

# Robust Optimal $H_{\infty}$ Control for Uncertain 2-D Discrete State-Delayed Systems Described by the General Model

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# Abstract

This paper investigates the problem of robust optimal  $H_{\infty}$  control for uncertain two-dimensional (2-D) discrete state-delayed systems described by the general model (GM) with norm-bounded uncertainties. A sufficient condition for the existence of  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controllers is established, based on linear matrix inequality (LMI) approach. Moreover, a convex optimization problem is developed to design a robust optimal  $H_{\infty}$  state feedback controller which minimizes the  $H_{\infty}$  noise attenuation level of the resulting closed-loop system. Finally, two illustrative examples are given to demonstrate the effectiveness of the proposed method.

# **Keywords**

2-D Discrete Systems, General Model,  $H_{\infty}$  Control, Linear Matrix Inequality, State Feedback, Uncertain System

# **1. Introduction**

Over the past decades, the problem of  $H_{\infty}$  control for 2-D discrete systems has drawn considerable attention. The main advantage of  $H_{\infty}$  control is that its performance specification takes into account the worst-case performance of the system in terms of the system energy gain [1]. Based on this idea, many important results have been obtained in the literature [2]-[5]. Among these results, the problem of  $H_{\infty}$  control and robust stabilization of 2-D discrete systems described by the Roesser model has been addressed in [2]. A solution to the problem of robust  $H_{\infty}$  control for uncertain 2-D discrete systems represented by the general model (GM) via output feedback controllers has been presented in [3]. A 2-D filtering approach, based on the 2-D bounded real lemma, with

an  $H_{\infty}$  performance measure for 2-D discrete systems described by the Fornasini-Marchesini (FM) second model has been developed in [4]. The dynamic output feedback  $H_{\infty}$  stabilization problem for a class of 2-D discrete switched systems represented by the FM second model has been addressed in [5].

It is well known that delay is encountered in many dynamic systems and is often a source of instability, thus, much attention has been focused on the problem of stability analysis and controller design for 2-D discrete state-delayed systems in the last few years [6]-[25]. In [6], the problem of stability analysis for 2-D discrete state-delayed systems in the GM has been considered and sufficient conditions for stability have been derived via Lyapunov approach. The problem of delay-dependent guaranteed cost control for uncertain 2-D discrete state-delayed system described by the FM second model has been presented in [7]. In [8], the problem of robust guaranteed cost control for uncertain 2-D discrete state-delayed systems described by the FM second model has been considered. Several corrections in the main results of [8] have been made in [9]. In [10], the guaranteed cost control problem via memory state feedback control laws for a class of uncertain 2-D discrete state-delayed systems described by the FM second model has been discussed. Robust reliable control of uncertain 2-D discrete switched state-delayed systems described by the Roesser model has been presented in [11]. The problem of positive real control for 2-D discrete state-delayed systems described by the FM second model via output feedback controllers has been addressed in [12]. In [13], the problem of delay-dependent  $H_{\infty}$  control for 2-D discrete state-delayed system described by the FM second model has been investigated. The problem of  $H_{\infty}$  control for 2-D discrete state-delayed systems described by the FM second model has been studied in [14] and a method to design an optimal  $H_{\infty}$  state feedback controller has been presented. Here, it may be mentioned that [14] considers the FM second model without uncertainties, but in the real world situation, the uncertainties in the system parameters cannot be avoided.

With this motivation, we consider the problem of robust optimal  $H_{\infty}$  control for uncertain 2-D discrete state-delayed systems described by the GM. The approach adopted in this paper is as follows: We first establish a sufficient condition for the existence of  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controllers in terms of a certain linear matrix inequality (LMI). Further, a convex optimization problem is introduced to select a robust optimal  $H_{\infty}$  state feedback controller which minimizes the  $H_{\infty}$  noise attenuation level  $\gamma$  of the closed-loop system. Finally, two illustrative examples are given to demonstrate the effectiveness of the proposed technique.

## 2. Problem Formulation and Preliminaries

The following notations are used throughout the paper:

- $R^n$  real vector space of dimension *n*.
- $R^{n \times m}$  set of  $n \times m$  real matrices.
- 0 null matrix or null vector of appropriate dimension.
- *I* identity matrix of appropriate dimension.
- $\boldsymbol{G}^{\mathrm{T}}$  transpose of matrix  $\boldsymbol{G}$ .
- $diag \{...\}$  stands for a block diagonal matrix.
- G > 0 matrix G positive definite symmetric.
- G < 0 matrix G negative definite symmetric.

Consider the uncertain 2-D discrete state-delayed systems described by the GM [26].

$$\mathbf{x}(i+1, j+1) = \overline{A}_{1}\mathbf{x}(i, j+1) + \overline{A}_{2}\mathbf{x}(i+1, j) + \overline{A}_{0}\mathbf{x}(i, j) + \overline{A}_{1d}\mathbf{x}(i-d_{1}, j+1) + \overline{A}_{2d}\mathbf{x}(i+1, j-d_{2}) + \overline{A}_{0d}\mathbf{x}(i-k_{1}, j-k_{2}) + \overline{B}_{1}\mathbf{w}(i, j+1) + \overline{B}_{2}\mathbf{w}(i+1, j) + \overline{B}_{0}\mathbf{w}(i, j) + \overline{C}_{1}\mathbf{u}(i, j+1) + \overline{C}_{2}\mathbf{u}(i+1, j) + \overline{C}_{0}\mathbf{u}(i, j),$$

$$(1a)$$

$$z(i, j) = Hx(i, j) + Lw(i, j), \qquad (1b)$$

where  $0 \le i, j \in \mathbb{Z}$  are horizontal and vertical coordinates,  $x(i, j) \in \mathbb{R}^n$ ,  $u(i, j) \in \mathbb{R}^m$  represent the state and control input, respectively,  $z(i, j) \in \mathbb{R}^p$  is the controlled output,  $w(i, j) \in \mathbb{R}^q$  is the noise input which belongs to  $\ell_2\{[0, \infty), [0, \infty)\}$  and

$$\overline{A}_{1} = (A_{1} + \Delta A_{1}), \ \overline{A}_{2} = (A_{2} + \Delta A_{2}), \ \overline{A}_{0} = (A_{0} + \Delta A_{0}), \ \overline{A}_{1d} = (A_{1d} + \Delta A_{1d}), 
\overline{A}_{2d} = (A_{2d} + \Delta A_{2d}), \ \overline{A}_{0d} = (A_{0d} + \Delta A_{0d}), \ \overline{B}_{1} = (B_{1} + \Delta B_{1}), \ \overline{B}_{2} = (B_{2} + \Delta B_{2}),$$

$$(1c)$$

$$\overline{B}_{0} = (B_{0} + \Delta B_{0}), \ \overline{C}_{1} = (C_{1} + \Delta C_{1}), \ \overline{C}_{2} = (C_{2} + \Delta C_{2}), \ \overline{C}_{0} = (C_{0} + \Delta C_{0}).$$

The matrices  $A_1, A_2, A_0, A_{1d}, A_{2d}, A_{0d} \in \mathbb{R}^{n \times n}$ ,  $B_1, B_2, B_0 \in \mathbb{R}^{n \times q}$ ,  $C_1, C_2, C_0 \in \mathbb{R}^{n \times m}$ ,  $H \in \mathbb{R}^{p \times n}$  and  $L \in \mathbb{R}^{p \times q}$  are known constant matrices representing the nominal plant;  $d_1, d_2, k_1$  and  $k_2$  are constant positive integers representing delays. The matrices  $\Delta A_1, \Delta A_2, \Delta A_0, \Delta B_1, \Delta B_2, \Delta B_0, \Delta C_1, \Delta C_2$  and  $\Delta C_0$  represent parameter uncertainties in the system matrices, which are assumed to be of the form

$$\begin{bmatrix} \Delta \mathbf{A}_1 & \Delta \mathbf{A}_2 & \Delta \mathbf{A}_0 & \Delta \mathbf{A}_{1d} & \Delta \mathbf{A}_{2d} & \Delta \mathbf{A}_{0d} \end{bmatrix} = \mathbf{H}_0 \mathbf{F} (i, j) \begin{bmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 & \mathbf{E}_{1d} & \mathbf{E}_{2d} & \mathbf{E}_{0d} \end{bmatrix},$$

$$\begin{bmatrix} \Delta \mathbf{B}_1 & \Delta \mathbf{B}_2 & \Delta \mathbf{B}_0 \end{bmatrix} = \mathbf{H}_0 \mathbf{F} (i, j) \begin{bmatrix} \mathbf{E}_4 & \mathbf{E}_5 & \mathbf{E}_6 \end{bmatrix}, \begin{bmatrix} \Delta \mathbf{C}_1 & \Delta \mathbf{C}_2 & \Delta \mathbf{C}_0 \end{bmatrix} = \mathbf{H}_0 \mathbf{F} (i, j) \begin{bmatrix} \mathbf{E}_7 & \mathbf{E}_8 & \mathbf{E}_9 \end{bmatrix},$$
(1d)

where  $H_0 \in R^{n \times k}$ ,  $E_1, E_2, E_3, E_{1d}, E_{2d}, E_{0d} \in R^{l \times n}$ ,  $E_4, E_5, E_6 \in R^{l \times q}$ , and  $E_7, E_8, E_9 \in R^{l \times m}$  are known structural matrices of uncertainty and  $F(i, j) \in R^{k \times l}$  is an unknown matrix representing parameter uncertainty which satisfies

$$\boldsymbol{F}^{\mathrm{T}}(i,j)\boldsymbol{F}(i,j) \leq \boldsymbol{I}$$
 (or equivalently,  $\|\boldsymbol{F}(i,j)\| \leq 1$ ). (1e)

