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Bayesian Prediction of Future Generalized Order Statistics from a Class of Finite Mixture Distributions

Abd EL-Baset A. Ahmad[1], Areej M. Al-Zaydi[2]

[1]Department of Mathematics, Assiut University, Assiut, Egypt
[2]Department of Mathematics, Taif University, Taif, Saudi Arabia
Email: abahmad2002@yahoo.com, aree.m.z@hotmail.com

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Abstract

This article is concerned with the problem of prediction for the future generalized order statistics from a mixture of two general components based on doubly type II censored sample. We consider the one sample prediction and two sample prediction techniques. Bayesian prediction intervals for the median of future sample of generalized order statistics having odd and even sizes are obtained. Our results are specialized to ordinary order statistics and ordinary upper record values. A mixture of two Gompertz components model is given as an application. Numerical computations are given to illustrate the procedures.

Keywords

Generalized Order Statistics, Bayesian Prediction, Heterogeneous Population, Doubly Type II Censored Samples, One- and Two-Sample Schemes

1. Introduction

Let the random variable (rv) T follows a class including some known lifetime models; its cumulative distribution function (CDF) is given by

$$F(t) = 1 - \exp\left[-\theta\lambda(t)\right], t > 0, (\theta > 0), \tag{1}$$

and its probability density function (PDF) is given by

$$f(t) = \theta\lambda'(t)\exp\left[-\theta\lambda(t)\right], t > 0, (\theta > 0), \tag{2}$$

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where $\lambda'(t)$ is the derivative of $\lambda(t)$ with respect to t and $\lambda(t) \equiv \lambda(t;\alpha)$ is a nonnegative continuous function of t and α may be a vector of parameters, such that

$\lambda(t) \to 0$ as $t \to 0^+$ and $\lambda(t) \to \infty$ as $t \to \infty$.

The reliability function (RF) and hazard rate function (HRF) are given, respectively, by

$$R(t) = \exp\left[-\theta\lambda(t)\right], \tag{3}$$

$$H(t) = \theta\lambda'(t), \tag{4}$$

where $H(.) = f(.)/R(.)$.

The general problem of statistical prediction may be described as that of inferring the value of unknown observable that belongs to a future sample from current available information, known as the informative sample. As in estimation, a predictor can be either a point or an interval predictor. The problem of prediction can be solved fully within Bayesian framework [1].

Prediction has been applied in medicine, engineering, business and other areas as well. For details on the history of statistical prediction, analysis, application and examples see for example [1] [2].

Bayesian prediction of future order statistics and records from different populations has been dealt with by many authors. Among others, [3] predicted observables from a general class of distributions. [4] obtained Bayesian prediction bounds under a mixture of two exponential components model based on type I censoring. [5] obtained Bayesian predictive survival function of the median of a set of future observations. Bayesian prediction bounds based on type I censoring from a finite mixture of Lomax components were obtained by [6]. [7] obtained Bayesian predictive density of order statistics based on finite mixture models. [8] obtained Bayesian interval prediction of future records. Based on type I censored samples, Bayesian prediction bounds for the s^{th} future observable from a finite mixture of two component Gompertz life time model were obtained by [9]. [10] considered Bayes inference under a finite mixture of two compound Gompertz components model. Bayesian prediction of future median has been studied by, among others, they were [5] [11] [12].

Recently, [13] introduced the generalized order statistics (GOS'S). Ordinary order statistics, ordinary record values and sequential order statistics were, among others, special cases of GOS'S. For various distributional properties of GOS'S, see [13]. The GOS'S have been considered extensively by many authors, among others, they were [14]-[33].

Mixtures of distributions arise frequently in life testing, reliability, biological and physical sciences. Some of the most important references that discuss different types of mixtures of distributions are a monograph by [34]-[36].

The PDF, CDF, RF and HRF of a finite mixture of two components of the class under study are given, respectively, by

$$f(t) = p_1 f_1(t) + p_2 f_2(t), \tag{5}$$

$$F(t) = p_1 F_1(t) + p_2 F_2(t), \tag{6}$$

$$R(t) = p_1 R_1(t) + p_2 R_2(t), \tag{7}$$

$$H(t) = f(t)/R(t), \tag{8}$$

where, for $j = 1,2$, the mixing proportions p_j are such that $0 \le p_j \le 1, p_1 + p_2 = 1$ and $f_j(t), F_j(t), R_j(t)$ are given from (1), (2), (3) after using θ_j and $\lambda_j(t)$ instead of θ and $\lambda(t)$.

The property of identifiability is an important consideration on estimating the parameters in a mixture of distributions. Also, testing hypothesis, classification of random variables, can be meaning fully discussed only if the class of all finite mixtures is identifiable. Idenifiability of mixtures has been discussed by several authors, including [37]-[39].

This article is concerned with the problem of obtaining Bayesian prediction intervals (BPI) for the future GOS'S from a mixture of two general components based on doubly type II censored sample. One- and two-sample prediction cases are treated in Sections 2 and 3, respectively. Bayesian prediction intervals for the median of future sample of GOS'S having odd and even sizes are obtained in Sections 4. A mixture of two Gompertz components is given as an application in Section 5. Finally, numerical computations are given in Section 6.

2. One Sample Prediction

Let $T_{s;n,m,k}, T_{s+1;n,m,k}, \cdots, T_{r;n,m,k}, 1 \le s < r \le n, k > 0$ be the $(r-s)$ GOS'S drawn from a mixture of two components of the class (2). Based on this doubly censored sample, the likelihood function can be written (see [27]) as

$$L(\theta|\boldsymbol{t}) = \begin{cases} c_1 \left\{ \prod_{i=s}^{r} \left[R(t_i) \right]^m f(t_i) \right\} \left[R(t_r) \right]^{\gamma_{r+1}} \times \sum_{\ell=0}^{s-1} \omega_\ell^{(s)} \left[R(t_s) \right]^{(s-\ell-1)(m+1)}, & m \ne -1, \\ c_2 \left[R(t_r) \right]^k \left[\ln R(t_s) \right]^{s-1} \prod_{i=s}^{r} H(t_i), & m = -1, \end{cases} \tag{9}$$

where $\boldsymbol{t} = (t_s, \cdots, t_r)$, $\theta \in \Theta$, Θ is the parameter space, and

$$\left. \begin{aligned} & c_1 = \frac{(-1)^{s-1} C_{r-1}}{(m+1)^{s-1} (s-1)!}, \quad c_2 = \frac{(-1)^{s-1} k^r}{(s-1)!}, \\ & C_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad \gamma_r = k + (n-r)(m+1), \\ & \omega_\ell^{(s)} = (-1)^\ell \binom{s-1}{\ell}. \end{aligned} \right\}$$

For definition and various distributional properties of GOS'S, see [13].

