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Global stability for delay SIR epidemic model with vertical transmission

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Abstract—A SIR epidemic model with delay, saturated contact rate and vertical transmission is considered. The basic reproduction number is calculated. It is shown that this number characterizes the disease transmission dynamics: if $R_0 < 1$, there only exists the disease-free equilibrium which is globally asymptotically stable; if $R_0 > 1$, there is a unique endemic equilibrium and the disease persists, sufficient cond- itions are obtained for the global asymptotic stability of the endemic equilibrium.

Keywords- delay; vertical transmission; saturated contact rate; global asymptotic stability; permanence

1. Introduction

In [1], Cooke formulated an SIR model with time delay

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)I(t-\tau) - \mu_1 S(t), \\ \dot{I}(t) = \beta S(t)I(t-\tau) - (\mu_2 + \gamma)I(t), \\ \dot{R}(t) = \gamma I(t) - \mu_3 R(t), \end{cases}$$

where $S(t)$ denote the number of susceptibles, $I(t)$ the number of infectives and the number of the removed. Λ is the recruitment rate. The force of infection at time t is given by $\beta S(t)I(t-\tau)$. β is the average number of contacts per infective per day and $\tau > 0$ is a fixed time during which the infectious agents develop in the vector, and it is only after that time that the infected vector can infect a susceptible human. μ_1, μ_2, μ_3 are positive constants repre- senting the death rates of susceptibles, infectives, and recovered, respectively. γ is the recovery rate of infectives.

Incidence rate plays an important role in the modelling of epidemic dynamics. The bilinear incidence rate and the standard incidence rate $\beta SI/N$ are mostly used. Different incidence rate have been used recently, such as $\beta S^p I^q$ [2], $\beta(N)SI/N$ [3], $\beta SI/(1+\alpha I)$ [4].

Hu et al [5] studied an SIR epidemic model with saturation incidence rate and vertical transmission as follows

$$\begin{cases} \dot{S} = -\beta SI/(1+\alpha S) + \mu(1-S) - \mu(1-\delta)I, \\ \dot{I} = \beta SI/(1+\alpha S) - (\mu\delta + \gamma)I(t). \end{cases} \quad (1)$$

In model (1), it is assumed that the total population is unity, i.e. $S + I + R = 1$, $\mu > 0$ is the birth rate (or death rate), $\beta S/(1+\alpha S)$ $(\beta > 0, \alpha \geq 0)$ is the saturation inci- dence rate, $\gamma > 0$ is the recovery rate, $1-\delta$ $(0 < \delta \leq 1)$ is the proportion of the vertical transmission.

For system (1), they obtained the disease-free equilibrium $E_0(1,0)$. The basic reproduction for (1) is

$$R_0 = \beta/[(1+\alpha)(\gamma + \mu\delta)].$$

If $R_0 > 1$, (1) has a unique endemic equilibrium $E^*(S^*, I^*)$ with $S^* = (\gamma + \mu\delta)/[\beta - \alpha(\gamma + \mu\delta)]$,
$I^* = \mu[\beta - (1+\alpha)(\gamma + \mu\delta)]/[(\gamma + \mu)(\beta - \alpha(\gamma + \mu\delta))]$.

Also, in [5], it is proved when $R_0 \leq 1$, is globally asym- ptotically stable; when $0 \leq \alpha \leq 1$, E^* is globally asymptotic- ally stable.

Motivated by the works [1] and [5], we now consider the delay effect and formulate the following model

$$\begin{cases} \dot{S} = -\beta SI(t-\tau)/(1+\alpha S) + \mu(1-S) - \mu(1-\delta)I, \\ \dot{I} = \beta S(t-\tau)/(1+\alpha S) - (\mu\delta + \gamma)I(t). \end{cases} \quad (2)$$

The initial condition of delay differential equations (2) is given as

$$S(\theta) = \phi_1(\theta) > 0, I(\theta) = \phi_2(\theta) > 0, \theta \in [-\tau, 0], \quad (3)$$

where $(\phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], R^2{}_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $R^2{}_+ = \{(x_1, x_2) : x_i \geq 0, i = 1, 2\}$.

System (2) has the same basic reproduction number and equilibria as those of (1).

2. Global Stability of the Equilibria and the Permanence of the System

Lemma 2.1. (see [6]) Consider the following equation

$$\dot{x}(t) = a\, x(t-\tau) - b\, x(t), \qquad (4)$$

where $a, b, \tau > 0$, $x(t) > 0$ for $-\tau \leq t \leq 0$. We have

(i) if $a < b$, then $\lim_{t\to+\infty} x(t) = 0;$

(ii) if $a > b$, then $\lim_{t\to+\infty} x(t) = +\infty.$

Theorem 2.1. The disease-free equilibrium $E_0(1,0)$ is globa-
lly asymptotically stable for $R_0 < 1$, and unstable for $R_0 > 1$.