It is assumed that the system (1) has a finite set of initial conditions [6], *i.e.*, there exist two positive integers  $r_1$  and  $r_2$ , such that

$$\begin{aligned} & \boldsymbol{x}(i, j) = \boldsymbol{h}_{ij}, \forall \ 0 \le j < r_1, \quad i = -d_1, -d_1 + 1, \cdots, 0 \\ & \boldsymbol{x}(i, j) = \boldsymbol{v}_{ij}, \forall \ 0 \le i < r_2, \quad i = -d_2, -d_2 + 1, \cdots, 0 \\ & \boldsymbol{x}(i, j) = \boldsymbol{w}_{ij}, \forall \ i = [-k_1, 0], \ j = [-k_2, 0] \\ & \boldsymbol{h}_{00} = \boldsymbol{v}_{00} = \boldsymbol{w}_{00}. \end{aligned}$$

$$(2)$$

**Definition 1** [14]. The system described by (1) is asymptotically stable if  $\lim X_r = 0$  with w(i, j) = 0, u(i, j) = 0 and the initial condition (2), where  $X_r = \sup\{||\mathbf{x}(i, j)||: i + j = r, i, j \in \mathbb{Z}\}$ . **Definition 2** [14]. Consider the system (1) with u(i, j) = 0 and the initial condition (2). Given a scalar

**Definition 2** [14]. Consider the system (1) with u(i, j) = 0 and the initial condition (2). Given a scalar  $\gamma > 0$ , and symmetric positive definite matrices  $Q_1, Q_2, Q_3, Z_1, Z_2, Z_3 \in \mathbb{R}^{n \times n}$ , the system (1) is said to have an  $H_{\infty}$  noise attenuation  $\gamma$  if it is robustly stable and satisfies

$$J = \sup_{0 \neq w \in \ell_{2}} \frac{\|\overline{z}\|_{2}^{2}}{\|\overline{w}\|_{2}^{2} + D_{1}(d_{1}, j) + D_{2}(i, d_{2}) + D_{3}(k_{1}, k_{2})} < \gamma^{2}$$
(3)  
where  $\|\overline{z}\|_{2}^{2} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} z(i, j+1) \\ z(i, j) \\ z(i, j) \end{bmatrix} \right\|^{2}, \quad \|\overline{w}\|_{2}^{2} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left\| \begin{bmatrix} w(i, j+1) \\ w(i, j) \\ \end{bmatrix} \right\|^{2}$ 
$$D_{1}(d_{1}, j) = \sum_{j=0}^{\infty} \left[ x^{T}(0, j+1)Q_{1}x(0, j+1) + \sum_{l=-d_{1}}^{-1} x^{T}(l, j+1)Z_{1}x(l, j+1) \right],$$
$$D_{2}(i, d_{2}) = \sum_{i=0}^{\infty} \left[ x^{T}(i+1, 0)Q_{2}x(i+1, 0) + \sum_{l=-d_{2}}^{-1} x^{T}(i+1, l)Z_{2}x(i+1, l) \right]$$
and
$$D_{3}(k_{1}, k_{2}) = \sum_{i=0}^{\infty} \left[ x^{T}(i, 0)Q_{3}x(i, 0) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{2}}^{-1} (x^{T}(i+m, n)Z_{3}x(i+m, n)) \right]$$
$$+ \sum_{j=0}^{\infty} \left[ x^{T}(0, j)Q_{3}x(0, j) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{2}}^{-1} (x^{T}(m, j+n)Z_{3}x(m, j+n)) \right].$$

The following well established lemmas are essential for the proof of our main results. **Lemma 1** [27]-[29]. Let  $A \in R^{n \times n}$ ,  $D \in R^{n \times k}$ ,  $E \in R^{l \times n}$  and  $Q = Q^{T} \in R^{n \times n}$  be given matrices. Then, there

(8)

exist a positive definite matrix P such that

$$\left[A + DFE\right]^{\mathrm{T}} P\left[A + DFE\right] - Q < 0 \tag{4}$$

for all F satisfying  $F^{T}(i, j)F(i, j) \leq I$ . if and only if there exists a scalar  $\varepsilon > 0$  such that

$$\begin{bmatrix} -\boldsymbol{P}^{-1} + \varepsilon \boldsymbol{D} \boldsymbol{D}^{\mathrm{T}} & \boldsymbol{A} \\ \boldsymbol{A}^{\mathrm{T}} & \varepsilon^{-1} \boldsymbol{E}^{\mathrm{T}} \boldsymbol{E} - \boldsymbol{Q} \end{bmatrix} < \boldsymbol{0}.$$
 (5)

**Lemma 2** [30]. For real matrices M, L, Q of appropriate dimension, where  $M = M^{T}$  and  $Q = Q^{T} > 0$  then  $M + L^{T}QL < 0$ , if and only if

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{L}^{\mathrm{T}} \\ \boldsymbol{L} & -\boldsymbol{Q}^{-1} \end{bmatrix} < \boldsymbol{0}$$
(6)

or equivalently

$$\begin{bmatrix} -\boldsymbol{Q}^{-1} & \boldsymbol{L} \\ \boldsymbol{L}^{\mathrm{T}} & \boldsymbol{M} \end{bmatrix} < \boldsymbol{0}.$$
 (7)

### 3. Main Results

## 3.1. Stability and $H_{\infty}$ Performance Analysis

The following theorem gives a sufficient condition for the system (1) to have a specified  $H_{\infty}$  noise attenuation.

**Theorem 1.**Consider the system (1) with u(i, j) = 0 and initial condition (2), for a given positive scalar  $\gamma$ , if there exist symmetric positive definite matrices  $P, P_1, P_2, R_1, R_2, R_3 \in \mathbb{R}^{n \times n}$ , satisfying  $P_1 > \gamma^2 Q_1$ ,  $P_2 > \gamma^2 Q_2$ ,  $0 < P - P_1 - P_2 > \gamma^2 Q_3$ ,  $R_1 < \gamma^2 Z_1$ ,  $R_2 < \gamma^2 Z_2$ , and  $R_3 < \gamma^2 Z_3$ , such that the following matrix inequality

$$\left( \begin{bmatrix} \overline{A}_{1}^{\mathrm{T}} \\ \overline{A}_{2}^{\mathrm{T}} \\ \overline{A}_{0}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{2}^{\mathrm{T}} \\ \overline{B}_{0}^{\mathrm{T}} \\ \overline{B}_{1}^{\mathrm{T}} \\ \overline{B}_{1$$

holds, then the system (1) is asymptotically stable and has a specified  $H_{\infty}$  noise attenuation  $\gamma$ .

**Proof:** To prove that the system (1) is asymptotically stable, we choose a Lyapunov-Krasovskii functional [14]

$$V(\mathbf{x}(i,j)) = V_1(\mathbf{x}(i,j)) + V_2(\mathbf{x}(i,j)) + V_3(\mathbf{x}(i,j)),$$
(9)

where

$$V_{1}(\mathbf{x}(i,j)) = \mathbf{x}^{\mathrm{T}}(i,j)\mathbf{P}_{1}\mathbf{x}(i,j) + \sum_{l=-d_{1}}^{-1} \mathbf{x}^{\mathrm{T}}(i+l,j)\mathbf{R}_{1}\mathbf{x}(i+l,j),$$
$$V_{2}(\mathbf{x}(i,j)) = \mathbf{x}^{\mathrm{T}}(i,j)\mathbf{P}_{2}\mathbf{x}(i,j) + \sum_{l=-d_{2}}^{-1} \mathbf{x}^{\mathrm{T}}(i,j+l)\mathbf{R}_{2}\mathbf{x}(i,j+l),$$
$$V_{3}(\mathbf{x}(i,j)) = \mathbf{x}^{\mathrm{T}}(i,j)(\mathbf{P}-\mathbf{P}_{1}-\mathbf{P}_{2})\mathbf{x}(i,j) + \sum_{m=-k_{1}}^{-1} \sum_{m=-k_{2}}^{-1} \mathbf{x}^{\mathrm{T}}(i+m,j+n)\mathbf{R}_{3}\mathbf{x}(i+m,j+n).$$

It is explicit that  $V(\mathbf{x}(i, j)) > 0$ .

