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On a Characterization of Zero-Inflated Negative Binomial Distribution

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Abstract

Zero-inflated negative binomial distribution is characterized in this paper through a linear differential equation satisfied by its probability generating function.

Keywords

Zero-Inflated Negative Binomial Distribution, Probability Distribution, Probability Generating Function, Linear Differential Equation

1. Introduction

Zero-inflated discrete distributions have paved ways for a wide variety of applications, especially count regression models. Nanjundan [1] has characterized a subfamily of power series distributions whose probability generating function (pgf) $f(s)$ satisfies the differential equation $(a+bs)f'(s)=cf(s)$, where $f'(s)$ is the first derivative of $f(s)$. This subfamily includes binomial, Poisson, and negative binomial distributions. Also, Nanjundan and Sadiq Pasha [2] have characterized zero-inflated Poisson distribution through a differential equation. In the similar way, Nagesh *et al*. [3] have characterized zero-inflated geometric distribution. Along the same lines, zero-inflated negative binomial distribution is characterized in this paper via a differential equation satisfied by its pgf.

A random variable X is said to have a zero-inflated negative binomial distribution, if its probability mass function is given by

$$p(x)=\begin{cases}\varphi+(1-\varphi)p^r & x=0\\ (1-\varphi)\binom{x+r-1}{x}p^r q^x & x=1,2,\cdots,\end{cases} \quad (1)$$

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where $0<\varphi<1$, $0<p<1$, $p+q=1$, and $r>0$.

The probability generating function of X is given by

$$f(s)=E(s^X)=\sum_{x=0}^{\infty}p(s)s^x,\ 0<s<1$$

$$=\varphi+(1-\varphi)p^r\sum_{x=0}^{\infty}\binom{x+r-1}{x}q^x \tag{2}$$

$$f(s)=\varphi+(1-\varphi)\frac{p^r}{(1-qs)^r}.$$

Hence the first derivative of $f(s)$ is given by

$$f'(s)=(1-\varphi)rq\frac{p^r}{(1-qs)^{r+1}}.$$

2. Characterization

The following theorem characterizes the zero-inflated negative binomial distribution.

Theorem 1 *Let X be a non-negative integer valued random variable with* $0<P(X=0)<1$. *Then X has a zero-inflated negative binomial distribution if and only if its pgf* $f(s)$ *satisfies*

$$f(s)=a+b(1+cs)f'(s), \tag{3}$$

where a, b, c are constants.

Proof. 1) Suppose that X has zero-inflated negative binomial distribution with the probability mass function specified in (1). Then its pgf can be expressed as

$$f(s)=\varphi+(1-\varphi)\frac{rqp^r(1-qs)}{rq(1-qs)^{r+1}}$$

$$f(s)=\varphi+\frac{1}{rq}(1-qs)f'(s). \tag{4}$$

Hence $f(s)$ in (4) satisfies (3) with $a=\varphi, b=1/rq, c=-q$.

2) Suppose that the pgf of X satisfies the linear differential equation in (3).

Writing the Equation (3) as

$$y=a+b(1+cx)\frac{\mathrm{d}y}{\mathrm{d}x},$$

we get

$$\frac{\mathrm{d}y}{y-a}=\frac{1}{bc}c\frac{\mathrm{d}x}{1+cx}.$$

On integrating both sides w.r.t. x, we get

$$\int\frac{\mathrm{d}y}{y-a}=\frac{1}{bc}\int\frac{c\,\mathrm{d}x}{1+cx}$$

$$\Rightarrow \log(y-a)=\frac{1}{bc}\log(1+ck)+k_1,\text{ where } k_1 \text{ is an arbitary constant.}$$

$$\therefore \log(y-a)=\log(1+ck)^{\frac{1}{bc}}+\log k,\text{ by writing } k_1=\log k$$

$$=\log\left[k(1+ck)^{\frac{1}{bc}}\right].$$

That is

$$y=a+k(1+cx)^{\frac{1}{bc}}.$$

The solution of the differential equation in (3) becomes

$$f(s) = a + k(1+cs)^{\frac{1}{bc}}. \quad (5)$$

If either b or c or both are equal to zero, then $\frac{1}{bc} = \infty$ and hence $f(s)$ has no meaning. Thus, both b and c are non-zero. Since $f(s)$ is a pgf, it is a power series of the type $p_0 + p_1 s + p_2 s^2 + \cdots$. When either $c > 0$ or $\frac{1}{bc}$ is not a negative integer, the expansion of the factor $(1+cs)^{\frac{1}{bc}}$ on the right hand side of (5) will have negative coefficients, which is not permissible because $f(s)$ is a pgf. Hence the equation in (5) can be written as

$$f(s) = a + k(1-ds)^{-N},$$

where N is a positive integer. Since $f(1) = 1$, $k = (1-a)(1-d)^N$.

Therefore

$$f(s) = a + (1-a)(1-d)^N (1-ds)^{-N}. \quad (6)$$

Hence $f(s)$ in (6) satisfies (2) with $a = \varphi$, $p = (1-d)$, $q = d$, and $N = r$.

This completes the proof of the theorem.

□

Also, it can be noted that when $N = r = 1$, the negative binomial distribution reduces to geometric distribution and the **Theorem 1** in Section 2 concurs with the characterization result of Nagesh *et al*. [3].

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