By substituting Equations (1) and (5) in Equation (9), we get

for $m \ne -1$,

$$\begin{aligned} L(\theta|\boldsymbol{t}) = c_1 & \left\{ \prod_{i=s}^{r} \left[p_1 R_1(t_i) + p_2 R_2(t_i) \right]^m \left[p_1 f_1(t_i) + p_2 f_2(t_i) \right] \right\} \\ & \times \left[p_1 R_1(t_r) + p_2 R_2(t_r) \right]^{\gamma_{r+1}} \sum_{\ell=0}^{s-1} \omega_\ell^{(s)} \left[p_1 R_1(t_s) + p_2 R_2(t_s) \right]^{(s-\ell-1)(m+1)}. \end{aligned} \tag{10}$$

And for $m = -1$,

$$L(\theta|\boldsymbol{t}) = c_2 \left[p_1 R_1(t_r) + p_2 R_2(t_r) \right]^k \left(\ln \left[p_1 R_1(t_s) + p_2 R_2(t_s) \right] \right)^{s-1} \times \prod_{i=s}^{r} \frac{\left[p_1 f_1(t_i) + p_2 f_2(t_i) \right]}{\left[p_1 R_1(t_i) + p_2 R_2(t_i) \right]}. \tag{11}$$

We shall use the conjugate prior density, that was suggested by [3], in the following form

$$\pi(\theta; \nu) \propto C(\theta; \nu) \exp\left[-D(\theta; \nu) \right], \theta = (p, \theta_1, \theta_2, \alpha_1, \alpha_2), \nu \in \Omega, \tag{12}$$

where Ω is the hyper parameter space.

Then the posterior PDF of θ, $\pi^*(\theta|\boldsymbol{t})$, is given by

$$\pi^*(\theta|\boldsymbol{t}) \propto \pi(\theta; \nu) L(\theta|\boldsymbol{t}). \tag{13}$$

Substituting from Equations (10) and (12) in Equation (13), for $m \ne -1$, the posterior PDF $\pi^*(\theta|\boldsymbol{t})$ takes the form

$$\begin{aligned} \pi^*(\theta|\boldsymbol{t}) \propto C(\theta; \nu) \exp\left[-D(\theta; \nu) \right] & \left\{ \prod_{i=s}^{r} \left[p_1 R_1(t_i) + p_2 R_2(t_i) \right]^m \left[p_1 f_1(t_i) + p_2 f_2(t_i) \right] \right\} \\ & \times \left[p_1 R_1(t_r) + p_2 R_2(t_r) \right]^{\gamma_{r+1}} \sum_{\ell=0}^{s-1} \omega_\ell^{(s)} \left[p_1 R_1(t_s) + p_2 R_2(t_s) \right]^{\delta_\ell^{(s)}}, \end{aligned} \tag{14}$$

where $\delta_\ell^{(s)} = (s-\ell-1)(m+1)$.

For $m = -1$, using Equations (11) and (12) in Equation (13), the posterior PDF can be written as

$$\pi^*(\theta|\boldsymbol{t}) \propto C(\theta;\nu)\exp[-D(\theta;\nu)][p_1R_1(t_r)+p_2R_2(t_r)]^k \times (\ln[p_1R_1(t_s)+p_2R_2(t_s)])^{s-1}\prod_{i=s}^{r}\frac{[p_1f_1(t_i)+p_2f_2(t_i)]}{[p_1R_1(t_i)+p_2R_2(t_i)]}. \tag{15}$$

Now, suppose that the first $(r-s)$ GOS'S $T_{s;n,m,k}, T_{s+1;n,m,k}, \cdots, T_{r;n,m,k}, 1 \le s < r \le n,$ have been formed and we wish to predict the future GOS'S $T_{r+1;n,m,k}, T_{r+2;n,m,k}, \cdots, T_{n;n,m,k}$. Let $T_a^* \equiv T_{r+a;n,m,k}$, $a = 1,2,\cdots,n-r$, the conditional PDF of the a^{th} future GOS given the past observations $\boldsymbol{t}$, can be written (see [27]) as

$$h(t_a^*|\theta,\boldsymbol{t}) \propto \begin{cases} \sum_{i=0}^{a-1}\omega_i^{(a)}[R(t_a^*)]^{\gamma_{r+a-i}-1}[R(t_r)]^{-\gamma_{r+a-i}} f(t_a^*), & m \ne -1, \\ \sum_{i=0}^{a-1}\omega_i^{(a)}[\ln R(t_a^*)]^i[\ln R(t_r)]^{a-i-1}[R(t_a^*)]^{k-1}[R(t_r)]^{-k} f(t_a^*), & m = -1, \end{cases} \tag{16}$$

where $\omega_i^{(a)} = (-1)^i\binom{a-1}{i}$.

When $m \ne -1$, substituting from Equations (1) and (5) in Equation (16), the conditional PDF takes the form

$$h(t_a^*|\theta,\boldsymbol{t}) \propto \sum_{i=0}^{a-1}\omega_i^{(a)}[p_1R_1(t_a^*)+p_2R_2(t_a^*)]^{\gamma_{r+a-i}-1} \times [p_1R_1(t_r)+p_2R_2(t_r)]^{-\gamma_{r+a-i}}[p_1f_1(t_a^*)+p_2f_2(t_a^*)], \tag{17}$$

In the case when $m = -1$; the conditional PDF takes the form

$$h(t_a^*|\theta,\boldsymbol{t}) \propto \sum_{i=0}^{a-1}\left\{\omega_i^{(a)}(\ln[p_1R_1(t_a^*)+p_2R_2(t_a^*)])^i(\ln[p_1R_1(t_r)+p_2R_2(t_r)])^{a-i-1}\right\} \times [p_1R_1(t_a^*)+p_2R_2(t_a^*)]^{k-1}[p_1R_1(t_r)+p_2R_2(t_r)]^{-k}[p_1f_1(t_a^*)+p_2f_2(t_a^*)], \tag{18}$$