The forward difference along any trajectory of the system (1) with u(i, j) = 0 and w(i, j) = 0 is given by  $\Delta V(i+1, j+1) = V_1(x(i+1, j+1)) + V_2(x(i+1, j+1)) + V_3(x(i+1, j+1))$ 

$$-V_{1}\left(\boldsymbol{x}\left(i,j+1\right)\right) - V_{2}\left(\boldsymbol{x}\left(i+1,j\right)\right) - V_{3}\left(\boldsymbol{x}\left(i,j\right)\right)$$

$$= \begin{bmatrix} \boldsymbol{x}\left(i,j+1\right) \\ \boldsymbol{x}\left(i+1,j\right) \\ \boldsymbol{x}\left(i-d_{1},j+1\right) \\ \boldsymbol{x}\left(i-d_{1},j+1\right) \\ \boldsymbol{x}\left(i-k_{1},j-k_{2}\right) \end{bmatrix}^{\mathrm{T}} \left( \begin{bmatrix} \overline{A}_{1}^{\mathrm{T}} \\ \overline{A}_{2}^{\mathrm{T}} \\ \overline{A}_{1d}^{\mathrm{T}} \\ \overline{A}_{2d}^{\mathrm{T}} \\ \overline{A}_{2d}^{$$

Applying Lemma 2 on matrix inequality (8), we obtain

-

$$\begin{bmatrix} \bar{A}_{1}^{\mathrm{T}} \\ \bar{A}_{2}^{\mathrm{T}} \\ \bar{A}_{0}^{\mathrm{T}} \\ \bar{A}_{0d}^{\mathrm{T}} \\ \bar{A}_{2d}^{\mathrm{T}} \\ \bar{A}_{0d}^{\mathrm{T}} \end{bmatrix} P \begin{bmatrix} \bar{A}_{1}^{\mathrm{T}} \\ \bar{A}_{2}^{\mathrm{T}} \\ \bar{A}_{1d}^{\mathrm{T}} \\ \bar{A}_{2d}^{\mathrm{T}} \\ \bar{A}_{2d}^{\mathrm{T}} \\ \bar{A}_{0d}^{\mathrm{T}} \end{bmatrix}^{1} + \begin{bmatrix} -P_{1} + R_{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -P_{2} + R_{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -P + P_{1} + P_{2} + R_{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -R_{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R_{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_{3} \end{bmatrix} < 0.$$
(11)

Thus, from (11), it implies that  $\Delta V(i+1, j+1) < 0$ . Hence, system (1) is asymptotically stable.

In order to establish the  $H_{\infty}$  performance of the system (1) with the control input u(i, j) = 0 for  $w(i, j) \in \ell_2\{[0, \infty), [0, \infty)\}$ , we consider

$$\Delta V(i+1,j+1) + \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z(i,j+1) \\ z(i,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z(i,j+1) \\ z(i,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \frac{A_{1}^{\mathrm{T}}}{A_{2}^{\mathrm{T}}} \\ \frac{A_{1}^{\mathrm{T}}}{A_{1}^{\mathrm{T}}} \\ \frac{A_{1}^{\mathrm{T}}}{B_{1}^{\mathrm{T}}} \\ \frac{A_{1}^{\mathrm{T}}}{B_{1}^{\mathrm{T}}} \\ \frac{B_{1}^{\mathrm{T}}}{B_{1}^{\mathrm{T}}} \\ \frac{B_{1$$

It follows from matrix inequality (8) that

$$\Delta V(i+1,j+1) + \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix} - \gamma^{2} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix} < 0.$$
(13)

Summing the inequality (13) over  $i, j = 0 \rightarrow \infty$ , we get

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \Delta V(i+1,j+1) + \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} z(i,j+1) \\ z(i+1,j) \\ z(i,j) \end{bmatrix} - \gamma^{2} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} w(i,j+1) \\ w(i+1,j) \\ w(i,j) \end{bmatrix} \right) < 0.$$
(14)

which implies

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V\left(i+1, j+1\right) + \left\|\overline{z}\right\|_{2}^{2} - \gamma^{2} \left\|\overline{w}\right\|_{2}^{2} < 0$$
(15)

Inequality (15) can be re-written as

$$\begin{aligned} \|\overline{\mathbf{z}}\|_{2}^{2} - \gamma^{2} \|\overline{\mathbf{w}}\|_{2}^{2} &< -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \Delta V(i+1,j+1) \\ &= \sum_{j=0}^{\infty} \left[ \left( \mathbf{x}^{\mathrm{T}}(0,j+1) \mathbf{P}_{1} \mathbf{x}(0,j+1) + \sum_{l=-d_{1}}^{-1} \mathbf{x}^{\mathrm{T}}(l,j+1) \mathbf{R}_{1} \mathbf{x}(l,j+1) \right) \right] \\ &+ \sum_{i=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}}(i+1,0) \mathbf{P}_{2} \mathbf{x}(i+1,0) + \sum_{l=-d_{2}}^{-1} \mathbf{x}^{\mathrm{T}}(i+1,l) \mathbf{R}_{2} \mathbf{x}(i+1,l) \right) \right] \\ &+ \sum_{i=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}}(i,0) (\mathbf{P} - \mathbf{P}_{1} - \mathbf{P}_{2}) \mathbf{x}(i,0) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{1}}^{-1} \mathbf{x}^{\mathrm{T}}(i+m,n) \mathbf{R}_{3} \mathbf{x}(i+m,n) \right) \right] \\ &+ \sum_{j=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}}(0,j) (\mathbf{P} - \mathbf{P}_{1} - \mathbf{P}_{2}) \mathbf{x}(0,j) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{1}}^{-1} \mathbf{x}^{\mathrm{T}}(m,j+n) \mathbf{R}_{3} \mathbf{x}(m,j+n) \right) \right] \end{aligned}$$

Since  $P_1 < \gamma^2 Q_1$ ,  $P_2 < \gamma^2 Q_2$ ,  $P - P_1 - P_2 < \gamma^2 Q_3$ ,  $R_1 < \gamma^2 Z_1$ ,  $R_2 < \gamma^2 Z_2$ , and  $R_3 < \gamma^2 Z_3$ , the inequality (16) leads to

$$\begin{aligned} \left\| \overline{\mathbf{z}} \right\|_{2}^{2} &\leq \gamma^{2} \left\{ \left\| \overline{\mathbf{w}} \right\|_{2}^{2} + \sum_{j=0}^{\infty} \left[ \left( \mathbf{x}^{\mathrm{T}} \left( 0, j+1 \right) \mathbf{Q}_{1} \mathbf{x} \left( 0, j+1 \right) + \sum_{l=-d_{1}}^{-1} \mathbf{x}^{\mathrm{T}} \left( l, j+1 \right) \mathbf{R}_{1} \mathbf{x} \left( l, j+1 \right) \right) \right] \\ &+ \sum_{i=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}} \left( i+1,0 \right) \mathbf{Q}_{2} \mathbf{x} \left( i+1,0 \right) + \sum_{l=-d_{2}}^{-1} \mathbf{x}^{\mathrm{T}} \left( i+1,l \right) \mathbf{R}_{2} \mathbf{x} \left( i+1,l \right) \right) \right] \\ &+ \sum_{i=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}} \left( i,0 \right) \mathbf{Q}_{3} \mathbf{x} \left( i,0 \right) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{2}}^{-1} \mathbf{x}^{\mathrm{T}} \left( i+m,n \right) \mathbf{R}_{3} \mathbf{x} \left( i+m,n \right) \right) \right] \right] \\ &+ \sum_{j=0}^{\infty} \left[ \left[ \left( \mathbf{x}^{\mathrm{T}} \left( 0,j \right) \mathbf{Q}_{3} \mathbf{x} \left( 0,j \right) + \sum_{m=-k_{1}}^{-1} \sum_{n=-k_{1}}^{-1} \mathbf{x}^{\mathrm{T}} \left( m,j+n \right) \mathbf{R}_{3} \mathbf{x} \left( m,j+n \right) \right) \right] \right]. \end{aligned}$$

Therefore, it follows from Definition 2 that the result of Theorem 1 is true. This completes the proof of Theorem 1.

When we consider the case of zero initial condition, then  $H_{\infty}$  performance measure (3) reduces to

$$\boldsymbol{J}_{0} = \sup_{0 \neq w \in I_{2}} \frac{\left\| \boldsymbol{\overline{z}} \right\|_{2}}{\left\| \boldsymbol{\overline{w}} \right\|_{2}} < \gamma.$$
(18)

Using the 2-D Parseval's theorem [31], equation (18) is equivalent to

$$\left\|\boldsymbol{G}\left(\boldsymbol{z}_{1},\boldsymbol{z}_{2}\right)\right\|_{\infty} = \sup_{\boldsymbol{\omega}_{1},\boldsymbol{\omega}_{2} \in [0,2\pi]} \sigma_{\max}\left[\boldsymbol{G}\left(\boldsymbol{e}^{j\boldsymbol{\omega}_{1}},\boldsymbol{e}^{j\boldsymbol{\omega}_{2}}\right)\right] < \gamma,$$
(19)

where  $\sigma_{\max}(.)$  represents the maximum singular value of the corresponding matrix and the transfer function from the noise input w(i, j) to the controlled output z(i, j) for the system (1) is

$$G(z_{1}, z_{2}) = H(z_{1}z_{2}I_{n} - z_{2}\overline{A}_{1} - z_{1}\overline{A}_{2} - \overline{A}_{0} - z_{1}^{-d_{1}}z_{2}\overline{A}_{1d} - z_{1}z_{2}^{-d_{2}}\overline{A}_{2d} - z_{1}^{-k_{1}}z_{2}^{-k_{2}}\overline{A}_{0d})^{-1} \times (z_{2}\overline{B}_{1} + z_{1}\overline{B}_{2} + \overline{B}_{0}) + L.$$
(20)

