

The Large Scale Instability in Rotating Fluid with Small Scale Force

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Abstract

In this paper, we find a new large scale instability in rotating flow forced turbulence. The turbulence is generated by a small scale external force at low Reynolds number. The theory is built on the rigorous asymptotic method of multi-scale development. The nonlinear equations for the instability are obtained at the third order of the perturbation theory. In this article, we explain the nonlinear stage of the instability and the generation vortex kinks.

Keywords

Large Scale Vortex Instability, Coriolis Force, Multi-Scale Development, Small Scale Turbulence, Vortex Kinks

1. Introduction

It is well known, that the rotating effects play an important role in many practical and theoretical applications for fluid mechanics [1] and are especially important for geophysics and astrophysics [2] [3] when one has to deal with rotating objects such as Earth, Jupiter and Sun. Rotating fluids could generate different waves and vortex motions, for example, gyroscopic waves, Rossby waves, internal waves, located vortices and coherent vortex structures [4]-[7]. Among the vortex structures, the most interesting are the large scale ones, since they carry out the efficient transport of energy and impulse. The structures which have characteristic scale much more than the scale of turbulence or of the external force which generates this turbulence, are understood as large scale ones. At present, we can state that there are a lot of instabilities which generate the large scale vortex structures (see for example [8]-[14]), in particular, in rotating fluid with the non-homogeneous turbulence [15]. In this work, we find the new large scale instability in rotating fluid, under impact of small external force which keeps up turbulent fluctuations. The nonlinear large scale helical vortex structures of Beltrami type or localized kinks

with internal helical structure appear as a result of the development of this instability in rotating fluid. We can consider that external small scale force substitutes the action of small scale turbulence. It is supposed that external force is in plane (*X*, *Y*), which is perpendicular to the rotation axis, for example, axis *Z* is directed along the vector of angular velocity of rotation Ω . Helical 2D field of velocity W_x , W_y turns around axis *Z*, when *Z* changes in the kink which links the hyperbolic point and the stable focus (**Figure 1**). Moreover, this field does some turns in the kink, which links instable and stable focuses (**Figure 2**). The found instability belongs to the class of instabilities called hydrodynamic α -effect. For these instabilities, the positive feedback between velocity components of W_x , W_y is typical.



Figure 1. The kink which connects the hyperbolic point with stable knot with D = 1, $C_1 = 0.04$, $C_2 = 0.04$. When approaching the stable knot one can see rotations of velocity field.



Figure 2. The kink which connects the instable and stable focuses with D = 1, $C_1 = 0.04$, $C_2 = 0.04$. One can see the internal helical structure of the kink.

$$\partial_T W_x - \Delta W_x - \alpha_y \frac{\partial}{\partial z} W_y = 0,$$

 $\partial_T W_y - \Delta W_y + \alpha_x \frac{\partial}{\partial z} W_x = 0,$

and leads to the instability. The α -effect is taking its origins from magnetic hydrodynamics, where it engenders the increase of large scale magnetic fields (see for example [16]). It was generalized later for ordinary hydrodynamics. For the time being some examples of hydrodynamics α -effect [8]-[14] are already known. From this point of view, in this work we found a new example of α -effect. The theory of this instability is developed rigourously using the method of asymptotic multi-scale development, similar to what was done by Frisch, She and Sulem for the theory of the AKA effect [13]. This method allows finding the equations for large scale perturbations as secular equations of asymptotical theory in order to calculate the Reynolds stress tensor and to find the instability. The small parameter of asymptotical development is the Reynolds number *R*, $R \ll 1$. Our paper is organised as follows: in Section 2 we formulate the problem and the main equations in rotating system of coordinates; in Section 3 we examine the principal scheme of the multi-scale development and we give the secular equations. In Section 4 we calculate the velocity field of zero approximation. In Section 5 we describe the calculations of the Reynolds stress and find the large scale instability. In Section 6 we discuss the saturation of the instability and find non linear stationary vortex structures. The results obtained are discussed in the conclusions given in Section 7.

2. The Main Equations and Formulation of the Problem

Let us examine the equations of motion for non-compressible rotating fluid with external force F_0 in rotating coordinates system:

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} + 2\mathbf{\Omega} \times \mathbf{V} = -\frac{1}{\rho_0}\nabla P + \mathbf{v}\Delta \mathbf{V} + \mathbf{F}_0, \tag{1}$$

$$div \mathbf{V} = \mathbf{0}.$$
 (2)

The external force F_0 is divergence-free. Here Ω -angular velocity of fluid rotation, ν -viscosity, ρ_0 -constant fluid density. Let us design characteristic amplitude of force f_0 , and its characteristic space and time scale λ_0 and t_0 respectively.