The predictive PDF of T_a^* given the past observations $\boldsymbol{t}$ is obtained from Equations (13), (17) and (18) and written as

$$f^*(t_a^*|\boldsymbol{t}) = \int_\theta h(t_a^*|\theta,\boldsymbol{t})\pi^*(\theta|\boldsymbol{t})\mathrm{d}\theta, \; t_a^* > t_r, \tag{19}$$

where for $m \ne -1$,

$$\begin{aligned} h(t_a^*|\theta,t)\pi^*(\theta|t) \propto{}& C(\theta;\nu)\exp[-D(\theta;\nu)][p_1f_1(t_a^*)+p_2f_2(t_a^*)] \\ &\times\left\{\prod_{i=s}^{r}[p_1R_1(t_i)+p_2R_2(t_i)]^m[p_1f_1(t_i)+p_2f_2(t_i)]\right\} \\ &\times\sum_{i=0}^{a-1}\sum_{\ell=0}^{s-1}\left\{\eta_{i,\ell}^{(a,s)}[p_1R_1(t_s)+p_2R_2(t_s)]^{\delta_\ell^{(s)}}[p_1R_1(t_r)+p_2R_2(t_r)]^{\delta_i^{(a)}}\right. \\ &\left.\times[p_1R_1(t_a^*)+p_2R_2(t_a^*)]^{\gamma_{r+a-i}-1}\right\}, \end{aligned} \tag{20}$$

where

$$\left.\begin{aligned} \eta_{i,\ell}^{(a,s)} &= (-1)^{i+\ell}\binom{a-1}{i}\binom{s-1}{\ell}, \\ \delta_i^{(a)} &= (a-i-1)(m+1), \\ \delta_\ell^{(s)} &= (s-\ell-1)(m+1). \end{aligned}\right\}$$

Also, for $m=-1$,

$$\begin{aligned} h\left(t_a^*|\theta,\boldsymbol{t}\right)\pi^*\left(\theta|\boldsymbol{t}\right) &\propto C(\theta;\nu)\exp\left[-D(\theta;\nu)\right]\left[p_1 f_1\left(t_a^*\right)+p_2 f_2\left(t_a^*\right)\right] \\ &\times\left[p_1R_1\left(t_a^*\right)+p_2R_2\left(t_a^*\right)\right]^{k-1}\left(\ln\left[p_1R_1\left(t_s\right)+p_2R_2\left(t_s\right)\right]\right)^{s-1} \\ &\times\left\{\prod_{i=s}^{r}\frac{\sum_{j=1}^{2}p_j f_j\left(t_i\right)}{\sum_{j=1}^{2}p_j R_j\left(t_i\right)}\right\}\sum_{i=0}^{a-1}\omega_i^{(a)}\left(\ln\left[p_1R_1\left(t_a^*\right)+p_2R_2\left(t_a^*\right)\right]\right)^{i} \\ &\times\left(\ln\left[p_1R_1\left(t_r\right)+p_2R_2\left(t_r\right)\right]\right)^{a-i-1}. \end{aligned} \tag{21}$$

It then follows that the predictive survival function is given, for the a^{th} future GOS, by

$$P\left[T_a^*>\nu|\boldsymbol{t}\right]=\int_\nu^\infty f^*\left(t_a^*|\boldsymbol{t}\right)\mathrm{d}t_a^*,\ \ \nu>t_r. \tag{22}$$

A $100\tau\%$ BPI for T_a^* is then given by

$$P\left[L(\boldsymbol{t})<T_a^*<U(\boldsymbol{t})\right]=\tau,$$

where $L(\boldsymbol{t})$ and $U(\boldsymbol{t})$ are obtained, respectively, by solving the following two equations

$$P\left[T_a^*>L(\boldsymbol{t})|\boldsymbol{t}\right]=\frac{1+\tau}{2}, \tag{23}$$

$$P\left[T_a^*>U(\boldsymbol{t})|\boldsymbol{t}\right]=\frac{1-\tau}{2}. \tag{24}$$

3. Two Sample Prediction

Suppose that $T_{s;n,m,k},T_{s+1;n,m,k},\cdots,T_{r;n,m,k},1\le s<r\le n$.

Be a doubly type II censored random sample drawn from a population whose CDF, $F(x)$ and PDF, $f(x)$ and let $Y_{1;N,M,K},Y_{2;N,M,K},\cdots,Y_{N;N,M,K},K\ge 1,M\ge -1$.

Be a second independent generalized ordered random sample (of size *N*) of future observations from the same distribution. Based on such a doubly type II censored sample, we wish to predict the future GOS $Y_b\equiv Y_{b;N,M,K},b=1,2,\cdots,N,$ in the future sample of size *N*.

It was shown by [32] that the PDF of GOS Y_b is in the form

$$h_{Y_b}^{\star}\left(y|\theta\right)\propto\begin{cases} f(y)\sum_{i=0}^{b-1}\omega_i^{(b)}\left[R(y)\right]^{\gamma_{b-i}^{\star}-1}, & M\ne -1,\\ \left[R(y)\right]^{K-1}\left[\ln R(y)\right]^{b-1}f(y), & M=-1,\end{cases} \tag{25}$$

where $\gamma_i^{\star}=K+(N-i)(M+1)$ and $\omega_i^{(b)}=(-1)^i\binom{b-1}{i}$.

Substituting from Equations (1) and (5) in (25), we have

$$h_{Y_b}^{\star}\left(y|\theta\right)\propto\begin{cases}\left[p_1f_1(y)+p_2f_2(y)\right]\sum_{i=0}^{b-1}\omega_i^{(b)}\left[p_1R_1(y)+p_2R_2(y)\right]^{\gamma_{b-i}^{\star}-1}, & M\ne -1,\\ \left[p_1R_1(y)+p_2R_2(y)\right]^{K-1}\left(\ln\left[p_1R_1(y)+p_2R_2(y)\right]\right)^{b-1}\left[p_1f_1(y)+p_2f_2(y)\right], & M=-1.\end{cases} \tag{26}$$