# 3.2. Robust Optimal $H_{\infty}$ Controller Design

Consider the system (1) and the following state feedback controller

$$\boldsymbol{u}(i,j) = \boldsymbol{K}\boldsymbol{x}(i,j). \tag{21}$$

(23)

Applying the controller (21) to system (1) results in the following closed-loop system:

$$\boldsymbol{x}(i+1,j+1) = (\bar{\boldsymbol{A}}_{1} + \bar{\boldsymbol{C}}_{1}\boldsymbol{K})\boldsymbol{x}(i,j+1) + (\bar{\boldsymbol{A}}_{2} + \bar{\boldsymbol{C}}_{2}\boldsymbol{K})\boldsymbol{x}(i+1,j) + (\bar{\boldsymbol{A}}_{0} + \bar{\boldsymbol{C}}_{0}\boldsymbol{K})\boldsymbol{x}(i,j) + \bar{\boldsymbol{A}}_{1d}\boldsymbol{x}(i-d_{1},j+1) + \bar{\boldsymbol{A}}_{2d}\boldsymbol{x}(i+1,j-d_{2}) + \bar{\boldsymbol{A}}_{0d}\boldsymbol{x}(i-k_{1},j-k_{2}) + \bar{\boldsymbol{B}}_{1}\boldsymbol{w}(i,j+1) + \bar{\boldsymbol{B}}_{2}\boldsymbol{w}(i+1,j) + \bar{\boldsymbol{B}}_{0}\boldsymbol{w}(i,j),$$
(22a)

$$\boldsymbol{z}(i,j) = \boldsymbol{H}\boldsymbol{x}(i,j) + \boldsymbol{L}\boldsymbol{w}(i,j).$$
(22b)

The following theorem presents a sufficient condition for the existence of a controller of the form (21) such that the closed-loop system (22) is asymptotically stable and the  $H_{\infty}$  norm of transfer function (20) from the noise input w(i, j) to the controlled output z(i, j) for the closed-loop system (22) is smaller than  $\gamma$ . Such controller is said to be a  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controller for system (1).

**Theorem 2.** Consider the system (1) and initial condition (2). Given scalars  $\gamma > 0$  and  $\varepsilon > 0$ , if there exist a matrix  $N \in R^{m \times n}$  and symmetric positive definite matrices  $\overline{P}, \overline{P}_1, \overline{P}_2, \overline{R}_1, \overline{R}_2, \overline{R}_3 \in R^{n \times n}$  such that

$\left[-\overline{P}_{1}+\right]$	$\overline{R}_1$	0		0		0	0	0	0	0	0	$\overline{P}A_1^{\mathrm{T}} + N^{\mathrm{T}}C_1^{\mathrm{T}}$	$\overline{P}H^{\mathrm{T}}$
*		$-\overline{P}_2 + \overline{R}$	2	0		0	0	0	0	0	0	$\overline{P}A_2^{\mathrm{T}} + N^{\mathrm{T}}C_2^{\mathrm{T}}$	0
*		*	$-\overline{P}$ +	$\overline{P}_1 + \overline{P}_2$	$+\overline{R}_3$	0	0	0	0	0	0	$\overline{P}A_0^{\mathrm{T}} + N^{\mathrm{T}}C_0^{\mathrm{T}}$	0
*		*		*	-	$\overline{R}_1$	0	0	0	0	0	$\overline{\boldsymbol{P}} \boldsymbol{A}_{1d}^{\mathrm{T}}$	0
*		*		*		* _	$\overline{R}_2$	0	0	0	0	$\overline{\boldsymbol{P}} \boldsymbol{A}_{2d}^{\mathrm{T}}$	0
*		*		*		*	* _	$-\overline{R}_3$	0	0	0	$\overline{\boldsymbol{P}} \boldsymbol{A}_{0d}^{\mathrm{T}}$	0
*		*		*		*	*	*	$-\gamma^2 I$	0	0	$\boldsymbol{B}_{1}^{\mathrm{T}}$	$\boldsymbol{L}^{\mathrm{T}}$
*		*		*		*	*	*	*	$-\gamma^2 I$	0	$\boldsymbol{B}_2^{\mathrm{T}}$	0
*		*		*		*	*	*	*	*	$-\gamma^2 I$	$\boldsymbol{B}_{0}^{\mathrm{T}}$	0
*		*		*		*	*	*	*	*	*	$-\overline{\boldsymbol{P}} + \varepsilon \boldsymbol{H}_0 \boldsymbol{H}_0^{\mathrm{T}}$	0
*		*		*		*	*	*	*	*	*	*	- <b>I</b>
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
*		*		*		*	*	*	*	*	*	*	*
_ *		*		*		*	*	*	*	*	*	*	*
0	0	$\overline{P}E_1^T$	$E_7^{\mathrm{T}}N$	0	0	0	0						
$PH^{\mathrm{T}}$	_0	$PE_2^1$	0	$E_8^{\mathrm{T}}N$	0	0	0						
0	PH	$PE_3^1$ =-T	0	0	$E_9^1 N$	0	0						
0	0	$PE_{1d}^1$	, <b>0</b>	0	0	0	0						
0	0	$PE_{2a}^{1}$	1 <b>0</b>	0	0	0	0						
0	0	$PE_{0i}^{T}$	1 <b>0</b>	0	0	0	0						
0 •T	0	$E_4^{+}$	0	0	0	0	0						
Ľ	0 7	$E_5$	0	0	U	U	U						
0			0	0	0	0	0	_0					
0	0	0	0	0	0	0	0	0,					
_1	0	0	0	0	0	0	0						
*	_1		0	0	0	0	0						
*	*	F <b>I</b>	0	0	0	0	0						
*	*	*	- <i>ε</i> I	0	0	0	0						
*	*	*	*	-ε <b>I</b>	Õ	Õ	Õ						
*	*	*	*	*	_ε <b>I</b>	Ũ	Ũ						
*	*	*	*	*	*	- <i>ε</i> I	0						
*	*	*	*	*	*	*	<i>−ε</i> <b>I</b>						

then the closed-loop system (22) has a specified  $H_{\infty}$  noise attenuation  $\gamma$  and controller (21) with

$$\boldsymbol{K} = \boldsymbol{N} \overline{\boldsymbol{P}}^{-1} \tag{24}$$

is a  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controller for the system (1).

**Proof**: Extending the matrix inequality (8) for the closed-loop system (22), we obtain

$\left( \right[$	$(\overline{A})$	$+\overline{C}_1 K$	$\begin{bmatrix} T \\ T \end{bmatrix}$	$\int (\overline{A}_1 + \overline{C}_1 K)$	$ \int_{T}^{T} \int_{T}^{T} $			
	$(\overline{A})$		z) <sup>T</sup>	$\left  \begin{pmatrix} 1 & 1 \\ \overline{A} & \overline{C} \end{pmatrix} \right  $	$\mathcal{T}$			
	$(A_2$	$_2 + \mathbf{C}_2 \mathbf{I}$	•)	$(\mathbf{A}_2 + \mathbf{C}_2\mathbf{A})$				
	$(\overline{A}_0)$	$+\overline{C}_0 H$	$(\mathbf{K})^{\mathbf{r}}$	$\left \left(\overline{A}_{0}+\overline{C}_{0}K\right)\right $				
		$ar{A}_{\!\!1d}^{\scriptscriptstyle\mathrm{T}}$	ļ	$\overline{A}_{1d}^{\mathrm{T}}$				
		$ar{A}_{2d}^{ ext{T}}$		$ar{m{A}}_{2d}^{ ext{T}}$				
		$ar{A}_{0d}^{ ext{T}}$		$ar{m{A}}_{0d}^{ ext{T}}$				
		$\overline{\pmb{B}}_1^{\mathrm{T}}$		$\overline{B}_{1}^{\mathrm{T}}$				
		$\overline{B}_{2}^{\mathrm{T}}$		$\overline{B}_{2}^{\mathrm{T}}$				
	-	$\boldsymbol{B}_{0}^{\mathrm{T}}$		$\begin{bmatrix} \mathbf{B}_0^{\mathrm{T}} \end{bmatrix}$				
	_ <b>-F</b>	$P_1 + R_1 + R_1$	$+ \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}$	H O	)	0		
		0		$-P_{2}+R_{2}$	$+ \boldsymbol{H}^{\mathrm{T}} \boldsymbol{H}$	0		
		0		0	) —	$P + P_1 + P_2 +$	$\boldsymbol{R}_3 + \boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}$	
		0		0		0		
+		0		0	)	0		
		0		0		0		
		$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{l}$	H	0		0		
		0		$L^{1}$	H	<b>0</b>	_	
	_	0		0		$L^{1}H$		
(	)	0	0	$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{L}$	0	0		
(	)	0	0	0	$\boldsymbol{H}^{\mathrm{T}}\boldsymbol{L}$	0		
(	)	0	0	0	0	$H^{\mathrm{T}}L$		
-1	$\mathbf{R}_{1}$	0	0	0	0	0		
(	)	- <b>R</b> <sub>2</sub>	0	0	0	0	< <b>0</b> .	(25)
(	)	0	$-\boldsymbol{R}_3$	<b>0</b>	0	0		
(	)	0	0	$L^{1}L-\gamma^{2}I$	<b>0</b> TT 2 -	0		
(	)	U	0	0	$L^{L}L-\gamma^{2}I$	<b>U</b>		
(	J	U	U	0	0	$L^{L}L-\gamma^{2}I$	J	