Then $F_0 = f_0 F_0 \left(\frac{x}{\lambda_0}, \frac{t}{t_0}\right)$. We will design the characteristic amplitude of velocity, generated by external

force as v_0 . We choose the dimensionless variables (t, x, V):

$$\begin{aligned} \mathbf{x} &\to \frac{\mathbf{x}}{\lambda_0}, \quad t \to \frac{t}{t_0}, \quad \mathbf{V} \to \frac{\mathbf{V}}{v_0}, \quad \mathbf{F}_0 \to \frac{\mathbf{F}_0}{f_0}, \quad P \to \frac{P}{\rho_0 P_0}, \\ t_0 &= \frac{\lambda_0^2}{\nu}, \quad P_0 = \frac{v_0 \nu}{\lambda_0}, \quad f_0 = \frac{v_0 \nu}{\lambda_0^2}, \quad v_0 = \frac{f_0 \lambda_0^2}{\nu}. \end{aligned}$$

Then, in dimensionless variables the Equation (1) takes forme:

$$\frac{\partial \boldsymbol{V}}{\partial t} + R(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} + \boldsymbol{D} \times \boldsymbol{V} = -\nabla P + \Delta \boldsymbol{V} + \boldsymbol{F}_{0}, \qquad (3)$$

 $R = \frac{\lambda_0 v_0}{v}$, $|D| = \sqrt{Ta}$ where R and $Ta = \frac{4\Omega^2 \lambda_0^4}{v^2}$ are respectively the Reynolds number and the Taylor num-

ber on scale λ_0 . Further we will consider the Reynolds number as small $R \ll 1$ and will construct on this small parameter the asymptitical development. Concerning the parameter D, we do not choose any range of values for the moment. Let us examine the following formulation of the problem. We consider the external force as being small scale and of high frequency. This force leads to small scale fluctuations in velocity. After averaging, these quickly oscillating fluctuations vanish. Nevertheless, due to small nonlinear interactions in some orders of perturbation theory, nonzero terms can occur after averaging. This means that they are not oscillatory, that is to

say, they are large scale. From a formal point of view, these terms are secular, *i.e.*, they create the conditions for the solvability of a large scale asymptotic development. So the purpose of this paper is to find and study the solvability equations, *i.e.*, the equations for large scale perturbations. Let us denote the small scale variables by $x_0 = (\mathbf{x}_0, t_0)$, and the large scale ones by $X = (\mathbf{X}, T)$. The small scale partial derivative operation $\frac{\partial}{\partial x_0^i}$, $\frac{\partial}{\partial t_0}$, and the large scale ones $\frac{\partial}{\partial \mathbf{X}}$, $\frac{\partial}{\partial T}$ are written, respectively, as ∂_i , ∂_i , ∇_i and ∂_T . To con-

struct a multi-scale asymptotic development we follow the method which is proposed in [16].

3. The Multi-Scale Asymptotic Development

Let us search for the solution to Equations (2) and (3) in the following form:

$$V(\mathbf{x},t) = \frac{1}{R} W_{-1}(X) + v_0(x_0) + Rv_1 + R^2 v_2 + R^3 v_3 + \cdots,$$
(4)

$$T(\mathbf{x},t) = \frac{1}{R}T_{-1}(X) + T_0(x_0) + RT_1 + R^2T_2 + R^3T_3 + \cdots,$$
(5)

$$P(\mathbf{x},t) = \frac{1}{R^3} P_{-3}(X) + \frac{1}{R^2} P_{-2}(X) + \frac{1}{R} P_{-1}(X) + P_0(x_0) + R(P_1 + \overline{P_1}(X)) + R^2 P_2 + R^3 P_3 + \cdots$$
(6)

Let us introduce the following equalities: $X = R^2 x_0$ and $T = R^4 t_0$ which lead to the expression for the space and time derivatives:

$$\frac{\partial}{\partial x^{i}} = \partial_{i} + R^{2} \nabla_{i}, \qquad (7)$$

$$\frac{\partial}{\partial t} = \partial_t + R^4 \partial_T, \qquad (8)$$

$$\frac{\partial^2}{\partial x^j \partial x^j} = \partial_{jj} + 2R^2 \partial_j \nabla_j + R^4 \partial_{jj}.$$
(9)