The predictive PDF of $Y_b,b=1,2,\cdots,N,$ given the past observation t is obtained from Equations (14), (15) and (26), and written as

$$f_{Y_b}^{\star}\left(y|\boldsymbol{t}\right)=\int_\theta h_{Y_b}^{\star}\left(y|\theta\right)\pi^*\left(\theta|\boldsymbol{t}\right)\mathrm{d}\theta,\ \ y>0, \tag{27}$$

where for $m\ne -1,M\ne -1$,

$$\begin{aligned} h^{\star}_{Y_b}(y|\theta)\pi^{*}(\theta|\boldsymbol{t}) &\propto C(\theta;\nu)\exp\left[-D(\theta;\nu)\right]\left\{\prod_{i=s}^{r}\left[p_1R_1(t_i)+p_2R_2(t_i)\right]^{m}\right. \\ &\times\left.\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{\gamma_{r+1}} \\ &\times\left[p_1f_1(y)+p_2f_2(y)\right]\sum_{i=0}^{b-1}\sum_{\ell=0}^{s-1}\left\{\eta_{i,\ell}^{(b,s)}\left[p_1R_1(y)+p_2R_2(y)\right]^{\gamma^{\star}_{b-i}-1}\right\} \end{aligned}, \tag{28}$$

where

$$\eta_{i,\ell}^{(b,s)}=(-1)^{i+\ell}\binom{b-1}{i}\binom{s-1}{\ell},$$

Also for $m=-1, M=-1$,

$$\begin{aligned} h^{\star}_{Y_b}(y|\theta)\pi^{*}(\theta|\boldsymbol{t}) &\propto C(\theta;\nu)\exp\left[-D(\theta;\nu)\right]\left(\ln\left[p_1R_1(y)+p_2R_2(y)\right]\right)^{b-1} \\ &\times\left[p_1R_1(y)+p_2R_2(y)\right]^{K-1}\left[p_1f_1(y)+p_2f_2(y)\right] \\ &\times\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{k}\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1} \\ &\times\left\{\prod_{i=s}^{r}\frac{\sum_{j=1}^{2}p_jf_j(t_i)}{\sum_{j=1}^{2}p_jR_j(t_i)}\right\} \end{aligned}. \tag{29}$$

Bayesian prediction bounds for Y_b, $b=1,2,\cdots,N$ are obtained by evaluating

$$P\left[Y_b>\nu|\boldsymbol{t}\right]=\int_{\nu}^{\infty}f^{\star}_{Y_b}(y|\boldsymbol{t})\,\mathrm{d}y,\ \nu>0. \tag{30}$$

A $100\tau\%$ BPI for Y_b is then given by

$$P\left[L(\boldsymbol{t})<Y_b<U(\boldsymbol{t})\right]=\tau,$$

where $L(\boldsymbol{t})$ and $U(\boldsymbol{t})$ are obtained, respectively, by solving the following two equations

$$P\left[Y_b>L(\boldsymbol{t})|\boldsymbol{t}\right]=\frac{1+\tau}{2}, \tag{31}$$

$$P\left[Y_b>U(\boldsymbol{t})|\boldsymbol{t}\right]=\frac{1-\tau}{2}. \tag{32}$$

4. Bayesian Prediction for the Future Median

The median of N observations, denoted by $\tilde{Y}_N$, is defined by

$$\tilde{Y}_N=\begin{cases} Y_{\varphi;N,M,K}, & N=2\varphi-1 \\ \frac{1}{2}\left[Y_{\varphi;N,M,K}+Y_{\varphi+1;N,M,K}\right], & N=2\varphi \end{cases},$$

where φ is a positive integer, $\varphi\geq 1$.

4.1. The Case of Odd Future Sample Size

The PDF of future median $\tilde{Y}_N$ takes the form (26) with $b=\varphi$ and $N=2\varphi-1$.

Substituting $b=\varphi$ in Equation (27), we obtain the predictive PDF $f^{\star}_{\tilde{Y}_N}(y|\boldsymbol{t})$ of the median of $N=2\varphi-1$ observations.

A $100\tau\%$ BPI for $\tilde{Y}_N$ is then given by

$$P\left[L(\boldsymbol{t})<\tilde{Y}_N<U(\boldsymbol{t})\right]=\tau,$$

where $L(\boldsymbol{t})$ and $U(\boldsymbol{t})$ are obtained, respectively, by solving the following two equations

$$P\left[\tilde{Y}_N > L(\boldsymbol{t})\middle|\boldsymbol{t}\right] = \frac{1+\tau}{2}, \tag{33}$$

$$P\left[\tilde{Y}_N > U(\boldsymbol{t})\middle|\boldsymbol{t}\right] = \frac{1-\tau}{2}, \tag{34}$$

where, for $\nu > 0, P\left[\tilde{Y}_N > \nu\middle|\boldsymbol{t}\right]$ is predictive survival function, given by Equation (30) with $b = \varphi$ and $N = 2\varphi - 1$.

4.2. The Case of Even Future Sample Size

The joint density function of two consecutive GOS $Y_{\varphi;N,M,K} \equiv Y_\varphi$ and $Y_{\varphi+1;N,M,K} \equiv Y_{\varphi+1}$ is given by

$$\hbar_{Y_\varphi,Y_{\varphi+1}}\left(y_\varphi, y_{\varphi+1}\middle|\theta\right) = \begin{cases} \dfrac{C_\varphi^\star}{(\varphi-1)!}\left[R\left(y_\varphi\right)\right]^M\left[h_M\left(F\left(y_\varphi\right)\right) - h_M(0)\right]^{\varphi-1} f\left(y_\varphi\right) \\ \qquad \times\left[R\left(y_{\varphi+1}\right)\right]^{\gamma_{\varphi+1}^\star-1} f\left(y_{\varphi+1}\right), \quad M \neq -1 \\ \dfrac{K^{\varphi+1}}{(\varphi-1)!} H\left(y_\varphi\right) H\left(y_{\varphi+1}\right)\left[S\left(y_\varphi\right)\right]^{\varphi-1}\left[R\left(y_{\varphi+1}\right)\right]^K, \quad M = -1 \end{cases}, \tag{35}$$

where

$$\left.\begin{aligned} &C_\varphi^\star = \prod_{j=1}^{\varphi+1}\gamma_j^\star,\ \gamma_j^\star = K + (N-j)(M+1), \\ &S(.) = -\ln\left(\bar{F}(.)\right), \end{aligned}\right\}$$

And

$$h_M\left(F\left(y_\varphi\right)\right) = \begin{cases} \dfrac{-\left(\bar{F}\left(y_\varphi\right)\right)^{M+1}}{M+1}, & M \neq -1 \\ -\ln\left(\bar{F}\left(y_\varphi\right)\right), & M = -1 \end{cases}.$$