Proof. The characteristic equation of the linearized system of (2) at $E_0(1,0)$ is

$$(\lambda + \mu)(\lambda + \gamma + \mu\delta - \beta e^{-\lambda\tau} / (1+\alpha)) = 0. \qquad (5)$$

For $\tau = 0$, we have that

$$(\lambda + \mu)(\lambda + \gamma + \mu\delta - \beta / (1+\alpha)) = 0.$$

Obviously, $\lambda_1 = -\mu, \lambda_2 = (\gamma + \mu\delta)(R_0 - 1)$. Hence, for $R_0 < 1$ the roots of (5) have negative real parts for $\tau = 0$.

Note that $\lambda = 0$ is not a root of (5) while $R_0 < 1$. If (5) has pure imaginary roots $\lambda = \pm i\omega (\omega > 0)$ for some $\tau > 0$, then we have from (5) that

$$\begin{cases} \gamma + \mu\delta = \beta\cos(\omega\tau) / (1+\alpha), \\ \omega = -\beta\sin(\omega\tau) / (1+\alpha). \end{cases}$$

This implies that $\omega^2 = (\gamma + \mu\delta)^2 (R_0{}^2 - 1) < 0$ for $R_0 < 1$. Therefore, any root of (5) must have a negative real part, and hence the disease-free equilibrium ξ is locally asymptotical- ly stable for any time delay $\tau \geq 0$.

Denote $h(\lambda) = \lambda + \gamma + \mu\delta - \beta e^{-\lambda\tau} / (1+\alpha)$, then for $R_0 > 1$, $h(0) = (\gamma + \mu\delta)(1 - R_0) < 0$, $\lim_{\lambda\to+\infty} h(\lambda) = +\infty$, thus $h(\lambda) = 0$ has one positive real root. Therefore, ξ is unstable for $R_0 > 1$.

Also we have

$$\dot{I}(t) \leq \beta I(t-\tau) / (1+\alpha) - (\gamma + \mu\delta) I(t).$$

If $R_0 < 1$, from Lemma 2.1 and the comparison theorem, we obtain $I(t) \to 0$ as $t \to +\infty$. Then the limit equation for $S(t)$ is $\dot{S} = \mu(1 - S)$, which implies $\lim_{t\to+\infty} S(t) = 1$. This completes the proof.

Theorem2.2. If $R_0 > 1$, the endemic equilibrium $E^*(S^*, I^*)$ is locally asymptotically stable.

Proof. It is clear that at $E^*(S^*, I^*)$ the associated transcend-dental characteristic is

$$\lambda^2 + A\lambda + B - (C\lambda + D)e^{-\lambda\tau} = 0, \qquad (6)$$

where $A = \gamma + \mu\delta + \mu + \beta I^* / (1 + \alpha S^*)^2$,

$$B = \mu(\gamma + \mu\delta) + (\mu + \gamma)\beta I^* / (1 + \alpha S^*)^2,$$

$$C = \gamma + \mu\delta, D = \mu(\gamma + \mu\delta).$$

For $\tau = 0$, we have from (6) that

$$\lambda^2 + (A - C)\lambda + B - D = 0. \qquad (7)$$

Since $A - C > 0$, $B - D > 0$, then from Routh-Hurwitz criterion it follows that for $\tau = 0$ both roots of (6) have negative real parts.

Since $B > D$, then $\lambda = 0$ is not a root of (6). If (6) has pure imaginary roots $\lambda = \pm i\omega (\omega > 0)$ for some $\tau > 0$, then we have from (6) that

$$\begin{cases} B - \omega^2 = C\omega\sin(\omega\tau) + D\cos(\omega\tau), \\ A\omega = C\omega\cos(\omega\tau) - D\sin(\omega\tau). \end{cases}$$

We have that

$$g(z) = z^2 + (A^2 - 2B - C^2)z + B^2 - D^2 = 0, \; z = \omega^2.$$

And $A^2 - 2B - C^2$

$$= \mu^2 + 2\mu\delta\beta I^* / (1 + \alpha S^*)^2 + (\beta I^*)^2 / (1 + \alpha S^*)^4 > 0,$$

since $B > D > 0$, then we have $B^2 - D^2 > 0$. Hence, we have that $g(z) > 0$ for any $z > 0$, this is a contradicts to $g(z) = 0$. This shows that all roots of the characteristic (6) have negative real parts for any time delay $\tau \geq 0$. This completes the proof.

Theorem 2.3. If $R_0 > 1$, the unique endemic equilibrium E^* is globally asymptotically stable whenever $I \leq I^*$, $S \leq S^*$ or $I \geq I^*, S \geq S^*$.