Using indicial notation, the system of equation can be written as

$$\begin{aligned} &\left(\partial_t + R^4 \partial_T\right) V^i + R \left(\partial_j + R^2 \nabla_j\right) \left(V^i V^j\right) + D^j \varepsilon_{ijk} V^k \\ &= \left(\partial_j + R^2 \nabla_j\right) P + \left(\partial_{jj} + 2R^2 \partial_j \nabla_j + R^4 \nabla_{jj}\right) V^i + F_0^i, \end{aligned}$$
(10)

$$\partial_t T - \partial_{jj} T = -V^z - R \partial_j \left(V^i T \right), \tag{11}$$

$$\left(\partial_i + R^2 \nabla_i\right) V^i = 0. \tag{12}$$

Substituting these expressions into the initial Equations (2) and (3) and then gathering together the terms of the same order, we obtain the equations of the multi-scale asymptotic development and write down the obtained equations up to order R^3 inclusive. In the order R^{-3} there is only the equation

$$\partial_i P_{-3} = 0 \Longrightarrow P_{-3} = P_{-3} \left(X \right). \tag{13}$$

In order R^{-2} we have the equation

$$\partial_i P_{-2} = 0 \Longrightarrow P_{-2} = P_{-2} \left(X \right). \tag{14}$$

In order R^{-1} we get a system of equations:

$$\partial_{t}W_{-1}^{i} - \partial_{jj}W_{-1}^{i} + D^{j}\varepsilon_{ijk}W_{-1}^{k} = -(\partial_{i}P_{-1} + \nabla_{i}P_{-3}) - \partial_{j}W_{-1}^{i}W_{-1}^{j},$$
(15)

$$\partial_i W_{-1}^i = 0.$$

The system of Equations (17) and (18) gives the secular terms

$$-\nabla_i P_{-3} = D^j \varepsilon_{ijk} W^k_{-1}, \tag{16}$$

which corresponds to a geostrophic equilibrum equation.

In zero order R^0 , we have the following system of equations:

$$\partial_{i}v_{0}^{i} - \partial_{jj}v_{0}^{i} + \partial_{j}\left(W_{-1}^{i}v_{0}^{j} + v_{0}^{i}W_{-1}^{j}\right) + D^{j}\varepsilon_{ijk}v_{0}^{k} = -\left(\partial_{i}P_{0} + \nabla_{i}P_{-2}\right) + F_{0}^{i},$$
(17)
$$\partial_{i}v_{0}^{i} = 0.$$

These equations give one secular equation:

$$\nabla P_{-2} = 0 \Longrightarrow P_{-2} = \text{Const.} \tag{18}$$

Let us consider the equations of the first approximation *R*:

$$\partial_{t}v_{1}^{i} - \partial_{jj}v_{1}^{i} + D^{j}\varepsilon_{ijk}v_{1}^{k} + \partial_{j}\left(W_{-1}^{i}v_{1}^{j} + v_{1}^{i}W_{-1}^{j} + v_{0}^{i}v_{0}^{j}\right) = -\nabla_{j}\left(W_{-1}^{i}W_{-1}^{j}\right) - \left(\partial_{i}P_{1} + \nabla_{i}P_{-1}\right),\tag{19}$$

$$\partial_i V_1^i + \nabla_i W_{-1}^i = 0. (20)$$

From this system of equations there follows the secular equations:

$$\nabla_i W^i_{-1} = 0, \tag{21}$$

$$\nabla_{j} \left(W_{-1}^{i} W_{-1}^{j} \right) = -\nabla_{i} P_{-1}.$$
(22)

The secular Equations (27) and (29) are satisfied by choosing the following geometry for the velocity field (Beltrami field):

$$W = (W_{-1}^{x}(Z), W_{-1}^{y}(Z), 0);$$
(23)

$$\nabla P_{-1} = 0 \Longrightarrow P_{-1} = \text{Const.}$$

In the second order R^2 , we obtain the equations

$$\partial_{t}v_{2}^{i} - \partial_{jj}v_{2}^{i} - 2\partial_{j}\nabla_{j}v_{0}^{i} + \partial_{j}\left(W_{-1}^{i}v_{2}^{j} + v_{2}^{i}W_{-1}^{j} + v_{0}^{i}v_{1}^{j} + v_{1}^{i}v_{0}^{j}\right) + D^{j}\varepsilon_{ijk}v_{2}^{k}$$

