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Entanglement Quantifier Based on Atomic Wehrl Entropy for Non-Linear Interaction between a Single Two-Level Atom and SU(1,1) Quantum System

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Abstract

In this paper, we study the dynamics of the atomic inversion, scaled atomic Wehrl entropy and marginal atomic Wehrl density for a single two-level atom interacting with SU(1,1) quantum system. We obtain the expectation values of the atomic variables using specific initial conditions. We examine the effects of different parameters on the scaled atomic Wehrl entropy and marginal atomic Wehrl density. We observe an interesting monotonic relation between the different physical quantities for different values of the initial atomic position and detuning parameter.

Keywords

Scaled Atomic Wehrl Entropy; Atomic Q-Function; Atomic Inversion

1. Introduction

Quantum entropy, is considered the main generalization of the Boltzmann classical entropy, proposed by von Neumann [1]. It has been applied, as a measure to many aspects in quantum information processing such as quantum entanglement, photocount statistics, quantum decoherence, quantum optical correlations, purity of the quantum states, accessible information in quantum measurement. The quantification of entanglement is necessary to understand and develop the quantum information theory. For this reason different entanglement measures have been used for the mixed and pure states such as concurrence [2]-[5], entanglement of formation [6] [7], and negativity [2] [3]. In this way, the concurrence and negativity are used as a good entanglement measure for

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mixed state, but the von Neumann entropy has been proposed for pure state entanglement [1], all these measures to test whether a given quantum state is separable or entangled. Also, some interesting physical phenomenon is observed as a result of entanglement measure, such as "entanglement sudden death" (ESD), entanglement sudden birth (ESB) [8]-[14].

The Wehrl entropy (WE) is more sensitive in distinguishing states than the von Neumann entropy since WE is a state dependent [15]. The concept of the Wehrl phase distribution (WPD) has been developed and shown that it serves as a measure of both noise (phase-space uncertainty) and phase randomization [16]. Furthermore, the WE has been applied to some dynamical systems, too. In this respect, the time evolution of the field WE for the Kerr-like medium has been discussed in [17] [18] showing that the FWE gives a clear signature for the formation of finite superposition of coherent states (cat-like states) as well as the number of coherent components taking part in the superposition. For the trapped ion system the WE gives an information on the dynamical properties and entanglement of the system [19] [20]. On the other hand some features of WE and WPD of a single-Cooper pair box placed inside a dissipative cavity have been discussed [21]. It is shown that phase damping leads to generating long living correlation of the system.

Different entanglement measures and quantifiers for mixed and pure states have been proposed, such as the negativity and atomic Wehrl entropy. The relation between mixed state entanglement and the atomic Wehrl entropy (AWE) has not been studied widely. However, there are some attempts to quantify the pure state entanglement by using AWE. In this context, the entanglement evaluation with AWE and atomic Fisher information has been investigated [22] [23]. It has been found that entanglement of a two-level atom can be measured by AWE and their marginal distribution. On the other hand, atomic Wehrl entropy was used as an entanglement measure for a mixed state two-level system in the presence of intrinsic decoherence [24]. It found that the information about entanglement is obtained by comparing the results for the atomic Wehrl entropy and negativity with the analytical results for a simple case.

Realistic quantum systems are not closed, which causes the rapid destruction of crucial quantum properties. Therefore, the unavoidable interaction between a quantum system, understanding the dynamics of entanglement measures and finding the correlation between different phenomenons may stimulate great interest. In the present article, our main interest is to investigate the evolution of the scaled atomic Wehrl entropy (AWE) of a single two-level atom and SU(1,1) quantum system in the presence of detuning parameter, which leads us to address the question: Can the AWE be used as a indicator of the entanglement and dynamical properties of the system in the presence of non-linear terms?

The article is organized as follows: In Section 2, we introduce the model of the single two-level and SU(1,1) quantum system in the presence of detuning parameter. The definitions of the scaled atomic Wehrl entropy, atomic inversion and marginal atomic Wehrl density are introduced in Section 3. We conclude the main results with some remarks in Section 4.