Proof. Denote $f(S,I) = \beta SI / (1 + \alpha S)$. Consider the following Lyapunov functional

$$\begin{aligned} V(t) = \; & S(t) - S^* - f(S^*, I^*) \int_{S^*}^{S(t)} 1 / f(\theta, I^*) d\theta \\ & + I - I^* - I^* \ln(I / I^*) + f(S^*, I^*) \\ & \times \int_{t-\tau}^{t} [I(\theta) / I^* - 1 - \ln(I(\theta) / I^*)] d\theta. \end{aligned}$$

Differentiating this function with respect to time yields

$$\begin{aligned} \dot{V}(t) = \; & \mu S^* (1 - S(t) / S^*)[1 - f(S^*, I^*) / f(S(t), I^*)] \\ & + \mu(1-\delta)(I^* - I(t))[1 - f(S^*, I^*) / f(S(t), I^*)] \\ & + f(S^*, I^*)[1 - f(S^*, I^*) / f(S(t), I^*) \\ & + \ln f(S^*, I^*) / f(S(t), I^*)] + f(S^*, I^*) \\ & \times [1 - (I^* / I(t))(f(S(t), I(t-\tau)) / f(S^*, I^*)) \\ & + \ln((I^* / I(t))(f(S(t), I(t-\tau)) / f(S^*, I^*)))], \end{aligned}$$

in the above equation, we have used the fact that

$$\begin{aligned} & \ln(I(t-\tau) / I(t)) \\ & = \ln(f(S^*, I^*) / f(S(t), I^*)) \\ & + \ln(I^* f(S(t), I(t-\tau)) / (I(t) f(S^*, I^*))) \\ & + \ln(I(t-\tau) f(S(t), I^*) / (I^* f(S(t), I(t-\tau)))) \end{aligned}$$

and $f(S(t), I^*) / f(S(t), I(t-\tau)) = I^* / I(t-\tau)$.

From the monotonicity of the function $f(S,I)$ with respect to S, we have

$$\mu S^* (1 - S(t) / S^*)[1 - f(S^*, I^*) / f(S(t), I^*)] \leq 0.$$

The conditions in this theorem implies that

$$\begin{aligned} & \mu(1-\delta)(I^* - I(t))[1 - f(S^*, I^*) / f(S(t), I^*)] \\ & = \mu(1-\delta)(I^* - I(t))(S(t) - S^*) / (S(t)(1 + \alpha S^*)) \leq 0. \end{aligned}$$

Since the function $H(x) = x - 1 - \ln(x) \leq 0$ for any $x > 0$, and $H(x) = 0$ if and only if $x = 1$. Therefore, $\dot{V}(t) \leq 0$ and the equality holds only at $S = S^*, I = I^*$. Hence, from Kuang [7] (1993, Corollary 5.2, p. 30), we have that E^* is globally asymptotically stable under the condition $R_0 > 1$. This completes the proof of Theorem 2.3.

From the above proof, we obtain the following result.

Corollary 2.1. If $R_0 > 1$, $\delta = 1$, then the unique endemic equilibrium E^* is globally asymptotically stable.

Corollary 2.1 shows that if there is no vertical transmission, when the endemic equilibrium exists, it is globally asymptotically stable.

We now consider the permanence of system (2). Using the same method as that in Theorem 3.2 of [8], we get the following result.

Theorem 2.4. If $R_0 > 1$, then there exists an $\varepsilon > 0$ such that every solution $(S(t), I(t))$ of system (2) with initial conditions (3) satisfies $\liminf_{t \to +\infty} I(t) \geq \varepsilon$.

It is easy to obtain that $\liminf_{t \to +\infty} S(t) \geq \mu\delta / (\mu + \beta)$, and also $S(t) \leq 1, I(t) \leq 1$, from Theorem 2.4, we have

Theorem 2.5. If $R_0 > 1$, then system (2) is permanent.

3. Discussion

In this paper, a SIR epidemic model with delay, saturated contact rate and vertical transmission is investigated. We obtained the basic reproduction number R_0. It is shown that if $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable; if $R_0 > 1$, there is a endemic equilibrium and the system is permanent. When $R_0 > 1$, in Theorem 2.3, by constructing a Lyapunov functional, sufficient conditions are obtained for the global asymptotic stability of the endemic equilibrium. We conjecture that for any $0 < \delta \leq 1$, when the endemic equilibrium exists, it is globally asymptotically stable. New technique is required to prove this.

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REFERENCES

[1] K. L. Cooke, Stability analysis for a vector disease model, Rocky Mountain J. Math., vol. 9, 1979, pp. 31-42.
[2] M. Y. Li, J. S. Muldowney, Global stability for the SEIR model in epidemiology, Math. Biosci., vol.125, 1995, pp. 155-168.
[3] J. Zhang, Z. E. Ma, Gobal dynamics of an SEIR epidemic moel with saturating contact rate, Mathy. Biosci., vol.185, 2003, pp. 15-32.
[4] R. Xu, Z. E. Ma, Global stability of a SIR epidemic model with nonlinear incidence rate and time delay, Nonlinear Anal. Real World Appli., vol. 10, 2009, pp. 3175-3189.
[5] X. L. Hu, F. G. Sun and C. X. Wang, Global analysis of SIR epidemic model with the aturated contact rate and vertical transmission, Basic Sciences Journal of Textile Universities, vol.23, 2010, pp. 120-122.
[6] Y. N. Xiao and L. S. Chen, Modeling and analysis of a predator-prey model with disease in the prey}, Math. Biosci., vol.171, 2001, pp. 59-82.
[7] Y. Kuang, Delay differential equations with applications in population dynamics, Boston: Academic Press, 1993.
[8] J. L. Liu, T. L. Zhang, J. X. Lu, An impulsive controlled eco-epidemic model with disease in the prey, J. Appl. Math. Comput., to appear.