$$= (24)$$

$$= -\nabla_{j} \Big(W_{-1}^{i} v_{0}^{j} + v_{0}^{i} W_{-1}^{j} \Big) - \Big(\partial_{i} P_{2} + \nabla_{i} P_{0} \Big),$$

$$\partial_i v_2 + \nabla_i v_0 = 0. \tag{25}$$

It is easy to see that there are no secular terms in this order. Let us come now to the most important order R^3 . In this order we obtain the equations

$$\partial_{t}v_{3}^{i} + \partial_{T}W_{-1}^{i} - \left(\partial_{jj}v_{3}^{i} + 2\partial_{j}\nabla_{j}v_{1}^{i} + \nabla_{jj}W_{-1}^{i}\right) + \nabla_{j}\left(W_{-1}^{i}v_{1}^{j} + v_{1}^{i}W_{-1}^{j} + v_{0}^{i}v_{0}^{j}\right) + \partial_{j}\left(W_{-1}^{i}v_{3}^{j} + v_{3}^{i}W_{-1}^{j} + v_{0}^{i}v_{2}^{j} + v_{2}^{i}v_{0}^{j} + v_{1}^{i}v_{1}^{j}\right) + D^{j}\varepsilon_{ijk}v_{3}^{k} = -\left(\partial_{i}P_{3} + \nabla_{i}\overline{P_{1}}\right),$$

$$(26)$$

$$\partial_i v_3 + \nabla_i v_1 = 0$$

From this we get the main secular equation:

$$\partial_T W^i_{-1} - \Delta W^i_{-1} + \nabla_k \overline{\left(v^k_0 v^i_0 \right)} = -\nabla_i \overline{P}_1.$$
⁽²⁷⁾

There is also an equation to find the pressure P_{-3} :

$$-\nabla_i P_{-3} = D^j \varepsilon_{iik} W^k_{-1}. \tag{28}$$

4. The Velocity Field in Zero Approximation

It is clear that the most important is Equation (36). In order to obtain these equations in closed form, we need to calculate the Reynolds stresses $\nabla_k \left(\overline{v_0^k v_0^i} \right)$. First of all we have to calculate the fields of zero approximation v_0^k . From the asymptotic development in zero order we have

$$\partial_t v_0^i - \partial_{jj} v_0^i + W_{-1}^k \partial_k v_0^i + D^j \varepsilon_{ijk} v_0^k = -\partial_i P_0 + F_0^i.$$
⁽²⁹⁾

Let us introduce the operator \hat{D}_0 :

$$\hat{D}_0 \equiv \partial_t - \partial_{jj} + W^k \partial_k.$$
(30)

Using \hat{D}_0 , were write Equations (29):

$$\hat{D}_{0}v_{0}^{i} + D^{j}\varepsilon_{ijk}v_{0}^{k} = -\partial_{i}P_{0} + F_{0}^{i}.$$
(31)

Pressure P_0 can be found from condition div V = 0.

$$P_0 = \frac{\left[\boldsymbol{D} \times \boldsymbol{\partial}\right]_i v_0^i}{\boldsymbol{\partial}^2}.$$
(32)

Let us introduce designations for operatores:

$$\hat{P}_{ij} = \partial_j \frac{\left[\boldsymbol{D} \times \boldsymbol{\partial}\right]_i}{\boldsymbol{\partial}^2}$$
(33)

and for velocities: $v_0^x = u_0$, $v_0^y = v_0$, $v_0^z = w_0$. Then excluding pressure from (31), we obtain the system of equations to find the velocity field of zero approximation:

$$\left(\widehat{D_0} + \widehat{P_{xx}}\right) u_0 + \left(\widehat{P_{yx}} - D_z\right) v_0 + \left(\widehat{P_{zx}} + D_y\right) w_0 = F_0^x, \left(\widehat{P_{xy}} + D_z\right) u_0 + \left(\widehat{D_0} + \widehat{P_{yy}}\right) v_0 + \left(\widehat{P_{zy}} - D_x\right) w_0 = F_0^y, \left(\widehat{P_{xz}} - D_y\right) u_0 + \left(\widehat{P_{yz}} + D_x\right) v_0 + \left(\widehat{D_0} + \widehat{P_{zz}}\right) w_0 = F_0^z.$$

$$(34)$$

For simplicity, we choose the systeme of coordinates so that the axis Z coincides with the direction of angular velocity of rotation Ω . Then $D_x = 0$, $D_y = 0$, $D_z = D$ In order to solve this system of equations we have to set the force in the explicit form.Let us choose now the external force in the rotating system of coordinates in the following form:

$$F_0^z = 0, F_{0\perp} = f_0 (i \cos \varphi_2 + j \cos \varphi_1); \varphi_1 = k_1 x - \omega_0 t, \varphi_2 = k_2 x - \omega_0 t, k_1 = k_0 (1, 0, 1), k_2 = k_0 (0, 1, 1).$$

It is obvious that divergence of this force us equal to zero. Thus, external force is given in plane (x, y), orthogonal to rotation axis.