Expanding $\left[h_M\left(F\left(y_\varphi\right)\right) - h_M(0)\right]^{\varphi-1}$ binomially and applying the transformation $\tilde{Y}_N = \dfrac{Y_\varphi + Y_{\varphi+1}}{2}$ and $Z = Y_\varphi$, the Jacobian of transformation is 2, we obtain

$$\hbar_{Z,\tilde{Y}_N}\left(z, y\middle|\theta\right) \propto \begin{cases} \displaystyle\sum_{j=0}^{\varphi-1}\omega_j^{(\varphi)}\left[R(z)\right]^{j(M+1)+M}\left[R(2y-z)\right]^{\gamma_{\varphi+1}^\star-1} f(z) f(2y-z), & M \neq -1 \\ H(z) H(2y-z)\left(\ln\left[R(z)\right]\right)^{\varphi-1}\left[R(2y-z)\right]^K, & M = -1 \end{cases}. \tag{36}$$

By substituting Equations (2), (4) and (5) in Equation (36) and integrating out *z*, yields the density function of $\tilde{Y}_N$, in the case of $M \neq -1$, as

$$\begin{aligned} \hbar_{\tilde{Y}_N}\left(y\middle|\theta\right) \propto \sum_{j=0}^{\varphi-1}\omega_j^{(\varphi)}\int_0^y &\left[p_1R_1(z) + p_2R_2(z)\right]^{j(M+1)+M}\left[p_1R_1(2y-z) + p_2R_2(2y-z)\right]^{\gamma_{\varphi+1}^\star-1} \\ &\times\left[p_1f_1(z) + p_2f_2(z)\right]\left[p_1f_1(2y-z) + p_2f_2(2y-z)\right]\mathrm{d}z, \quad y > 0. \end{aligned} \tag{37}$$

In the case $M = -1$, we have

$$\begin{aligned} \hbar_{\tilde{Y}_N}\left(y\middle|\theta\right) \propto \int_0^y \sum_{j=1}^{2}\frac{p_jf_j(z)}{p_jR_j(z)}\sum_{j=1}^{2}\frac{p_jf_j(2y-z)}{p_jR_j(2y-z)}&\left(\ln\left[p_1R_1(z) + p_2R_2(z)\right]\right)^{\varphi-1} \\ \times\left[p_1R_1(2y-z) + p_2R_2(2y-z)\right]^K \mathrm{d}z,& \quad y > 0. \end{aligned} \tag{38}$$

The predictive density function of the future median of $N=2\varphi$ observations is given by

$$f^{\star}_{\tilde{Y}_N}(y|\boldsymbol{t})=\int_{\theta}\hbar_{\tilde{Y}_N}(y|\theta)\pi^*(\theta|\boldsymbol{t})\mathrm{d}\theta,\ \ y>0, \tag{39}$$

where $\pi^*(\theta|\boldsymbol{t})$ and $\hbar_{\tilde{Y}_N}(y|\theta)$ are given by Equations (13) and (37), (38), respectively. It then follows that the predictive survival function is given, for $\tilde{Y}_N$, by

$$P\left[\tilde{Y}_N>v|\boldsymbol{t}\right]=\int_{v}^{\infty}f^{\star}_{\tilde{Y}_N}(y|t)\mathrm{d}y,\ \ v>0. \tag{40}$$

The lower and upper bound of $100\tau\%$ BPI for $\tilde{Y}_N$ can be obtained by solving Equations (33) and (34), numerically.

5. Example

Gompertz Components

Suppose that, for $j=1,2$ and $t>0,\theta_j=1,\lambda_j(t)=\dfrac{1}{\alpha_j}\left[\mathrm{e}^{\alpha_j t}-1\right]$ so $\lambda_j(t)=\mathrm{e}^{\alpha_j t}$.

In this case, the j^{th} subpopulation is Gompertz distribution with parameter $\alpha_j>0$. Let p,α_1 and α_2 are independent random variables such that $p\sim Beta(b_1,b_2)$ and for $j=1,2$, α_j to follow a left truncated exponential density with parameter d_j ($LTE(d_j)$), as used by [40]. A joint prior density function is then given by

$$\pi(\theta;v)\propto p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right], \tag{41}$$

where $0<p_1<1,\alpha_j>1,(b_1,b_2,d_1,d_2)>0$ and $p_2=1-p_1$.

5.1.1. One Sample Prediction

For $m\neq-1,m=-1$ substituting $\lambda_j(t)$, $\lambda_j(t)$.

And Equation (41) in Equation (22) and solving, numerically, Equations (23) and (24) we can obtain the lower and upper bounds of BPI.

Special Cases

1) Upper order statistics

The predictive PDF (19), when $m=0$ and $k=1$ becomes

$$\begin{aligned}f^*(t_a^*|\boldsymbol{t})=&K_1\int_0^1\int_0^{\infty}\int_0^{\infty}p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\\&\times\left[p_1f_1(t_a^*)+p_2f_2(t_a^*)\right]\left\{\prod_{i=s}^{r}\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\\&\times\sum_{i=0}^{a-1}\sum_{\ell=0}^{s-1}\left\{\eta_{i,\ell}^{(a,s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{s-\ell-1}\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{a-i-1}\right.\\&\left.\times\left[p_1R_1(t_a^*)+p_2R_2(t_a^*)\right]^{n-r-a+i}\right\}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p\end{aligned}, \tag{42}$$

where

$$K_1^{-1}=\int_{t_r}^{\infty}\int_{\theta}h(t_a^*|\theta,\boldsymbol{t})\pi^*(\theta|\boldsymbol{t})\mathrm{d}\theta\mathrm{d}t_a^*.$$

Substituting from Equation (42) in Equation (22) and solving Equations (23) and (24), numerically, we can obtain the bounds of BPI.

2) Upper record values

When $m=-1,k=1(\gamma_r=1)$, the predictive PDF (19) becomes

$$f^{*}\left(t_a^{*}\middle|t\right)=K_2\int_0^1\int_0^{\infty}\int_0^{\infty}p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{\left(\alpha_j-1\right)}{d_j}\right]\left[p_1f_1\left(t_a^{*}\right)+p_2f_2\left(t_a^{*}\right)\right]$$
$$\times\left(\ln\left[p_1R_1\left(t_s\right)+p_2R_2\left(t_s\right)\right]\right)^{s-1}\left\{\prod_{i=s}^{r}\frac{\sum_{j=1}^{2}p_jf_j\left(t_i\right)}{\sum_{j=1}^{2}p_jR_j\left(t_i\right)}\right\}\sum_{i=0}^{a-1}\omega_i^{(a)}\quad,\qquad(43)$$
$$\times\left(\ln\left[p_1R_1\left(t_a^{*}\right)+p_2R_2\left(t_a^{*}\right)\right]\right)^{i}\left(\ln\left[p_1R_1\left(t_r\right)+p_2R_2\left(t_r\right)\right]\right)^{a-i-1}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p$$

where

$$K_2^{-1}=\int_{t_r}^{\infty}\int_{\theta}h\left(t_a^{*}\middle|\theta,\boldsymbol{t}\right)\pi^{*}\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta\mathrm{d}t_a^{*}.$$

Substituting from Equation (43) in Equation (22) and solving Equations (23) and (24), numerically, we can obtain the bounds of BPI.