2. The System Hamiltonian

The Hamiltonian which describe the interaction between a single two-level atom and $SU(1,1)$ quantum system take the following form

$$H=\omega_1 k_z^{(1)}+\omega_2 k_z^{(2)}+\Omega_1 S_{11}+\Omega_2 S_{22}+\lambda\left(k_-^{(1)}k_-^{(2)}S_{12}+k_+^{(1)}k_+^{(2)}S_{21}\right), \tag{1}$$

where ω is the frequency of the system, Ω_j is the energy and S_{ij} are elements of the $SU(1,1)$ group obeying the following commutation relation

$$\left[S_{ij},S_{kl}\right]=S_{il}\delta_{kj}-S_{kj}\delta_{il}, \tag{2}$$

while $k_\pm$ and k_z satisfy the following commutation relation

$$\left[k_z,k_\pm\right]=\pm k_\pm,\left[k_-,k_+\right]=2k_z \text{ and } \left[S_{ij},k_{\pm,z}\right]=0. \tag{3}$$

The Heisenberg equation of motion for any operator O is given by

$$\mathrm{i}\frac{dO}{dt}=\left[O,H\right],\ (\hbar=1) \tag{4}$$

thus, the equations of motion for S_{ij} and k_z are given by

$$\mathrm{i}\frac{\mathrm{d}S_{11}}{\mathrm{d}t}=\left[S_{11},H\right]=\lambda\left\{k_{-}^{(1)}k_{-}^{(2)}S_{12}-k_{+}^{(1)}k_{+}^{(2)}S_{21}\right\}, \tag{5}$$

$$\mathrm{i}\frac{\mathrm{d}S_{22}}{\mathrm{d}t}=\left[S_{22},H\right]=-\lambda\left\{k_{-}^{(1)}k_{-}^{(2)}S_{12}-k_{+}^{(1)}k_{+}^{(2)}S_{21}\right\}, \tag{6}$$

$$\mathrm{i}\frac{\mathrm{d}k_z^{(1)}}{\mathrm{d}t}=\left[k_z^{(1)},H\right]=-\lambda\left\{k_{-}^{(1)}k_{-}^{(2)}S_{12}-k_{+}^{(1)}k_{+}^{(2)}S_{21}\right\}, \tag{7}$$

$$\mathrm{i}\frac{\mathrm{d}k_z^{(2)}}{\mathrm{d}t}=\left[k_z^{(2)},H\right]=-\lambda\left\{k_{-}^{(1)}k_{-}^{(2)}S_{12}-k_{+}^{(1)}k_{+}^{(2)}S_{21}\right\}, \tag{8}$$

$$\mathrm{i}\frac{\mathrm{d}S_{11}}{\mathrm{d}t}-\mathrm{i}\frac{\mathrm{d}S_{22}}{\mathrm{d}t}+\mathrm{i}\frac{\mathrm{d}k_z^{(1)}}{\mathrm{d}t}+\mathrm{i}\frac{\mathrm{d}k_z^{(2)}}{\mathrm{d}t}=0 \tag{9}$$

then

$$S_{11}-S_{22}+k_z^{(1)}+k_z^{(2)}=\text{constant of motion}$$

In this case, we have $N_1=k_z^{(1)}+S_{11}$, $N_2=k_z^{(2)}-S_{22}$ are constant of motion. Therefore, the Hamiltonian takes the following form