The solution for equations system (34) can be found easily in accordance with Cramer's Rule:

$$u_0 = \frac{\Delta_1}{\Delta}, v_0 = \frac{\Delta_2}{\Delta}, w_0 = \frac{\Delta_3}{\Delta}.$$
(35)

Here Δ is the determinant of the system (34):

$$\Delta = \begin{vmatrix} \widehat{D_0} + \widehat{P_{xx}} & \widehat{P_{yx}} - D & \widehat{P_{zx}} \\ \widehat{P_{xy}} + D & \widehat{D_0} + \widehat{P_{yy}} & \widehat{P_{zy}} \\ \widehat{P_{xz}} & \widehat{P_{yz}} & \widehat{D_0} + \widehat{P_{zz}} \end{vmatrix},$$
(36)

$$\Delta_{1} = \begin{vmatrix} F_{0}^{x} & \widehat{P_{yx}} - D & \widehat{P_{zx}} \\ F_{0}^{y} & \widehat{D_{0}} + \widehat{P_{yy}} & \widehat{P_{zy}} \\ 0 & \widehat{P_{yz}} & \widehat{D_{0}} + \widehat{P_{zz}} \end{vmatrix},$$
(37)

$$\Delta_{2} = \begin{vmatrix} \widehat{D_{0}} + \widehat{P_{xx}} & F_{0}^{x} & \widehat{P_{zx}} \\ \widehat{P_{xy}} + D & F_{0}^{y} & \widehat{P_{zy}} \\ \widehat{P_{xz}} & 0 & \widehat{D_{0}} + \widehat{P_{zz}} \end{vmatrix},$$
(38)

$$\Delta_{3} = \begin{vmatrix} \widehat{D_{0}} + \widehat{P_{xx}} & \widehat{P_{yx}} - D & F_{0}^{x} \\ \widehat{P_{xy}} + D & \widehat{D_{0}} + \widehat{P_{yy}} & F_{0}^{y} \\ \widehat{P_{xz}} & \widehat{P_{yz}} & 0 \end{vmatrix}.$$
(39)

After writing down the determinants in the explicit form, we obtain:

$$u_{0} = \frac{1}{\Delta} \left[\left(\widehat{P}_{0} + \widehat{P}_{yy} \right) \left(\widehat{D}_{0} + \widehat{P}_{zz} \right) - \left(\widehat{P}_{yz} \right) \left(\widehat{P}_{zy} \right) \right] F_{0}^{x} + \frac{1}{\Delta} \left[\left(\widehat{P}_{zx} \right) \left(\widehat{P}_{yz} \right) - \left(\widehat{P}_{yx} - D \right) \left(\widehat{D}_{0} + \widehat{P}_{zz} \right) \right] F_{0}^{y},$$

$$(40)$$

$$v_{0} = \frac{1}{\Delta} \left[\left(\widehat{P_{xz}} \right) \left(\widehat{P_{zy}} \right) - \left(\widehat{P_{xy}} + D \right) \left(\widehat{D_{0}} + \widehat{P_{zz}} \right) \right] F_{0}^{x} + \frac{1}{\Delta} \left[\left(\widehat{D_{0}} + \widehat{P_{xx}} \right) \left(\widehat{D_{0}} + \widehat{P_{zz}} \right) - \left(\widehat{P_{xz}} \right) \left(\widehat{P_{zx}} \right) \right] F_{0}^{y},$$

$$(41)$$

$$w_{0} = \frac{1}{\Delta} \left[\left(\widehat{P_{xy}} + D \right) \left(\widehat{P_{yz}} \right) - \left(\widehat{P_{xz}} \right) \left(\widehat{D_{0}} + \widehat{P_{yy}} \right) \right] F_{0}^{x} + \frac{1}{\Delta} \left[\left(\widehat{P_{xz}} \right) \left(\widehat{P_{yx}} - D \right) - \left(\widehat{D_{0}} + \widehat{P_{xx}} \right) \left(\widehat{P_{yz}} \right) \right] F_{0}^{y}.$$