5.1.2. Two Sample Prediction

For $M\neq-1$ and $m\neq-1, M=-1$ and $m=-1$, substituting $\lambda_j\left(t\right)$, $\lambda_j\left(t\right)$ and Equation (41) in Equation (30) and solving, numerically, Equations (31) and (32) we can obtain the lower and upper bounds of BPI.

Special Cases

1) Upper order statistics

Substituting $M=m=0$ and $K=k=1$ in Equation (27), we have

$$f_{Y_b}^{\star}\left(y\middle|t\right)=B_1\int_0^1\int_0^{\infty}\int_0^{\infty}p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{\left(\alpha_j-1\right)}{d_j}\right]\left\{\prod_{i=s}^{r}\left[p_1f_1\left(t_i\right)+p_2f_2\left(t_i\right)\right]\right\}$$
$$\times\left[p_1R_1\left(t_r\right)+p_2R_2\left(t_r\right)\right]^{n-r}\left[p_1f_1\left(y\right)+p_2f_2\left(y\right)\right]\sum_{i=0}^{b-1}\sum_{\ell=0}^{s-1}\left\{\eta_{i,\ell}^{(b,s)}\quad,\qquad(44)\right.$$
$$\left.\times\left[p_1R_1\left(t_s\right)+p_2R_2\left(t_s\right)\right]^{s-\ell-1}\left[p_1R_1\left(y\right)+p_2R_2\left(y\right)\right]^{N-b+i}\right\}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p$$

where

$$B_1^{-1}=\int_0^{\infty}\int_{\theta}h_{Y_b}^{\star}\left(y\middle|\theta\right)\pi^{*}\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta\mathrm{d}y.$$

To obtain $100\tau\%$ BPI for $Y_b, b=1,2,\cdots,N$, we solve Equations (31) and (32), numerically.

2) Upper record values

In Equation (27), by putting $K=k=1$, the predictive PDF of Y_b takes the form

$$f_{Y_b}^{\star}\left(y\middle|\boldsymbol{t}\right)=B_2\int_0^1\int_0^{\infty}\int_0^{\infty}p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{\left(\alpha_j-1\right)}{d_j}\right]\left(\ln\left[p_1R_1\left(y\right)+p_2R_2\left(y\right)\right]\right)^{b-1}$$
$$\times\left[p_1f_1\left(y\right)+p_2f_2\left(y\right)\right]\left[p_1R_1\left(t_r\right)+p_2R_2\left(t_r\right)\right]\quad,\qquad(45)$$
$$\times\left(\ln\left[p_1R_1\left(t_s\right)+p_2R_2\left(t_s\right)\right]\right)^{s-1}\left\{\prod_{i=s}^{r}\frac{\sum_{j=1}^{2}p_jf_j\left(t_i\right)}{\sum_{j=1}^{2}p_jR_j\left(t_i\right)}\right\}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p$$

where

$$B_2^{-1}=\int_0^{\infty}\int_{\theta}h_{Y_b}^{\star}\left(y\middle|\theta\right)\pi^{*}\left(\theta\middle|\boldsymbol{t}\right)\mathrm{d}\theta\mathrm{d}y.$$

Substituting from Equation (45) in Equation (30) and solving Equations (31) and (32), numerically, we can obtain the bounds of BPI.

5.1.3. Prediction for the Future Median (the Case of Odd *N*)

Special Cases

1) Upper order statistics

Substituting $\lambda_j(t)$, $\lambda_j(t)$, $C(\theta;\nu)$ and $D(\theta;\nu)$ in Equation (27) with $b=\varphi$ and $N=2\varphi-1$ and by putting $M=m=0$ and $K=k=1$, we have

$$\begin{aligned} f_{\tilde{Y}_N}^{\star}(y|\mathbf{t}) = B_1^{\star}\int_0^1\int_0^{\infty}\int_0^{\infty} p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right] \\ \times\left\{\prod_{i=s}^{r}\left[p_1f_1(t_i)+p_2f_2(t_i)\right]\right\}\left[p_1R_1(t_r)+p_2R_2(t_r)\right]^{n-r} \\ \times\left[p_1f_1(y)+p_2f_2(y)\right]\sum_{i=0}^{\varphi-1}\sum_{\ell=0}^{s-1}\left\{\eta_{i,\ell}^{(\varphi,s)}\left[p_1R_1(t_s)+p_2R_2(t_s)\right]^{s-\ell-1}\right. \\ \left.\times\left[p_1R_1(y)+p_2R_2(y)\right]^{N-\varphi+i}\right\}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p \end{aligned}, \qquad (46)$$

where

$$B_1^{\star-1}=\int_0^{\infty}\int_{\theta}h_{\tilde{Y}_N}^{\star}(y|\theta)\pi^{*}(\theta|\mathbf{t})\mathrm{d}\theta\mathrm{d}y.$$

To obtain $100\tau\%$ BPI for $\tilde{Y}_N$, we solve Equations (33) and (34), numerically.

2) Upper record values

The predictive PDF (27), when $K=k=1$, becomes

$$\begin{aligned} f_{\tilde{Y}_N}^{\star}(y|t) = B_2^{\star}\int_0^1\int_0^{\infty}\int_0^{\infty} p_1^{b_1-1}p_2^{b_2-1}\exp\left[-\sum_{j=1}^{2}\frac{(\alpha_j-1)}{d_j}\right]\left(\ln\left[p_1R_1(y)+p_2R_2(y)\right]\right)^{\varphi-1} \\ \times\left[p_1f_1(y)+p_2f_2(y)\right]\left[p_1R_1(t_r)+p_2R_2(t_r)\right] \\ \times\left(\ln\left[p_1R_1(t_s)+p_2R_2(t_s)\right]\right)^{s-1}\times\left\{\prod_{i=s}^{r}\frac{\sum_{j=1}^{2}p_jf_j(t_i)}{\sum_{j=1}^{2}p_jR_j(t_i)}\right\}\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}p \end{aligned}, \qquad (47)$$

where

$$B_2^{\star-1}=\int_0^{\infty}\int_{\theta}h_{\tilde{Y}_N}^{\star}(y|\theta)\pi^{*}(\theta|\mathbf{t})\mathrm{d}\theta\mathrm{d}y.$$

To obtain $100\tau\%$ BPI for $\tilde{Y}_N$, we solve Equations (33) and (34), numerically.