Applying Lemma 1 on (25), we get

$\left[-\boldsymbol{P}^{-1}+\boldsymbol{arepsilon}\boldsymbol{H}_{0}^{\mathrm{T}}\boldsymbol{H}_{0}^{\mathrm{T}} ight]$	0 0	<b>0</b> $A_1 + C$	$C_1 K$	$\boldsymbol{A}_2 + \boldsymbol{C}_2 \boldsymbol{K}$
*	<i>–I</i> 0	0 <i>H</i>		0
*	* - <b>I</b>	0 0		H
*	* * * - <i>I</i>			0
*	* * * * <b>-P</b> <sub>1</sub>		$+\varepsilon^{-1}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{E}_{7}^{\mathrm{T}}\boldsymbol{E}_{7}\boldsymbol{K}$	$\boldsymbol{arepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_2$
*	* *	* *		$-\boldsymbol{P}_{2}+\boldsymbol{R}_{2}+\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{2}^{\mathrm{T}}\boldsymbol{E}_{2}+\boldsymbol{\varepsilon}^{-1}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{E}_{8}^{\mathrm{T}}\boldsymbol{E}_{8}\boldsymbol{K}$
*	* *	* *		*
*	* *	* *		*
*	* *	* *		*
*	* *	* *		*
*	* *	* *		*
*	* *	* *		*
*	* *	* *		*
	$\boldsymbol{A}_0 + \boldsymbol{C}_0 \boldsymbol{K}$		$A_{1d}$	$A_{2d}$
	0		0	0
	0		0	0
	H		0	0
	$\varepsilon^{-1}\boldsymbol{E}_{1}^{\mathrm{T}}\boldsymbol{E}_{3}$	$arepsilon^{-1}$	$\mathbf{\mathcal{E}}_{1}^{\mathrm{T}}\mathbf{\mathcal{E}}_{7}$ $\mathbf{\mathcal{E}}_{1}^{\mathrm{T}}$	${}^{1}E_{1}^{T}E_{8}$
	$\varepsilon^{-1}\boldsymbol{E}_{2}^{\mathrm{T}}\boldsymbol{E}_{3}$	$\mathcal{E}^{-1}$	$\mathbf{\mathcal{E}}_{2}^{\mathrm{T}}\mathbf{\mathcal{E}}_{7}$ $\mathbf{\mathcal{E}}_{7}^{\mathrm{T}}$	$^{-1}E_2^{\mathrm{T}}E_8$
$-P + P_1 + P_2 + R_3$	$+\varepsilon^{-1}E_3^{-1}E_3$	$+\varepsilon^{-1}K^{T}E_{9}^{T}E_{9}K \qquad \varepsilon^{-1}$	${}^{1}\boldsymbol{E}_{3} {}^{1}\boldsymbol{E}_{7} \qquad \boldsymbol{\varepsilon}^{-}$	$^{-1}E_3 E_8$
	*	$-R_{1} + d$	$\varepsilon^{-1} E_{1d}^{-1} E_{1d} = \varepsilon^{-1}$	${}^{1}E_{1d}I_{8}$
	*		* $-R_2 +$	$\varepsilon^{-1} \boldsymbol{E}_{2d}^{1} \boldsymbol{E}_{2d}$
	*		*	*
	*		*	*
	*		<b>本</b>	т 
	*	D	*	*
$A_{0d}$	<b>B</b> <sub>1</sub>	<b>B</b> <sub>2</sub>	<b>B</b> <sub>0</sub>	
0		0	0	
0	U		0	
$\mathbf{U}$	$\mathbf{U}$	$\mathbf{U}$	$\mathbf{L}$	
$\boldsymbol{\mathcal{E}}  \boldsymbol{\mathcal{L}}_1 \ \boldsymbol{\mathcal{L}}_9$ $\boldsymbol{\mathcal{L}}_1^{-1} \boldsymbol{\mathcal{F}}^{\mathrm{T}} \boldsymbol{\mathcal{F}}$	$\boldsymbol{\varepsilon}  \boldsymbol{E}_1$	$\mathbf{E}_4 \qquad \mathbf{\mathcal{E}}  \mathbf{E}_1  \mathbf{E}_5 \\ \mathbf{F} \qquad \mathbf{c}^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	$\boldsymbol{\mathcal{E}}  \boldsymbol{\mathcal{L}}_1  \boldsymbol{\mathcal{L}}_6$ $\boldsymbol{\mathcal{L}}_{-1}  \boldsymbol{\mathcal{F}}^{\mathrm{T}}  \boldsymbol{\mathcal{F}}$	
$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_2 \boldsymbol{E}_9$	$c^{-1} \boldsymbol{F}^{\mathrm{T}}$	$\mathbf{E}_4 \qquad \mathbf{\mathcal{E}}  \mathbf{E}_2  \mathbf{E}_5$ $\mathbf{F} \qquad \mathbf{c}^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_2 \boldsymbol{E}_6$	
$c^{-1} \boldsymbol{E}_3 \boldsymbol{E}_9$	$c^{-1}E^{\mathrm{T}}$	$\mathbf{E}_4 \qquad \mathbf{c}  \mathbf{E}_3  \mathbf{E}_5$ $\mathbf{E}_3  \mathbf{c}^{-1} \mathbf{E}^{\mathrm{T}} \mathbf{E}_5$	$\boldsymbol{c}  \boldsymbol{E}_3 \boldsymbol{E}_6$ $\boldsymbol{c}^{-1} \boldsymbol{F}^{\mathrm{T}} \boldsymbol{F}$	
$\varepsilon^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	$\varepsilon^{-1} \mathbf{E}^{\mathrm{T}}$	$E \qquad e^{-1}E^{\mathrm{T}}E$	$\varepsilon \mathbf{L}_{1d} \mathbf{L}_{6}$ $\varepsilon^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	
$-\mathbf{R}_{2d} + \varepsilon^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	$\varepsilon^{-1} \mathbf{E}_{2d}$	$\boldsymbol{E}_{4} \qquad \boldsymbol{\varepsilon}  \boldsymbol{E}_{2d} \boldsymbol{E}_{5}$ $\boldsymbol{E}_{5} \qquad \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}^{\mathrm{T}} \boldsymbol{E}$	$\varepsilon \mathbf{E}_{2d} \mathbf{E}_{6}$ $\varepsilon^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	
* 0d • 0d	$-\gamma^2 \mathbf{I} + \varepsilon^{-1}$	$\boldsymbol{E}_{4}^{\mathrm{T}}\boldsymbol{E}_{4} \qquad \boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{0d}^{\mathrm{T}}\boldsymbol{E}_{5}$	$\varepsilon^{-1} \boldsymbol{F}^{\mathrm{T}} \boldsymbol{F}$	(26)
*	*	$-\gamma^2 \mathbf{I} + \varepsilon^{-1} \mathbf{E}_{\pm}^{\mathrm{T}} \mathbf{E}_{\pm}$	$\varepsilon^{-1}\boldsymbol{E}_{4}^{\mathrm{T}}\boldsymbol{E}_{5}$	
*	*	*	$-\gamma^2 \mathbf{I} + \varepsilon^{-1} \mathbf{E}_5^{\mathrm{T}} \mathbf{E}_6$	
			, 6 2 6	