$$(42)$$

$$\Delta = \left(\widehat{D_0} + \widehat{P_{xx}}\right) \left[\left(\widehat{D_0} + \widehat{P_{yy}}\right) \left(\widehat{D_0} + \widehat{P_{zz}}\right) - \left(\widehat{P_{yz}}\right) \left(\widehat{P_{zy}}\right) \right] - \left(\widehat{P_{yx}} - D\right) \left[\left(\widehat{P_{xy}} + D\right) \left(\widehat{D_0} + \widehat{P_{zz}}\right) - \left(\widehat{P_{xz}}\right) \left(\widehat{P_{zy}}\right) \right] + \left(\widehat{P_{zx}}\right) \left[\left(\widehat{P_{xy}} + D\right) \left(\widehat{P_{yz}}\right) - \left(\widehat{D_0} + \widehat{P_{yy}}\right) \left(\widehat{P_{xz}}\right) \right].$$
(43)

In order to calculate the expressions (40)-(43) we present the external force in complex form:

$$F_0^x = \frac{f_0}{2} \left(e^{i\phi_2} + e^{-i\phi_2} \right), \quad F_0^y = \frac{f_0}{2} \left(e^{i\phi_1} + e^{-i\phi_1} \right).$$
(44)

Then all operators in formulae (40)-(42) act from the left on their eigenfunctions. In particular:

$$\widehat{D}_{0}e^{i\varphi_{2}} = e^{i\varphi_{2}}\widehat{D}_{0}(\boldsymbol{k}_{2},-\omega_{0}), \quad \widehat{D}_{0}e^{i\varphi_{1}} = e^{i\varphi_{1}}\widehat{D}_{0}(\boldsymbol{k}_{1},-\omega_{0}),
\Delta e^{i\varphi_{2}} = e^{i\varphi_{2}}\Delta(\boldsymbol{k}_{2},-\omega_{0}), \quad \Delta e^{i\varphi_{1}} = e^{i\varphi_{1}}\Delta(\boldsymbol{k}_{1},-\omega_{0}).$$
(45)

To simplify the formulae, let us choose $k_0 = 1$, $\omega_0 = 1$. We will designate

$$\widehat{D_0}(k_2, -\omega_0) = 2 + i(w_y - 1) = A_y, \ \widehat{D_0}(k_1, -\omega_0) = 2 + i(w_x - 1) = A_x.$$
(46)

Before doing further calculations, we have to note that some components of tensors $\hat{P}_{ij}(\mathbf{k}_1)$ and $\hat{P}_{ij}(\mathbf{k}_2)$ vanish. Let us write the non-zero components only:

$$\hat{P}_{yx}(\boldsymbol{k}_{1}) = \frac{1}{2}D, \quad \hat{P}_{xz}(\boldsymbol{k}_{2}) = -\frac{1}{2}D, \quad \hat{P}_{xy}(\boldsymbol{k}_{2}) = -\frac{1}{2}D, \quad \hat{P}_{yz}(\boldsymbol{k}_{1}) = \frac{1}{2}D.$$
(47)

Taking into account the formulae (45)-(47), we can find the determinant:

$$\Delta(\boldsymbol{k}_{1}) = A_{x}^{3} + \frac{1}{2}D^{2}A_{x}, \ \Delta(\boldsymbol{k}_{2}) = A_{y}^{3} + \frac{1}{2}D^{2}A_{y}.$$
(48)

In a similar way we find velocity field of zero approximation:

$$u_{0} = f_{0} \frac{A_{y}}{2} \frac{e^{i\phi_{2}}}{A_{y}^{2} + \frac{1}{2}D^{2}} + f_{0} \frac{D}{4} \frac{e^{i\phi_{1}}}{A_{x}^{2} + \frac{1}{2}D^{2}} + C.C.,$$
(49)

$$v_{0} = -f_{0} \frac{D}{4} \frac{e^{i\varphi_{2}}}{A_{y}^{2} + \frac{1}{2}D^{2}} + f_{0} \frac{A_{x}}{2} \frac{e^{i\varphi_{1}}}{A_{x}^{2} + \frac{1}{2}D^{2}} + C.C.,$$
(50)

$$w_0 = f_0 \frac{D}{4} \frac{e^{i\phi_2}}{A_y^2 + \frac{1}{2}D^2} - f_0 \frac{D}{4} \frac{e^{i\phi_1}}{A_x^2 + \frac{1}{2}D^2} + C.C..$$
(51)