5.1.4. Prediction for the Future Median (the Case of Even *N*)

Special Cases

1) Upper order statistics

The predictive PDF and survival function of $\tilde{Y}_N$ can be obtained by substituting $M=m=0$ and $K=k=1$ in Equations (39) and (40), respectively.

2) Upper record values

The predictive PDF and survival function of $\tilde{Y}_N$ can be obtained by substituting $K=k=1$ in Equations (39) and (40), respectively.

To obtain $100\tau\%$ BPI for future median of ordinary order statistics or ordinary upper record values.

We solve Equations (33) and (34), numerically.

6. Numerical Computations

In this section, 95% BPI for future observations from a mixture of two $Gomp(\alpha_j), j=1,2$, components are obtained by considering one sample and two sample schemes.

6.1. One Sample Prediction

In this subsection, we compute 95% BPI for $T_a^{*}, a=1,2$, in the two cases ordinary order statistics and ordinary upper record values according to the following steps:

1) For a given values of the prior parameters (b_1, b_2) generate a random value p from the $Beta(b_1, b_2)$ distribution.
2) For a given values of the prior parameters d_j, for $j = 1, 2$, generate a random value α_j from the $LTE(d_j)$ distribution.
3) Using the generated values of p, α_1 and α_2, we generate a random sample from a mixture of two $Gomp(\alpha_j)$ components, $j = 1, 2$, as follows:

- generate two observations u_1, u_2 from $\text{Uniform}(0,1)$;
- if $u_1 \le p$, then $t = \frac{1}{\alpha_1}\log\left[1 - \alpha_1 \log(1 - u_2)\right]$, otherwise $t = \frac{1}{\alpha_2}\log\left[1 - \alpha_2 \log(1 - u_2)\right]$;
- repeat above steps n times to get a sample of size n;
- the sample obtained in above steps is ordered.

4) Using the generated values of p, α_1 and α_2, we generate upper record values of size $n = 12$ from a mixture of two $Gomp(\alpha_j), j = 1, 2$, components.
5) The 95% BPI for the future observations are obtained by solving numerically, Equations (23) and (24) with $\tau = 0.95$. Different sample size n and the censored size are considered.

6.2. Two Sample Prediction

In this subsection, we compute 95% BPI for two sample prediction in the two cases ordinary order statistics and ordinary upper record values according to the following steps:

1) For a given values of the prior parameters (b_1, b_2) generate a random value p from the $Beta(b_1, b_2)$ distribution.
2) For a given values of the prior parameters d_j, for $j = 1, 2$, generate a random value α_j from the $LTE(d_j)$ distribution.
3) Using the generated values of p, α_1 and α_2, we generate a doubly type II sample from a mixture of two $Gomp(\alpha_j), j = 1, 2$, components.
4) The 95% BPI for the observations from a future independent sample of size N are obtained by solving numerically, Equations (31) and (32) with $\tau = 0.95$.
5) Generate 10,000 samples each of size N from a mixture of two $Gomp(\alpha_j)$ components, then calculate the coverage percentage of Y_b.
6) Different sample sizes n and N are considered.

6.3. Prediction for the Future Median

In this subsection, 95% BPI for the median of N future observations are obtained when the underlying population distribution is a mixture of two Gompertz components in the two cases ordinary order statistics and ordinary upper record values according to the following steps:

1) For a given values of the prior parameters (b_1, b_2) generate a random value p from the $Beta(b_1, b_2)$ distribution.
2) For a given values of the prior parameters d_j, for $j = 1, 2$, generate a random value α_j from the $LTE(d_j)$ distribution.
3) Using the generated values of p, α_1 and α_2, we generate a doubly type II sample from a mixture of two $Gomp(\alpha_j), j = 1, 2$, components.
4) The 95% BPI for the median of N of future observations are obtained by solving numerically, Equations (33) and (34) with $\tau = 0.95$ for different values of N, when $N = 2\varphi - 1$ is odd and $N = 2\varphi$ is even.
5) Generate 10,000 samples each of size N from a mixture of two $Gomp(\alpha_j)$ components, then calculate the coverage percentage of $\tilde{Y}_N$.
6) The prediction are conducted on the basis of a doubly type II censored samples and type II censored samples.

The computational (our) results were computed by using Mathematica 6.0. When the prior parameters chosen as b_1 = 1.5, b_2 = 2, d_1 = 1, d_2 = 2 which yield the generated values of $p = 0.516065, \alpha_1 = 1.46186$, $\alpha_2 = 3.1847$. In **Tables 1-4**, 95% BPI for future observations are computed in case of the one and two

Table 1. 95% BPI for future order statistics T_a^*, $a=1,2$, when $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

Case (n,r)	T_a^*	$S=1$		$S=2$	
		(L,U)	Length	(L,U)	Length
(10, 7)	T_1^*	(0.562965, 0.744602)	0.181638	(0.443015, 0.618514)	0.175499
	T_2^*	(0.748781, 1.53094)	0.782164	(0.569291, 1.30527)	0.735977
(15, 10)	T_1^*	(0.47374, 0.548465)	0.0747253	(0.40578, 0.480169)	0.0743882
	T_2^*	(0.587001, 0.901358)	0.314357	(0.494304, 0.804901)	0.310597
(20, 15)	T_1^*	(0.719253, 0.774191)	0.0549385	(0.601514, 0.65858)	0.0570667
	T_2^*	(0.866788, 1.14337)	0.276584	(0.740253, 1.01961)	0.279359
(50, 35)	T_1^*	(0.789649, 0.797004)	0.00735491	(0.555976, 0.563516)	0.00754019
	T_2^*	(0.791368, 0.883652)	0.0922842	(0.559453, 0.816295)	0.256842

Table 2. 95% BPI for the future upper record values $T_a^*, a=1,2$, when $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

r	T_a^*	$S=1$		$S=2$	
		(L,U)	Length	(L,U)	Length
5	T_1^*	(1.47048, 2.71682)	1.24633	(0.783431, 1.80835)	1.02492
	T_2^*	(1.50643, 3.48872)	1.98229	(0.803645, 2.5354)	1.73175
8	T_1^*	(1.38189, 1.94658)	0.564687	(1.80196, 2.44531)	0.643359
	T_2^*	(1.36459, 2.32526)	0.960663	(1.8222, 2.82042)	0.998218
10	T_1^*	(1.93128, 2.45302)	0.52174	(1.90637, 2.49514)	0.58877
	T_2^*	(1.91008, 2.76182)	0.851738	(1.92627, 2.83318)	0.906915

