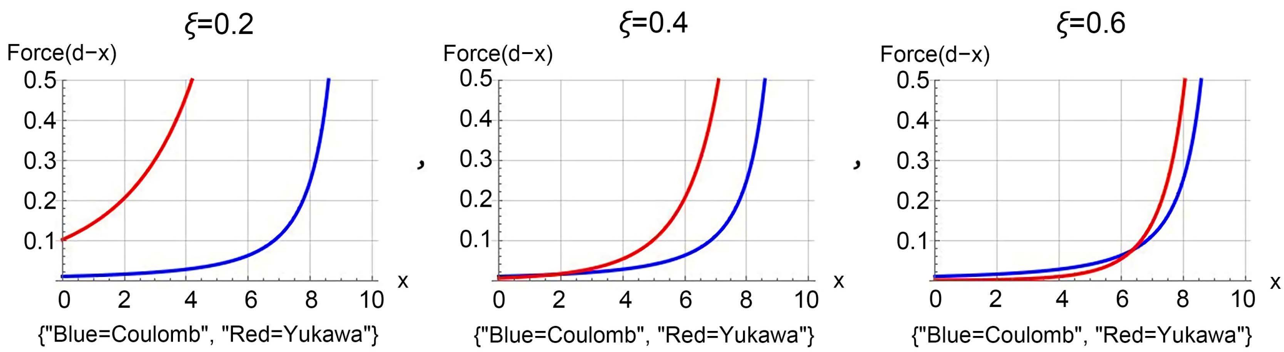

Figure 2. The vertical axis is the value of the repulsive force acting on the marbles. The horizontal axis is the instantaneous separation distance between the marbles. The *d* is the initial separation distance between the marbles, *d-x* is their actual effective separation distance. The impact of the scale factor ξ is displayed by the plot label.

Figure 2 shows while the marbles are closing in the repulsive force between them is gaining strength. And that the larger the ξ the longer the effectiveness of the Yukawa force. As shown for a certain value of ξ, *i.e.*, 0.6 the Coulomb and the Yukawa forces are closely comparable; at a certain separation distance, they cross over signifying the same strength. Concluding, **Figure 2** suggests the resulting oscillations due to various values of ξ ought to be compared. This is reported in the upcoming paragraphs.

For the Coulomb force the equation of motion for the sliding marble is,

$$\ddot{x}(t)+\alpha\frac{1}{\left[d-x(t)\right]^2}-c=0 \tag{3}$$

In (3), $\ddot{x}$ is the acceleration, $\alpha=\frac{kq^2}{m}$ and $c=g\sin(\theta)$, respectfully. For a set of reasonable practical values such as, $\{m,q,d,\theta\}=\{2\text{ grams},4\ \mu\text{C},10\text{ m},30^\circ\}$ these yields $\{\alpha,c\}=\{72\text{ m}^3\cdot\text{s}^{-2},5\text{ m}\cdot\text{s}^{-2}\}$.

For the Yukawa force equation of motion is,

$$\ddot{Y}(t)+\alpha\left\{1+\frac{1}{\xi\left[d-Y(t)\right]}\right\}\frac{1}{\xi\left[d-Y(t)\right]}\mathrm{e}^{-\xi\left[d-Y(t)\right]}-c=0 \tag{4}$$

As previously noted for comprehensive comparison the strength of the forces, *a*, *c* in (3-4) are set the same.

Applying *Mathematica* [4] with initial conditions: $\{x(0),\dot{x}(0)\}=\{Y(0),\dot{Y}(0)\}=\{0,0\}$ plots of the solutions of (3-4) are shown in **Figure 3**.

The Black curve is the solution of (3). It shows the sliding marble begins from the far initial distance approaches the bottom marble yet doesn't reach it, stops momentarily, rebounds climbing up the ramp. It repeats the process, meaning it oscillates. The Magenta and the Red curves have almost the same characters. Meaning the Yukawa force causes oscillations. Published literature hardly has discussed this issue. The latter two mentioned curves reveal the impact of the ξ. The Magenta curve with $\xi = 0.4$ is affiliated with the short-ranged and strong Yukawa force c.f. **Figure 2**. This value of ξ short-cuts the minimum distance of approach. By the same token, the red curve with $\xi = 0.6$ is affiliated with the long-ranged and weak Yukawa force. As such the minimum distance of approach is longer than the magenta curve. And at last, the green curve is indicative of an oscillation, however, the $\xi = 0.2$ corresponding to the strongest of the set provides a force pushing the marble up the ramp opposing the notion of the minimum distance of approach. Nonetheless, it inherits the oscillations.