$$\begin{aligned}
&H=\omega_1k_z^{(1)}+\omega_2k_z^{(2)}+\Omega_1S_{11}+\Omega_2S_{22}+\lambda\left(k_{-}^{(1)}k_{-}^{(2)}S_{12}+k_{+}^{(1)}k_{+}^{(2)}S_{21}\right),\\
&H_0=\omega_1k_z^{(1)}+\omega_2k_z^{(2)}+\Omega_1S_{11}+\Omega_2S_{22},\\
&H_0=\omega_1\left(N_1-S_{11}\right)+\omega_2(N_2+S_{22})+\Omega_1S_{11}+\Omega_2S_{22},\\
&H_0=\omega_1N_1+\omega_2N_2+\left(\Omega_1-\omega_1\right)S_{11}+\left(\Omega_2+\omega_2\right)S_{22},\\
&H_0=\omega_1N_1+\omega_2N_2+\left(\Omega_1-\omega_1\right)\left\{\frac{1}{2}\left(S_{11}-S_{22}\right)+\frac{1}{2}\left(S_{11}+S_{22}\right)\right\}\\
&\quad+\left(\Omega_2+\omega_2\right)\left\{\frac{1}{2}\left(S_{11}+S_{22}\right)-\frac{1}{2}\left(S_{11}-S_{22}\right)\right\},\\
&H_0=\omega_1N_1+\omega_2N_2+\frac{\left(\Omega_1-\omega_1-\Omega_2-\omega_2\right)}{2}\left(S_{11}-S_{22}\right),\\
&H_0=\omega_1N_1+\omega_2N_2+\frac{\Delta}{2}\left(S_{11}-S_{22}\right),\\
&H=\omega_1N_1+\omega_2N_2+\frac{\Delta}{2}\left(S_{11}-S_{22}\right)+H_{\mathrm{int}},\\
&H=N\ +C+H_{\mathrm{int}}\ ,
\end{aligned} \tag{10}$$

where $C=\frac{\Delta}{2}\left(S_{11}-S_{22}\right)+H_{\mathrm{int}}$ with $\Delta=\Omega_1-\omega_1-\Omega_2-\omega_2$. We note that $\left[N,C\right]=0$, therefore $\left[N,H\right]=0=\left[H,C\right]$, *i.e.* N and C are constants of motion, where $N=\omega_1N_1+\omega_2N_2$ the time evolution operator is defined

$$U\left(t\right)=\exp\left(-\mathrm{i}Ht\right), \tag{11}$$

thus

$$U\left(t\right)=\exp\left(-\mathrm{i}\omega_1N_1t\right)\exp\left(-\mathrm{i}\omega_2N_2t\right)\exp\left(-\mathrm{i}Ct\right), \tag{12}$$

where

$$\exp\left(-\mathrm{i}\omega_1 N_1 t\right)=\begin{bmatrix}\exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}+\frac{1}{2}\right)t\right] & 0\\ 0 & \exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}-\frac{1}{2}\right)t\right]\end{bmatrix},$$

$$\exp\left(-\mathrm{i}\omega_2 N_2 t\right)=\begin{bmatrix}\exp\left[-\mathrm{i}\omega_2\left(k_z^{(2)}+\frac{1}{2}\right)t\right] & 0\\ 0 & \exp\left[-\mathrm{i}\omega_2\left(k_z^{(2)}-\frac{1}{2}\right)t\right]\end{bmatrix}, \tag{13}$$

$$C=\begin{bmatrix}\frac{\Delta}{2} & \lambda k_-^{(1)}k_-^{(2)}\\ \lambda k_+^{(1)}k_+^{(2)} & -\frac{\Delta}{2}\end{bmatrix}, \tag{14}$$

$$C^2=\begin{bmatrix}\mu_1^2 & 0\\ 0 & \mu_2^2\end{bmatrix}, \tag{15}$$

where

$$\begin{aligned}\mu_1^2&=\frac{\Delta^2}{4}+\lambda^2 k_-^{(1)}k_+^{(1)}k_-^{(2)}k_+^{(2)}=\frac{\Delta^2}{4}+\nu_1,\\ \nu_1&=\lambda^2 k_-^{(1)}k_+^{(1)}k_-^{(2)}k_+^{(2)},\\ \mu_2^2&=\frac{\Delta^2}{4}+\lambda^2 k_+^{(1)}k_-^{(1)}k_+^{(2)}k_-^{(2)}=\frac{\Delta^2}{4}+\nu_2,\\ \nu_2&=\lambda^2 k_+^{(1)}k_-^{(1)}k_+^{(2)}k_-^{(2)},\end{aligned} \tag{16}$$

we note that

$$k_-^{(1)}k_-^{(2)}\mu_2^2=\mu_1^2 k_-^{(1)}k_-^{(2)}, \tag{17}$$

also

$$k_+^{(1)}k_+^{(2)}\mu_1^2=\mu_2^2 k_+^{(1)}k_+^{(2)}, \tag{18}$$

$$C^3=\begin{bmatrix}\frac{\Delta}{2}\mu_1^2 & \mu_1^2\lambda k_-^{(1)}k_-^{(2)}\\ \mu_2^2\lambda k_+^{(1)}k_+^{(2)} & -\frac{\Delta}{2}\mu_2^2\end{bmatrix}, \tag{19}$$