Applying Lemma 2 in (26), we obtain

$\left[\left(-\boldsymbol{P}_{1}+\boldsymbol{R}_{1}+\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}\right)\right]$	$\boldsymbol{E}_{1}^{\mathrm{T}}\boldsymbol{E}_{1}+\boldsymbol{\varepsilon}^{-1}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{E}_{7}^{\mathrm{T}}\boldsymbol{E}_{7}$	K)	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_2$						
	*	$\left(-\boldsymbol{P}_{2}+\boldsymbol{R}_{2}+\boldsymbol{\varepsilon}\right)$	$\varepsilon^{-1} \boldsymbol{E}_{2}^{\mathrm{T}} \boldsymbol{E}_{2} + \varepsilon^{-1} \boldsymbol{K}^{\mathrm{T}} \boldsymbol{E}_{2}$	$E_8^{\mathrm{T}} E_8 K$	()				
	*		*	0 0	)				
	*		*						
	*		*						
	*		*						
	*		*						
	*		*						
	*		*						
	*		*						
	*		*						
	*		*						
L	* -1 <b>-</b> 1 -		-1 - 1 - 1	_1	<b>P</b> T P		_	laTa	
	$\varepsilon^{-1}E_1^TE_3$		$\varepsilon^{-1}E_1^TE_7$	$\mathcal{E}^{-1}$	$\boldsymbol{E}_1^{T}\boldsymbol{E}_8$ $\boldsymbol{E}^{T}\boldsymbol{E}$		3	${}^{1}\boldsymbol{E}_{1}^{T}\boldsymbol{E}_{9}$	
	$\mathcal{E} \mathbf{E}_2 \mathbf{E}_3$		$\mathcal{E} \mathbf{E}_2 \mathbf{E}_7$	E .	$\boldsymbol{E}_2 \boldsymbol{E}_8$		E	$\boldsymbol{E}_{2}\boldsymbol{E}_{9}$	
$\left(-\boldsymbol{P}+\boldsymbol{P}_{1}+\boldsymbol{P}_{2}+\boldsymbol{K}\right)$	$E_3 + \varepsilon E_3 E_3 + \varepsilon E_3$	$(E_9^*E_9K)$	$\varepsilon E_3 E_7$	8	$E_3 E_8$		Е	$E_{3}E_{9}$	
	*	$\left(-\boldsymbol{R}_{1}\right)$	$+ \varepsilon^{-1} \boldsymbol{E}_{1d}^{\mathrm{T}} \boldsymbol{E}_{1d} \Big)$	$\mathcal{E}^{-1}$	$\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{8}$		$\mathcal{E}^{-}$	${}^{1}\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{9}$	
	*		* (-	$R_2 + \epsilon$	$\boldsymbol{E}^{-1}\boldsymbol{E}_{2d}^{\mathrm{T}}$	$\boldsymbol{E}_{2d}$	$\mathcal{E}^{-}$	${}^{1}\boldsymbol{E}_{2d}^{\mathrm{T}}\boldsymbol{E}_{9}$	
	*		*		*		$(-R_3 +$	$\varepsilon^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{0d}$	)
	*		*		*			*	
	*		*		*			*	
	*		*		*			*	
	*		*		*			*	
	*		*		*			*	
	*		*		*			*	
	*		*		*	_	1	*	
$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_4$	$\varepsilon^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_5$	$\varepsilon^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_6$	$\left(\boldsymbol{A}_{1}^{\mathrm{T}}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{C}_{1}^{\mathrm{T}}\right)$	$\boldsymbol{H}^{\mathrm{T}}$	0	0			
$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_2^{\mathrm{T}} \boldsymbol{E}_4$	$\varepsilon^{-1} \boldsymbol{E}_2^{\mathrm{T}} \boldsymbol{E}_5$	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_2^{\mathrm{T}} \boldsymbol{E}_6$	$\left(\boldsymbol{A}_{2}^{\mathrm{T}}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{C}_{2}^{\mathrm{T}}\right)$	0	$\boldsymbol{H}^{\mathrm{T}}$	0			
$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_3^{\mathrm{T}} \boldsymbol{E}_4$	$\varepsilon^{-1} \boldsymbol{E}_3^{\mathrm{T}} \boldsymbol{E}_5$	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_3^{\mathrm{T}} \boldsymbol{E}_6$	$\left(\boldsymbol{A}_{0}^{\mathrm{T}}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{C}_{0}^{\mathrm{T}}\right)$	0	0	$\boldsymbol{H}^{\mathrm{T}}$			
$\boldsymbol{arepsilon}^{-1} \boldsymbol{E}_{1d}^{\mathrm{T}} \boldsymbol{E}_{4}$	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{1d}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{1d}^{\mathrm{T}} \boldsymbol{E}_{6}$	$oldsymbol{A}_{1d}^{ ext{T}}$	0	0	0			
$\boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{4}$	$\varepsilon^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\varepsilon^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{6}$	$oldsymbol{A}_{2d}^{ extsf{T}}$	0	0	0			
$\varepsilon^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{4}$	$\varepsilon^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\varepsilon^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{6}$	$A_{0d}^{\mathrm{T}}$	0	0	0	< 0.		
$-\gamma^2 \boldsymbol{I} + \varepsilon^{-1} \boldsymbol{E}_4^{\mathrm{T}} \boldsymbol{E}_4$	$\varepsilon^{-1} \boldsymbol{E}_{4}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\varepsilon^{-1} \boldsymbol{E}_{4}^{\mathrm{T}} \boldsymbol{E}_{6}$	$B_1^{\mathrm{T}}$	$\boldsymbol{L}^{\mathrm{T}}$	0	0			
*	$-\gamma^2 \boldsymbol{I} + \varepsilon^{-1} \boldsymbol{E}_5^{-1} \boldsymbol{E}_5$	$\varepsilon^{-1}E_5^{1}E_6$	$B_2^{T}$	0	$L^{1}$	0 • T			
*	*	$-\gamma^{2}I + \varepsilon^{-1}E_{6}E_{6}$	$B_0^{-1}$	0	0	Ľ			(27)
*	*	*	$-\mathbf{r} + \mathcal{E}\mathbf{H}_0\mathbf{H}_0^+$	U 7	U A	U A			
*	*	*	*	*	_I	0 0			
*	*	*	*	*	*	-I			
						-	L		

(28)

$\int \boldsymbol{P}^{-1} \left( -\boldsymbol{P}_1 + \boldsymbol{R}_1 + \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_1 + \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_1 \right)$	$-\varepsilon^{-1}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{E}_{7}^{\mathrm{T}}\boldsymbol{E}_{7}\boldsymbol{K})\boldsymbol{P}^{-1}$	1	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_1^{\mathrm{T}} \boldsymbol{E}_2 \boldsymbol{P}^{-1}$	
*		$P^{-1}(-P_2 + R_2 +$	$\varepsilon^{-1} \boldsymbol{E}_2^T \boldsymbol{E}_2 + \varepsilon^{-1} \boldsymbol{K}^T \boldsymbol{I}$	$E_8^T E_8 K ) P^{-1}$
*		(	*	~ ~ )
*			*	
*			*	
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*			*	
*			*	
*			*	
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$\mathbf{P}^{-1}\varepsilon^{-1}\mathbf{I}$	$E_{1}^{\mathrm{T}}E_{2}P^{-1}$	$P^{-1}$	$\varepsilon^{-1} \boldsymbol{E}_{1}^{\mathrm{T}} \boldsymbol{E}_{2} \boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{1}^{\mathrm{T}} \boldsymbol{E}_{2} \boldsymbol{P}^{-1}$
$P^{-1}\varepsilon^{-1}H$	$E_{2}^{T}E_{3}P^{-1}$	$P^{-1}$	$\varepsilon^{-1}\boldsymbol{E}_{2}^{\mathrm{T}}\boldsymbol{E}_{7}^{\mathrm{T}}\boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2}^{\mathrm{T}} \boldsymbol{E}_{8}^{\mathrm{T}} \boldsymbol{P}^{-1}$
$\boldsymbol{P}^{-1}\left(-\boldsymbol{P}+\boldsymbol{P}_{1}+\boldsymbol{P}_{2}+\boldsymbol{R}_{3}+\boldsymbol{\varepsilon}^{-1}\right)$	${}^{1}\boldsymbol{E}_{3}^{\mathrm{T}}\boldsymbol{E}_{3} + \varepsilon^{-1}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{E}_{9}^{\mathrm{T}}\boldsymbol{E}_{9}$	$E_{9}K)P^{-1}$ $P^{-1}$	$\varepsilon^{-1}\boldsymbol{E}_{3}^{\mathrm{T}}\boldsymbol{E}_{7}\boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{3}^{\mathrm{T}}\boldsymbol{E}_{8}\boldsymbol{P}^{-1}$
•	*	$\boldsymbol{P}^{-1}\left(-\boldsymbol{R}\right)$	$\mathbf{P}^{-1} \mathbf{E}_{1d}^{\mathrm{T}} \mathbf{E}_{1d} \mathbf{P}^{-1}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{1d}^{\mathrm{T}} \boldsymbol{E}_{8} \boldsymbol{P}^{-1}$
	*	X	*	$\boldsymbol{P}^{-1}\left(-\boldsymbol{R}_{2}+\varepsilon^{-1}\boldsymbol{E}_{2d}^{\mathrm{T}}\boldsymbol{E}_{2d}\right)\boldsymbol{P}^{-1}$
	*		*	*
	*		*	*
	*		*	*
	*		*	*
	*		*	*
	*		*	*
	*		*	*
$\mathbf{D}^{-1} \mathbf{c}^{-1} \mathbf{E}^{\mathrm{T}} \mathbf{E} \mathbf{D}^{-1}$	$\mathbf{D}^{-1} \mathbf{c}^{-1} \mathbf{E}^{\mathrm{T}} \mathbf{E}$	$\mathbf{D}^{-1} \mathbf{e}^{-1} \mathbf{E}^{\mathrm{T}} \mathbf{E}$	$\mathbf{D}^{-1} \mathbf{c}^{-1} \mathbf{E}^{\mathrm{T}} \mathbf{E}$	Ť
$\mathbf{P}  \mathbf{\mathcal{E}}  \mathbf{E}_1  \mathbf{E}_9 \mathbf{P}$ $\mathbf{P}^{-1} \mathbf{c}^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}  \mathbf{P}^{-1}$	$\boldsymbol{P}  \boldsymbol{\varepsilon}  \boldsymbol{E}_1  \boldsymbol{E}_4 \\ \boldsymbol{P}^{-1}  \boldsymbol{\varepsilon}^{-1}  \boldsymbol{F}^{\mathrm{T}}  \boldsymbol{F}$	$\mathbf{P}  \mathbf{\mathcal{E}}  \mathbf{L}_1  \mathbf{L}_5 \\ \mathbf{P}^{-1} \mathbf{c}^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	$\mathbf{P}  \mathbf{\mathcal{E}}  \mathbf{E}_1  \mathbf{E}_6 \\ \mathbf{P}^{-1} \mathbf{c}^{-1} \mathbf{F}^{\mathrm{T}} \mathbf{F}$	
$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{2}^{T}\boldsymbol{E}_{3}\boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{2}^{\mathrm{T}}\boldsymbol{E}_{4}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{2}^{\mathrm{T}}\boldsymbol{E}_{2}$	$\boldsymbol{P}^{-1}\varepsilon^{-1}\boldsymbol{E}_{1}^{\mathrm{T}}\boldsymbol{E}_{2}$	
$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{0}\boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{4}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{5}$	$\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{1d}^{\mathrm{T}}\boldsymbol{E}_{\epsilon}$	
$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{9}^{\mathrm{P}} \boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{4}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{2d}^{\mathrm{T}} \boldsymbol{E}_{6}^{\mathrm{T}}$	
$\boldsymbol{P}^{-1}\left(-\boldsymbol{R}_{3}+\boldsymbol{\varepsilon}^{-1}\boldsymbol{E}_{0d}^{\mathrm{T}}\boldsymbol{E}_{0d}\right)\boldsymbol{P}^{-1}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{4}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\boldsymbol{P}^{-1} \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_{0d}^{\mathrm{T}} \boldsymbol{E}_{6}$	
*	$-\gamma^2 \boldsymbol{I} + \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_4^{\mathrm{T}} \boldsymbol{E}_4$	$\varepsilon^{-1} \boldsymbol{E}_{4}^{\mathrm{T}} \boldsymbol{E}_{5}$	$\varepsilon^{-1} \boldsymbol{E}_{4}^{\mathrm{T}} \boldsymbol{E}_{6}$	
*	*	$-\gamma^2 \boldsymbol{I} + \boldsymbol{\varepsilon}^{-1} \boldsymbol{E}_5^{\mathrm{T}} \boldsymbol{E}_5$	$\varepsilon^{-1} \boldsymbol{E}_5^{\mathrm{T}} \boldsymbol{E}_6$	
*	*	*	$-\gamma^2 \boldsymbol{I} + \varepsilon^{-1} \boldsymbol{E}_6^{\mathrm{T}} \boldsymbol{E}_6$	
*	*	*	*	
*	*	*	*	
*	*	*	*	
$\boldsymbol{P}^{-1}\left(\boldsymbol{\Lambda}^{\mathrm{T}}+\boldsymbol{K}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\right) \boldsymbol{P}^{-1}\boldsymbol{H}^{\mathrm{T}}$	0 0	]		
$\boldsymbol{P}^{-1} \left( \boldsymbol{A}_{1}^{\mathrm{T}} + \boldsymbol{K}^{\mathrm{T}} \boldsymbol{C}_{1}^{\mathrm{T}} \right)  \boldsymbol{0}$	$P^{-1}H^{\mathrm{T}}$ 0			
$P^{-1}\left(\mathbf{A}_{2}^{\mathrm{T}}+\mathbf{K}^{\mathrm{T}}\mathbf{C}_{2}^{\mathrm{T}}\right) = 0$	$\mathbf{D} = \mathbf{P}^{-1} \mathbf{H}^{\mathrm{T}}$			
$P^{-1}A^{\mathrm{T}} = 0$	0 0			
$P^{-1}A_{1d}^{\mathrm{T}}$ 0	0 0			
$\boldsymbol{P}^{-1}\boldsymbol{A}_{0d}^{\mathrm{T}}$ <b>0</b>	0 0			
$\boldsymbol{B}_{1}^{\mathrm{T}}$ $\boldsymbol{L}^{\mathrm{T}}$	0 0	< 0.		
$\boldsymbol{B}_2^{\mathrm{T}}$ <b>0</b>	$\boldsymbol{L}^{\mathrm{T}}$ <b>0</b>			
$\boldsymbol{B}_0^{\mathrm{T}}$ <b>0</b>	$0 L^{\mathrm{T}}$			
$-\boldsymbol{P}^{-1} + \varepsilon \boldsymbol{H}_0 \boldsymbol{H}_0^{1} \qquad \boldsymbol{0}$	0 0			
* -I * *				
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Denoting  $\overline{P} = P^{-1}$ ,  $\overline{P}_1 = \overline{P}P_1\overline{P}$ ,  $\overline{P}_2 = \overline{P}P_2\overline{P}$ ,  $\overline{R}_1 = \overline{P}R_1\overline{P}$ ,  $\overline{R}_2 = \overline{P}R_2\overline{P}$ ,  $\overline{R}_3 = \overline{P}R_3\overline{P}$ , and  $N = K\overline{P}$  in (28), the equivalence of (28) and (23) follows trivially from Lemma 2. This completes the proof of Theorem 2.