Table 3. 95% *BPI* and *PC* for the future order statistics $Y_b, b=1,2$, when $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

N (n,r)	Y_b	$S=1$		$S=2$	
		(L,U) Length	PC	(L,U) Length	PC
10 (20, 15)	Y_1	(0.00253099, 0.357705) 0.355174	97.13	(0.00253088, 0.354125) 0.351594	97.32
	Y_2	(0.0249745, 0.561048) 0.536074	97.50	(0.0250174, 0.552829) 0.527811	97.55
10 (20, 18)	Y_1	(0.00253086, 0.352991) 0.35046	96.72	(0.00253091, 0.353966) 0.351435	96.99
	Y_2	(0.0250486, 0.550266) 0.525218	97.33	(0.0250276, 0.552571) 0.527543	97.29
15 (30, 22)	Y_1	(0.00168778, 0.244183) 0.242495	96.79	(0.00168788, 0.244617) 0.242929	96.53
	Y_2	(0.018541, 0.382363) 0.363821	96.59	(0.0194943, 0.384252) 0.364758	96.26
15 (30, 27)	Y_1	(0.00168747, 0.241649) 0.239962	96.56	(0.0016876, 0.243342) 0.241654	96.76
	Y_2	(0.0163242, 0.374301) 0.357976	97.29	(0.0172091, 0.379132) 0.361923	96.92

sample predictions, respectively. In **Table 5** and **Table 6**, 95% BPI for the medians of future samples with odd or even sizes are computed. Our results are specialized to ordinary order statistics and ordinary upper record values.

Table 4. 95% *BPI* and *PC* for future ordinary upper record values $Y_b, b=1,2$, when $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

Case N (n,r)	Y_b	$S=1$ (L,U) Length	$S=1$ PC	$S=2$ (L,U) Length	$S=2$ PC
6 (8, 5)	Y_1	(0.0244019, 1.26507) 1.24067	97.19	(0.0242337, 1.26247) 1.23824	97.20
	Y_2	(0.184594, 1.79401) 1.60942	97.02	(0.177304, 1.8683) 1.69099	97.44
6 (8, 7)	Y_1	(0.0243355, 1.17383) 1.14949	96.89	(0.0245445, 1.25834) 1.23379	96.91
	Y_2	(0.179104, 1.49161) 1.31251	97.36	(0.188427, 1.62202) 1.43359	97.37
8 (10, 7)	Y_1	(0.0240923, 1.1295) 1.10541	96.68	(0.0244941, 1.20562) 1.18112	96.92
	Y_2	(0.169789, 1.47476) 1.30497	96.85	(0.187398, 1.59348) 1.40609	96.03
8 (10, 9)	Y_1	(0.0239792, 1.08241) 1.05843	96.36	(0.0237464, 1.11903) 1.09529	96.75
	Y_2	(0.161039, 1.35434) 1.1933	97.08	(0.153654, 1.41319) 1.25954	97.43

Table 5. (Ordinary order statistics) 95% *BPI* and *PC* for future median $\tilde{Y}_N$ when $N=2\varphi-1$ is odd or 2φ, is even and $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

$n=30$ r	$\tilde{Y}_N$	$S=1$ (L,U) Length	$S=1$ PC	$S=2$ (L,U) Length	$S=2$ PC
18	$\tilde{Y}_5$	(0.150064, 1.7981) 1.64804	96.70	(0.149628, 1.81497) 1.66534	96.55
	$\tilde{Y}_4$	(0.15732, 1.87535) 1.71803	86.29	(0.156637, 1.89352) 1.73689	85.81
22	$\tilde{Y}_5$	(0.150599, 1.78292) 1.63232	96.50	(0.150879, 1.77517) 1.62429	96.70
	$\tilde{Y}_4$	(0.158553, 1.85921) 1.70066	85.77	(0.159278, 1.85101) 1.69173	85.63
27	$\tilde{Y}_5$	(0.151131, 1.76607) 1.61494	96.39	(0.15093, 1.77256) 1.62163	96.69
	$\tilde{Y}_4$	(0.160396, 1.84172) 1.68132	85.25	(0.159707, 1.8485) 1.68879	85.26

Table 6. (Ordinary upper record values) 95% *BPI* and *PC* for future median $\tilde{Y}_N$ when $N=2\varphi-1$ is odd or 2φ, is even and $b_1=1.5, b_2=2, d_1=1, d_2=2$ and the generated parameters $(p=0.516065, \alpha_1=1.46186, \alpha_2=3.1847)$.

$n=10$ r	$\tilde{Y}_N$	$S=1$ (L,U) Length	$S=1$ PC	$S=2$ (L,U) Length	$S=2$ PC
5	$\tilde{Y}_3$	(0.160855, 2.18738) 2.02653	98.29	(0.145888, 2.22403) 2.07814	98.77
	$\tilde{Y}_2$	(0.133759, 1.70622) 1.57246	84.56	(0.148567, 1.75259) 1.60402	83.42
7	$\tilde{Y}_3$	(0.170573, 2.00065) 1.83008	98.14	(0.148896, 1.98626) 1.83736	98.66
	$\tilde{Y}_2$	(0.142946, 1.5994) 1.45645	84.74	(0.140236, 1.60923) 1.469	84.34
9	$\tilde{Y}_3$	(0.200611, 1.52718) 1.32657	96.87	(0.196739, 1.51186) 1.31512	96.75
	$\tilde{Y}_2$	(0.116557, 1.29989) 1.18334	86.34	(0.118077, 1.27117) 1.1531	86.46

6.4. Conclusions

1) Bayes prediction intervals for future observations are obtained using a one-sample and two-sample schemes based on a finite mixture of two Gompertz components model. Our results are specialized to ordinary order statistics and ordinary upper record values.
2) Bayesian prediction intervals for the medians of future samples with odd or even sizes are obtained based on a finite mixture of two Gompertz components model. Our results are specialized to ordinary order statistics and ordinary upper record values.
3) It is evident from all tables that the lengths of the BPI decrease as the sample size increase.
4) In general, if the sample size n and censored size r are fixed the lengths of the BPI increase by increasing s.
5) For fixed sample size n, censored size r and s, the lengths of the BPI increase by increasing a or b.
6) The percentage coverage improves by the use of a large number of observed values.

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