5. Reynolds Stress and Large Scale Instability

To close the Equations (27) we have to calculate the Reynolds stresses w_0u_0 and w_0v_0 . These terms are easily calculated with help of formulae (49)-(51). As a result we obtain:

 $\overline{w_{0}u_{0}} = \frac{f_{0}^{2}}{2} \frac{D}{\left|A_{y}^{2} + \frac{1}{2}D^{2}\right|^{2}} - \frac{f_{0}^{2}}{8} \frac{D^{2}}{\left|A_{x}^{2} + \frac{1}{2}D^{2}\right|^{2}},$ $\overline{w_{0}v_{0}} = -\frac{f_{0}^{2}}{8} \frac{D^{2}}{\left|A_{y}^{2} + \frac{1}{2}D^{2}\right|^{2}} - \frac{f_{0}^{2}}{2} \frac{D}{\left|A_{x}^{2} + \frac{1}{2}D^{2}\right|^{2}}.$ (52)

Now Equations (27) are closed and take form:

$$\partial_T W_x - \Delta W_x + \frac{\partial}{\partial z} \overline{w_0 u_0} = 0,$$

$$\partial_T W_y - \Delta W_y - \frac{\partial}{\partial z} \overline{w_0 v_0} = 0.$$
(53)

We calculate the modules and write the Reynolds stresses (52) in the explicit form:

$$\overline{w_{0}u_{0}} = \frac{f_{0}^{2}}{2} \frac{D}{16(w_{y}-1)^{2} + \left[4 + \frac{1}{2}D^{2} - (w_{y}-1)^{2}\right]^{2}} - \frac{f_{0}^{2}}{8} \frac{D^{2}}{16(w_{x}-1)^{2} + \left[4 + \frac{1}{2}D^{2} - (w_{x}-1)^{2}\right]^{2}},$$

$$\overline{w_{0}v_{0}} = -\frac{f_{0}^{2}}{8} \frac{D^{2}}{16(w_{y}-1)^{2} + \left[4 + \frac{1}{2}D^{2} - (w_{y}-1)^{2}\right]^{2}} - \frac{f_{0}^{2}}{2} \frac{D}{16(w_{x}-1)^{2} + \left[4 + \frac{1}{2}D^{2} - (w_{x}-1)^{2}\right]^{2}}.$$
(54)

With small W_x , W_y Reynolds stresses (52) can be expanded in a series in the small parameters W_x , W_y . Taking into account the formula:

$$\frac{1}{\left|A_{x,y}^{2} + \frac{1}{2}D^{2}\right|^{2}} = \text{Const.} - \frac{32(D^{2} - 10)}{\left[(D^{2} + 6)^{2} + 64\right]^{2}}w_{x,y} + \cdots$$

We obtain the linearized Equations (53):

$$\frac{\partial}{\partial T}W_{x} - \frac{\partial^{2}}{\partial z^{2}}W_{x} - \frac{\alpha f_{0}^{2}D}{2}\frac{\partial}{\partial z}W_{y} + \frac{\alpha f_{0}^{2}D^{2}}{8}\frac{\partial}{\partial z}W_{x} = 0,$$

$$\frac{\partial}{\partial T}W_{y} - \frac{\partial^{2}}{\partial z^{2}}W_{y} + \frac{\alpha f_{0}^{2}D^{2}}{8}\frac{\partial}{\partial z}W_{y} + \frac{\alpha f_{0}^{2}D}{2}\frac{\partial}{\partial z}W_{x} = 0.$$

$$\alpha = \frac{32(10 - D^{2})}{\left[\left(D^{2} + 6\right)^{2} + 64\right]^{2}}.$$
(55)

We will search for the solution of linear system (55) in the form:

$$W_{x}, W_{y} \sim \exp(\gamma T + ikZ).$$
(56)

We substitute (56) in Equation (55) and obtain the dispersion equation:

$$\gamma = -ik\frac{\alpha f_0^2 D^2}{8} \pm k\frac{\alpha f_0^2 D}{2} - k^2.$$
(57)

The dispersion Equation (57) shows that equation system (55) has instable oscillatory solutions with oscillatory frequency $\omega = k \frac{\alpha f_0^2 D^2}{8}$ and instability growth rate $\gamma = k \frac{\alpha f_0^2 D}{2} - k^2$. The instability is large scale because the instable term dominates over dissipation on large scales: $\frac{\alpha f_0^2 D}{2} > k$. The maximum growth rate of

instability is equal to $\gamma_{\text{max}} = \frac{\alpha^2 f_0^4 D^2}{16}$, and is achieved on the wave vector $k_{\text{max}} = \frac{\alpha f_0^2 D}{4}$. As a result of the development of instability the large scale helical circular polarized vortices of Beltrami type are generated in the system.