**Remark 1.** Note that, if there is no uncertainty in system (1) and we set  $A_0 = A_{0d} = B_0 = C_0 = 0$ , then LMI (23) coincides with the criteria for the existence of  $H_{\infty}$  state feedback controllers for 2-D discrete state-delayed system given in [14].

Theorem 2 presents a method of designing a set of  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controllers (if they exist) in terms of feasible solutions to the LMI (23). In particular, the robust optimal  $H_{\infty}$  controller which minimizes the  $H_{\infty}$  noise attenuation  $\gamma$  of the closed-loop system (22) can be determined by solving a certain optimization problem. Based on Theorem 2, the design problem of a robust optimal  $H_{\infty}$  controller can be formulated as

minimize 
$$\gamma^2$$
 (29)

s.t. (23).

### **4. Illustrative Examples**

In this section, two examples illustrating the effectiveness of our proposed method are presented.

*Example* 4.1: Consider an uncertain 2-D discrete state-delayed system given by (1) and initial condition (2) with

$$\begin{aligned} \mathbf{A}_{1} &= \begin{bmatrix} 0.6 & 1 \\ 0.02 & 0 \end{bmatrix}, \ \mathbf{A}_{2} &= \begin{bmatrix} 0 & 0 \\ 0.1 & 0.6 \end{bmatrix}, \ \mathbf{A}_{0} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.21 \end{bmatrix}, \ \mathbf{C}_{1} &= \begin{bmatrix} 0 \\ 0.002 \end{bmatrix}, \ \mathbf{C}_{2} &= \begin{bmatrix} 0 \\ 0.04 \end{bmatrix}, \\ \mathbf{C}_{0} &= \begin{bmatrix} 0 \\ 0.04 \end{bmatrix}, \ \mathbf{A}_{1d} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.03 \end{bmatrix}, \ \mathbf{A}_{2d} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.09 \end{bmatrix}, \ \mathbf{A}_{0d} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.02 \end{bmatrix}, \ \mathbf{B}_{1} &= \begin{bmatrix} 0 \\ 0.04 \end{bmatrix}, \\ \mathbf{B}_{2} &= \begin{bmatrix} 0 \\ 0.03 \end{bmatrix}, \ \mathbf{B}_{0} &= \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}, \ \mathbf{H} &= \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \ \mathbf{H}_{0} &= \begin{bmatrix} 0.001 & 0.002 \\ 0 & 0.03 \end{bmatrix}, \ \mathbf{L} &= 0.5, \\ \mathbf{E}_{1} &= \begin{bmatrix} 0.03 & 0 \\ 0 & 0.03 \end{bmatrix}, \ \mathbf{E}_{2} &= \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \end{bmatrix}, \ \mathbf{E}_{3} &= \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \mathbf{E}_{1d} &= \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}, \\ \mathbf{E}_{2d} &= \begin{bmatrix} 0.06 & 0 \\ 0 & 0.06 \end{bmatrix}, \ \mathbf{E}_{0d} &= \begin{bmatrix} 0.08 & 0 \\ 0 & 0.08 \end{bmatrix}, \ \mathbf{E}_{4} &= \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \ \mathbf{E}_{5} &= \begin{bmatrix} -0.007 \\ 0 \end{bmatrix}, \ \mathbf{E}_{6} &= \begin{bmatrix} -0.7 \\ 0 \end{bmatrix}, \\ \mathbf{E}_{7} &= \begin{bmatrix} -0.001 \\ 0 \end{bmatrix}, \ \mathbf{E}_{8} &= \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, \ \mathbf{E}_{9} &= \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}, \ \mathbf{d}_{1} &= 3, \ \mathbf{d}_{2} &= 4, \ \mathbf{k}_{1} &= 3, \ \mathbf{k}_{2} &= 2. \end{aligned}$$

We wish to design a robust optimal  $H_{\infty}$  controller for the above system. Using the Matlab LMI toolbox [30] [32], it is found that the optimization problem (29) is feasible for the present example and the optimal solution is given by

$$\overline{P} = \begin{bmatrix} 31.3617 & -14.0417 \\ -14.0417 & 11.9625 \end{bmatrix}, \ \overline{P}_1 = \begin{bmatrix} 11.7299 & -2.2597 \\ -2.2597 & 3.1922 \end{bmatrix}, \ \overline{P}_2 = \begin{bmatrix} 9.4330 & -5.6712 \\ -5.6712 & 4.2997 \end{bmatrix}, \ \overline{R}_1 = \begin{bmatrix} 3.6479 & -2.5450 \\ -2.5450 & 1.9116 \end{bmatrix}, \ \overline{R}_2 = \begin{bmatrix} 5.9778 & -4.5516 \\ -4.5516 & 3.6430 \end{bmatrix}, \ \overline{R}_3 = \begin{bmatrix} 2.7307 & -1.8462 \\ -1.8462 & 1.3727 \end{bmatrix}, \ N = \begin{bmatrix} 115.1984 & -143.0584 \end{bmatrix}, \ \varepsilon = 4.2278 \times 10^4, \ \gamma = 0.5036.$$

Thus, the robust optimal  $H_{\infty}$  state feedback controller is obtained as

$$\boldsymbol{K} = \begin{bmatrix} -3.5435 & -16.1184 \end{bmatrix}. \tag{32}$$

**Figure 1** shows the frequency response from noise input w(i, j) to the controlled output z(i, j) for the closed-loop system (22) over all frequencies *i.e.*  $|G(e^{j\omega_1}, e^{j\omega_2})|$ ,  $0 \le \omega_1 \le 2\pi$ ,  $0 \le \omega_2 \le 2\pi$ . The peak value of the frequency response is 0.5029, which is lower than the specified level of attenuation  $\gamma = 0.5036$ .