$$C^4=\begin{bmatrix}\mu_1^4 & 0\\ 0 & \mu_2^4\end{bmatrix}, \tag{20}$$

$$\exp\left(-\mathrm{i}Ct\right)=I+\frac{-\mathrm{i}Ct}{1!}+\frac{\left(-\mathrm{i}Ct\right)^2}{2!}+\frac{\left(-\mathrm{i}Ct\right)^3}{3!}+\cdots=I-\mathrm{i}Ct-\mathrm{i}\frac{C^2t^2}{2!}+\mathrm{i}\frac{C^3t^3}{3!}+\cdots, \tag{21}$$

$$\exp\left(-\mathrm{i}Ct\right)=\begin{bmatrix}\cos\mu_1 t-\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_1 t}{\mu_1} & -\mathrm{i}\lambda\frac{\sin\mu_1 t}{\mu_1}k_-^{(1)}k_-^{(2)}\\ -\mathrm{i}\lambda\frac{\sin\mu_2 t}{\mu_2}k_+^{(1)}k_+^{(2)} & \cos\mu_2 t+\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_2 t}{\mu_2}\end{bmatrix}, \tag{22}$$

for simplicity we can write

$$U(t)=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \tag{23}$$

where

$$\begin{aligned}
F_{11} &= \exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}+\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}+\frac{1}{2}\right)t\right]\left(\cos\mu_1 t-\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_1 t}{\mu_1}\right),\\
F_{12} &= -\mathrm{i}\lambda\exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}+\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}+\frac{1}{2}\right)t\right]\frac{\sin\mu_1 t}{\mu_1}k_-^{(1)}k_-^{(2)},\\
F_{21} &= -\mathrm{i}\lambda\exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}-\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}-\frac{1}{2}\right)t\right]\frac{\sin\mu_2 t}{\mu_2}k_+^{(1)}k_+^{(2)},\\
F_{22} &= \exp\left[-\mathrm{i}\omega_1\left(k_z^{(1)}-\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}-\frac{1}{2}\right)t\right]\left(\cos\mu_2 t+\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_2 t}{\mu_2}\right).
\end{aligned} \tag{24}$$

The time evolution for the expectation value of any operator can be calculate through the following relation

$$\langle O(t)\rangle=\langle\Psi(t)|O(t)|\Psi(t)\rangle=\langle\Psi(0)|U^{\dagger}(t)O(t)U(t)|\Psi(0)\rangle, \tag{25}$$

Now the initial state of the system can be written as

$$|\Psi(0)\rangle=|\Psi(0)\rangle_{SU(2)}\otimes|\Psi(0)\rangle_{SU(1,1)}=\left(\cos\frac{\theta}{2}|e\rangle+\sin\frac{\theta}{2}|g\rangle\right)|m,k\rangle, \tag{26}$$

where

$$\begin{aligned}
k_z^{(1)}|m_1,m_2,k_1,k_2\rangle &= (m_1+k_1)|m_1,m_2,k_1,k_2\rangle,\\
k_z^{(2)}|m_1,m_2,k_1,k_2\rangle &= (m_2+k_2)|m_1,m_2,k_1,k_2\rangle,\\
k_+^{(1)}|m_1,m_2,k_1,k_2\rangle &= \sqrt{(m_1+k_1)(m_1+2k_1)}\,|m_1+1,m_2,k_1,k_2\rangle,\\
k_+^{(2)}|m_1,m_2,k_1,k_2\rangle &= \sqrt{(m_2+k_2)(m_2+2k_2)}\,|m_1,m_2+1,k_1,k_2\rangle,\\
k_-^{(1)}|m_1,m_2,k_1,k_2\rangle &= \sqrt{m_1(m_1+2k_1-1)}\,|m_1-1,m_2,k_1,k_2\rangle,\\
k_-^{(2)}|m_1,m_2,k_1,k_2\rangle &= \sqrt{m_2(m_2+2k_2-1)}\,|m_1,m_2-1,k_1,k_2\rangle,
\end{aligned} \tag{27}$$