6. Saturation of Instability and Nonlinear Vortex Structures

It is clear that with increasing of amplitude nonlinear terms decrease and instability becomes saturated. Consequently stationary nonlinear vortex structures are formed. To find these structures let us choose for Equations

(54) $\frac{\partial}{\partial T} = 0$ and integrate equations one time over Z. We obtain the system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}Z}W_{x} = \overline{w_{0}u_{0}} + C_{1},$$

$$\frac{\mathrm{d}}{\mathrm{d}Z}W_{y} = \overline{w_{0}v_{0}} + C_{2}.$$
(58)

From Equations (58) follows:

$$\frac{\mathrm{d}w_x}{\mathrm{d}w_y} = \frac{w_0 u_0 + C_1}{w_0 v_0 + C_2},\tag{59}$$

1

After integrating the system of Equations (59) we obtain:

$$\int \overline{w_0 v_0} dw_x + C_2 w_x = \int \overline{w_0 u_0} dw_y + C_1 w_y.$$
(60)

Integrals in expression (60) are calculated in elementary functions (see [17]), which give the expression for first integral of motion *J* of Equations (59):

$$J = \frac{D^2}{8} \frac{w_x}{\left[4 + \frac{1}{2}D^2 - (w_y - 1)^2\right]^2 + 16(w_y - 1)^2} + \frac{D}{2^{5/2}(8 + D^2)} \ln \frac{(w_x - 1)^2 + (w_x - 1)D\sqrt{2} + 4 + \frac{1}{2}D^2}{(w_x - 1)^2 - (w_x - 1)D\sqrt{2} + 4 + \frac{1}{2}D^2} + \frac{D}{8(8 + D^2)} \operatorname{arctg} \frac{(w_x - 1)^2 - 4 - \frac{1}{2}D^2}{4(w_x - 1)} - \frac{D^2}{8} \frac{w_y}{\left[4 + \frac{1}{2}D^2 - (w_x - 1)^2\right]^2 + 16(w_x - 1)^2} + \frac{D}{8(8 + D^2)} \ln \frac{(w_y - 1)^2 + (w_y - 1)D\sqrt{2} + 4 + \frac{1}{2}D^2}{(w_y - 1)^2 - (w_y - 1)D\sqrt{2} + 4 + \frac{1}{2}D^2} + \frac{D}{8(8 + D^2)} \operatorname{arctg} \frac{(w_y - 1)^2 - 4 - \frac{1}{2}D^2}{4(w_y - 1)} + C_1w_y + C_2w_x.$$

Equations (58) can be easily calculated numerically using standard tools. In particular, this allows to construct phase portrait of the dynamical system (58) (Figure 3) and to get the most interesting solutions which link singular points on phase plane. See for example Figure 1, where the hyperbolic singular point is connected with



the stable knot and **Figure 2**, where the solution connects instable and stable focuses. All these solutions correspond to the large scale localized vortex structures of kink type with rotation, generated by the instability which has been found in this work.

7. Conclusions and Discussion of the Results

In this work we find the new large scale instability in rotating fluid. It is supposed that the small scale vortex external force in rotating coordinates system acts on fluid which maintains the small velocity field fluctuations (small scale turbulence with small Reynolds number R, $R \ll 1$). For the real applications this Reynolds number should be calculated with help of the turbulent viscosity. The asymptotic development of motion equations by small Reynolds number allows obtaining motion equations for the large scale. These equations are of the hydrodynamic α -effect type, in which velocity components W_x , W_y are connected by the positive feedback. This may result in the appearance of the large scale vortex instability. The large scale vortices of Beltrami type are formed due to this instability in rotating fluid with small scale exterior force. With further increase of amplitude, the instability stabilizes and passes to stationary mode. In this mode the nonlinear stationary vortex structures form. Different vortex kinks belong to the most interesting structures. These kinks link stationary points of dynamical system (58). Kinks which link the hyperbolic point with the stable knot rotate around the stable knot as shown on **Figure 1**. In the kink which links instable and stable focuses, vector field turns around two singular points, see **Figure 2**.

Let us note that unlike previous works about hydrodynamic α -effect in rotating fluid, the use of the asymptotic development allows constructing naturally the nonlinear theory and studying the stationary nonlinear vortex kinks.

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