Example 4.2: Consider the thermal processes in chemical reactors, heat exchangers and pipe furnaces [33]



[34], which can be expressed by the following partial differential equation.

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - a_0 T(x,t) - a_1 T(x - x_{d_1},t) - a_2 T(x,t-\tau_1) - a_3 T(x - x_{d_2},t-\tau_2) + bu(x,t), \quad (33)$$

where T(x,t) is the temperature at space  $x \in [0, x_f]$  and time  $t \in [0, \infty]$ , u(x,t) is the input function,  $\tau_1$  and  $\tau_2$  are the time delays,  $x_{d_1}$  and  $x_{d_2}$  are the space delays, and  $a_0$ ,  $a_1$ ,  $a_2$ ,  $a_3$ , b are the real coefficients. Taking

$$T(i, j) = T(i\Delta x, j\Delta t), \quad u(i, j) = u(i\Delta x, j\Delta t)$$
(34)

$$\frac{\partial T(x,t)}{\partial x} \approx \frac{T(i,j) - T(i-1,j)}{\Delta x}, \quad \frac{\partial T(x,t)}{\partial t} \approx \frac{T(i,j+1) - T(i,j)}{\Delta t}$$
(35)

(33) can be written in the following form:

$$T(i, j+1) = \left(1 - \frac{\Delta t}{\Delta x} - a_0 \Delta t\right) T(i, j) + \frac{\Delta t}{\Delta x} T(i-1, j) - a_1 \Delta t T(i-d_1, j) - a_2 \Delta t T(i, j-d_2)$$

$$-a_3 \Delta t T(i-k_1, j-k_2) + b \Delta t u(i, j),$$
(36)

where  $d_1 = int(x_{d_1}/\Delta x)$ ,  $d_2 = int(\tau_1/\Delta t + 1)$ ,  $k_1 = int(x_{d_2}/\Delta x)$  and,  $k_2 = int(\tau_2/\Delta t + 1)$ , int(.) is the integer function.

It is assumed that the surface of the heat exchanger is insulated and the heat flow through it is in steady state condition, then we could take the boundary conditions as  $\frac{\partial T(x,t)}{\partial x} = 0$  and  $\frac{\partial T(x,t)}{\partial t} = 0$ , respectively.

Denoting  $x^{\mathrm{T}}(i, j) = [T^{\mathrm{T}}(i-1, j) \quad T^{\mathrm{T}}(i, j)]$ , it is easy to verify that (36) can be converted into the following 2-D state-delayed GM:

$$\mathbf{x}(i+1,j+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x}(i,j+1) + \begin{bmatrix} 0 & 0 \\ \frac{\Delta t}{\Delta x} & 1 - \frac{\Delta t}{\Delta x} - a_0 \Delta t \end{bmatrix} \mathbf{x}(i+1,j) + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\Delta t}{\Delta x} \end{bmatrix} \mathbf{x}(i,j) + \begin{bmatrix} 0 & 0 \\ 0 & -a_1 \Delta t \end{bmatrix} \mathbf{x}(i-d_1,j+1) + \begin{bmatrix} 0 & 0 \\ 0 & -a_2 \Delta t \end{bmatrix} \mathbf{x}(i+1,j-d_2) + \begin{bmatrix} 0 & 0 \\ 0 & -a_2 \Delta t \end{bmatrix} \mathbf{x}(i-k_1,j-k_2) + \begin{bmatrix} 0 \\ b \Delta t \end{bmatrix} \mathbf{u}(i+1,j).$$
(37)

Let  $\Delta t = 0.1$ ,  $\Delta x = 0.4$ ,  $a_0 = 1$ ,  $a_1 = -0.3$ ,  $a_1 = -0.3$ ,  $a_3 = -0.2$ , b = 0.4,  $d_1 = 3$ ,  $d_2 = 2$ ,  $k_1 = 2$ ,  $k_2 = 1$  and the initial state satisfies the condition (2) with  $r_1 = 3$ ,  $r_2 = 2$ . To consider the problem of  $H_{\infty}$  disturbance attenuation, the thermal process is modeled in the form (1) with

$$\boldsymbol{B}_{1} = \begin{bmatrix} 0\\ 0.004 \end{bmatrix}, \boldsymbol{B}_{2} = \begin{bmatrix} 0\\ 0.004 \end{bmatrix}, \boldsymbol{B}_{0} = \begin{bmatrix} 0\\ 0.004 \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \boldsymbol{L} = 0.5.$$
(38)

It is also assumed that the above system is subjected to the parameter uncertainties of the form (1c) and (1d) with

$$\mathbf{H}_{0} = \begin{bmatrix} 0.001 & 0.002 \\ 0 & 0 \end{bmatrix}, \ \mathbf{E}_{1} = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}, \ \mathbf{E}_{2} = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}, 
 \mathbf{E}_{3} = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.001 \end{bmatrix}, \ \mathbf{E}_{1d} = \begin{bmatrix} 0.009 & 0 \\ 0 & 0.009 \end{bmatrix}, \ \mathbf{E}_{2d} = \begin{bmatrix} 0.006 & 0 \\ 0 & 0.006 \end{bmatrix}, 
 \mathbf{E}_{0d} = \begin{bmatrix} 0.008 & 0 \\ 0 & 0.008 \end{bmatrix}, \ \mathbf{E}_{4} = \begin{bmatrix} 0.001 \\ 0 \end{bmatrix}, \ \mathbf{E}_{5} = \begin{bmatrix} -0.007 \\ 0 \end{bmatrix}, \ \mathbf{E}_{6} = \begin{bmatrix} -0.007 \\ 0 \end{bmatrix}, 
 \mathbf{E}_{7} = \begin{bmatrix} -0.001 \\ 0 \end{bmatrix}, \ \mathbf{E}_{8} = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}, \ \mathbf{E}_{9} = \begin{bmatrix} 0.003 \\ 0 \end{bmatrix}.$$
(39)

Now, using the Matlab LMI toolbox [30] [32], it is found that the optimization problem (29) is feasible for the considered system and the optimal solution is obtained as

$$\overline{P} = \begin{bmatrix} 2.8488 & -0.7935 \\ -0.7935 & 0.8105 \end{bmatrix}, \ \overline{P}_1 = \begin{bmatrix} 1.0044 & -0.4373 \\ -0.4373 & 0.5537 \end{bmatrix}, \ \overline{P}_2 = \begin{bmatrix} 0.8480 & -0.2275 \\ -0.2275 & 0.1723 \end{bmatrix}, \overline{R}_1 = \begin{bmatrix} 0.2319 & -0.1091 \\ -0.1091 & 0.0720 \end{bmatrix}, \ \overline{R}_2 = \begin{bmatrix} 0.3077 & -0.1819 \\ -0.1819 & 0.1424 \end{bmatrix}, \ \overline{R}_3 = \begin{bmatrix} 0.2023 & -0.0915 \\ -0.0915 & 0.0594 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} -9.0892 & -8.2432 \end{bmatrix}, \ \varepsilon = 7.7138 \times 10^3, \ \gamma = 0.5003.$$

Thus, the robust optimal  $H_{\infty}$  state feedback controller is given as

$$\boldsymbol{K} = \begin{bmatrix} -8.2820 & -18.2795 \end{bmatrix}. \tag{41}$$



**Figure 2** shows the frequency response from noise input w(i, j) to the controlled output z(i, j) for the closed-loop system (22) over all frequencies *i.e.*  $|G(e^{j\omega_1}, e^{j\omega_2})|$ ,  $0 \le \omega_1 \le 2\pi$ ,  $0 \le \omega_2 \le 2\pi$ . The peak value of the frequency response is 0.5002, which is lower than the above obtained specified level of attenuation  $\gamma = 0.5003$ .

### **5.** Conclusion

In this paper, the problem of robust optimal  $H_{\infty}$  control for a class of uncertain 2-D discrete state-delayed systems described by the GM has been studied. A sufficient condition for the existence of  $\gamma$ -suboptimal robust  $H_{\infty}$  state feedback controller has been derived in terms of the feasible solutions to a certain LMI. The desired robust optimal  $H_{\infty}$  controller has been obtained by solving a convex optimization problem. Finally, two illustrative examples have been provided to demonstrate the applicability of the proposed approach.

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