$$|\Psi(t)\rangle=U(t)|\Psi(0)\rangle=\left(F_{11}\cos\frac{\theta}{2}+F_{12}\sin\frac{\theta}{2}\right)|m_1,m_2k_1,k_2\rangle|e\rangle+\left(F_{21}\cos\frac{\theta}{2}+F_{22}\sin\frac{\theta}{2}\right)|m_1,m_2k_1,k_2\rangle|g\rangle, \tag{28}$$

Substituting from Equation (26) in Equation (28), then the final form of the wave function can be written as

$$\begin{aligned}
|\Psi(t)\rangle=&\left\{\left(\cos\mu_1 t-\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_1 t}{\mu_1}\right)\cos\frac{\theta}{2}-\mathrm{i}\lambda\frac{\sin\mu_1}{\mu_1}k_-^{(1)}k_-^{(2)}\sin\frac{\theta}{2}\right\}\mathrm{e}^{-\mathrm{i}\omega_1\left(k_z^{(1)}+\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}+\frac{1}{2}\right)t}|m_1,m_2,k_1,k_2\rangle|e\rangle\\
&-\lambda\left\{\mathrm{i}\frac{\sin\mu_2 t}{\mu_2}k_+^{(1)}k_+^{(2)}\cos\frac{\theta}{2}-\left(\cos\mu_2 t+\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_2 t}{\mu_2}\right)\sin\frac{\theta}{2}\right\}\mathrm{e}^{-\mathrm{i}\omega_1\left(k_z^{(1)}-\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}-\frac{1}{2}\right)t}|m_1,m_2,k_1,k_2\rangle|g\rangle.
\end{aligned} \tag{29}$$

Then the wave function can be written in the form

$$|\Psi(t)\rangle=A(t)|m_1,m_2,k_1,k_2\rangle|e\rangle+B(t)|m_1,m_2,k_1,k_2\rangle|g\rangle, \tag{30}$$

and consequently the density matrix $\rho(t)=|\Psi(t)\rangle\langle\Psi(t)|$ becomes

$$\begin{aligned}
\rho(t)=&\{A(t)|m_1,m_2,k_1,k_2\rangle|e\rangle\langle e|\langle k_2,k_1,m_2,m_1|A^{*}(t)+B(t)|m_1,m_2,k_1,k_2\rangle|g\rangle\langle g|\langle k_2,k_1,m_2,m_1|B^{*}(t)\\
&+B(t)|m_1,m_2,k_1,k_2\rangle|g\rangle\langle e|\langle k_2,k_1,m_2,m_1|A^{*}(t)+A(t)|m_1,m_2,k_1,k_2\rangle|e\rangle\langle g|\langle k_2,k_1,m_2,m_1|B^{*}(t)\},
\end{aligned} \tag{31}$$

where

$$A(t)=\mathrm{e}^{-\mathrm{i}\omega_1\left(k_z^{(1)}+\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}+\frac{1}{2}\right)t}\left\{\left(\cos\mu_1 t-\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_1 t}{\mu_1}\right)\cos\frac{\theta}{2}-\mathrm{i}\lambda\frac{\sin\mu_1 t}{\mu_1}k_-^{(1)}k_-^{(2)}\sin\frac{\theta}{2}\right\},$$
$$B(t)=\mathrm{e}^{-\mathrm{i}\omega_1\left(k_z^{(1)}-\frac{1}{2}\right)t-\mathrm{i}\omega_2\left(k_z^{(2)}-\frac{1}{2}\right)t}\left\{\left(\cos\mu_2 t+\frac{\mathrm{i}\Delta}{2}\frac{\sin\mu_2 t}{\mu_2}\right)\sin\frac{\theta}{2}-\mathrm{i}\lambda\frac{\sin\mu_2 t}{\mu_2}k_+^{(1)}k_+^{(2)}\cos\frac{\theta}{2}\right\}, \tag{32}$$