Given the descriptive paragraph, we are guided to investigate the velocity character of each case. See **Figure 4**.

Phase Diagram:

Having the position of the sliding marbles vs. time we form the needed acceleration. Consequently, by eliminating the time between the pairs, $\{x(t),\dot{x}(t)\}$ and $\{\dot{x}(t),\ddot{x}(t)\}$ we display the corresponding phase diagrams. These are depicted in **Figure 5**.

As shown irrespective of the nonlinearity of the forces all three paired phase diagrams are *closed*. As expected, the first two pairs are somewhat alike. This is expected because the Coulomb and Yukawa forces have much in common, c.f. **Figure 2**. In these diagrams, their adjusted force parameters are counted for the aspect ratio. The last pair is self-descriptive.

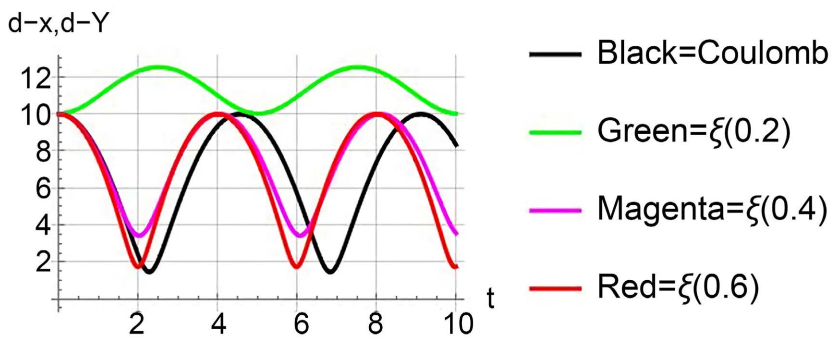

Figure 3. The black curve is the profile of Coulomb oscillations. The other three are the profiles due to Yukawa oscillations. The vertical and the horizontal axes are the actual separation distances of the corresponding cases and the oscillating times, respectively.

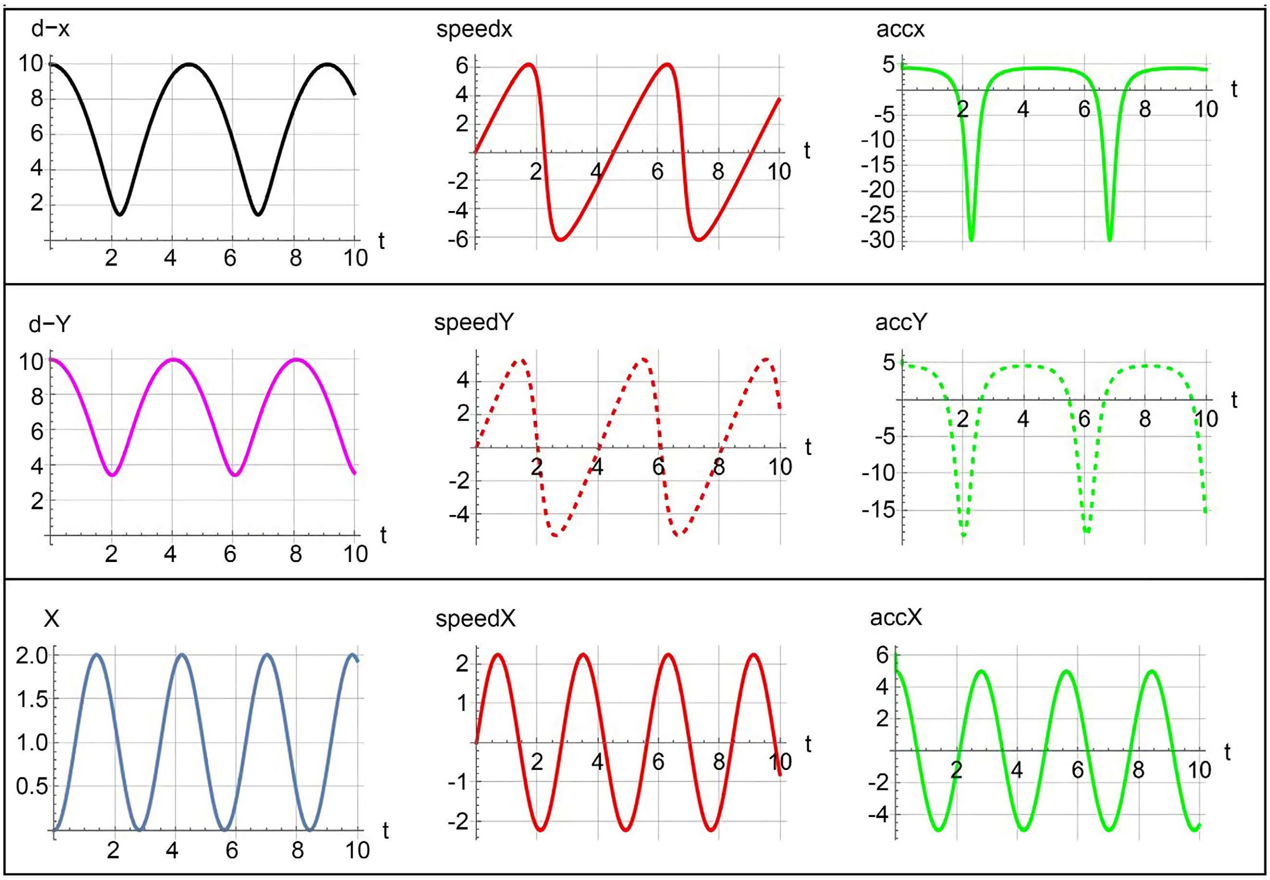

Figure 4. Rows correspond to Coulomb, Yukawa, and Hooke's forces. Left to right of each row corresponds to position, velocity, and acceleration vs. time. The vertical axis of the first, the second and the third column are the actual separation distances, the speeds, and the accelerations of the associated cases, respectively. The horizontal axis of all cases is the times.

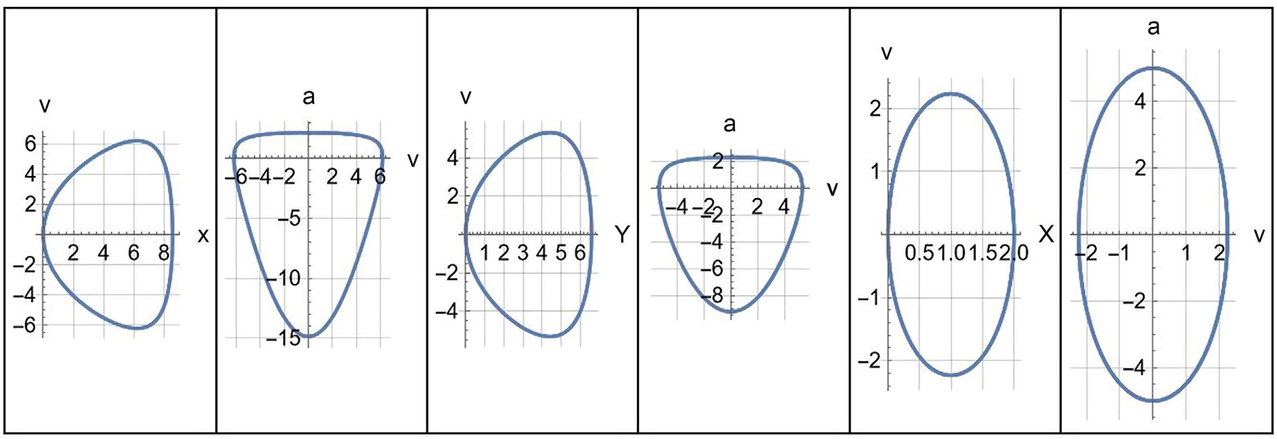

Figure 5. Phase diagrams of the Coulomb, Yukawa, and Hooke's forces. Paired diagrams of each case are display of the $\{x, v\}$ and $\{v, a\}$. Except for the familiar last pair, the other two pairs reveal the impact of the nonlinear forces.

3. Conclusion

We set an objective to compare the impact of the Coulomb and the Yukawa force for oscillations of paired massive point-like objects. The first force is somewhat familiar, the interaction is in proportion to $1/\text{dist}^2$, yet the character of the second one being confined to the short-range nucleonic force is confined to the mass of the exchanged mesons. No interest has been shown in the literature concerning their potential induced oscillations. In our investigation systematically we showed the similarities and differences of the impact of these forces. We have not applied quantum mechanical nonrelativistic Schrodinger physics, we strict our analysis to classical physics. The assumption is made concerning the impact of the Yukawa force beyond the short range. By solving the associated nonlinear equation of motion, we have shown the impact of these nonlinear forces on the classic two-body oscillations.

Acknowledgements

The author acknowledges the John T. and Page S. Smith Professorship funds for completing and publishing this work.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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