and

$$|A(t)|^2+|B(t)|^2=1. \tag{33}$$

Thus the expectation value for any operator can be calculated through

$$\langle O(t)\rangle=\langle\Psi(0)|O(t)|\Psi(0)\rangle=\langle\Psi(t)|O(0)|\Psi(t)\rangle, \tag{34}$$

where $|\Psi(0)\rangle$ and $|\Psi(t)\rangle$ are defined by Equations (26) and (30). With the help of Equations (32), (34), the expectation values for the atomic operators σ_x, σ_y and σ_z

$$\langle\sigma_x(t)\rangle=\frac{\mathrm{i}}{2}\sin\theta\left\{\cos\mu_1 t\cos\mu_2 t-\frac{\Delta^2}{4}\frac{\sin\mu_1 t\sin\mu_2 t}{\mu_1\mu_2}+\frac{\mathrm{i}\Delta}{2}\left(\frac{\sin\mu_1 t\cos\mu_2 t}{\mu_1}+\frac{\cos\mu_1 t\sin\mu_2 t}{\mu_2}\right)\right\}\sin(\omega_1+\omega_2)t \tag{35}$$

$$\langle\sigma_y(t)\rangle=\frac{i}{2}\sin\theta\left\{\cos\mu_1 t\cos\mu_2 t-\frac{\Delta^2}{4}\frac{\sin\mu_1 t\sin\mu_2 t}{\mu_1\mu_2}+\frac{i\Delta}{2}\left(\frac{\sin\mu_1 t\cos\mu_2 t}{\mu_1}+\frac{\cos\mu_1 t\sin\mu_2 t}{\mu_2}\right)\right\} \tag{36}$$
$$\times\cos(\omega_1+\omega_2)t,$$

$$\langle\sigma_z(t)\rangle=\cos^2\frac{\theta}{2}\left[\left(\cos^2\mu_1 t+\frac{\Delta^2}{4}\frac{\sin^2\mu_1 t}{\mu_1^2}\right)-\lambda^2\frac{\sin^2\mu_1 t}{\mu_1^2}(m_1+1)(m_1+2k_1)(m_2+1)(m_2+2k_2)\right]$$
$$+\sin^2\frac{\theta}{2}\left[2\lambda^2\frac{\sin^2\mu_1 t}{\mu_1^2}m_1(m_1+2k_1-1)m_2(m_2+2k_2-1)-1\right]. \tag{37}$$

Where we have used the abbreviations

$$\mu_1=\sqrt{\frac{\Delta^2}{4}+\lambda^2(m+1)(m+2k)},$$
$$\mu_2=\sqrt{\frac{\Delta^2}{4}+\lambda^2 m(m+2k-1)}. \tag{38}$$

3. Scaled Atomic Wehrl Entropy and Marginal Atomic Q-Function

In this section, we will use the scaled atomic Wehrl entropy as an entanglement quantifier between single two level atom and SU(1,1) quantum system.

The scaled atomic Wehrl entropy can be written in terms of the atomic Q function as [25] [26]:

(39)

In the above equation $S_q(\Theta,\Phi,t)=-Q_A(\Theta,\Phi,t)\ln Q_A(\Theta,\Phi,t)$ is the atomic Wehrl density and $Q_A(\Theta,\Phi,t)$ is the atomic Q-function which is defined as [27] [28]

$$Q_A(\Theta,\Phi,t)=\frac{1}{2\pi}\langle\Theta,\Phi|\hat{\rho}_A(t)|\Theta,\Phi\rangle, \tag{40}$$

where $\rho_A(t)$, is the reduced density of the atom and $|\Theta,\Phi\rangle$ is the atomic coherent state which is defined in the following form

$$|\Theta,\Phi\rangle=\cos(\Theta/2)|\mathrm{e}\rangle+\sin(\Theta/2)\mathrm{e}^{\mathrm{i}\Phi}|g\rangle, \tag{41}$$

where Θ and Φ is the atomic phase space parameters. Then the atomic Q-function can be written in terms of the expectation values of the atomic variables σ_x, σ_y and σ_z as follows

$$Q_A(\Theta,\Phi,t)=\frac{1}{2\pi}\left\{1+\langle\sigma_z(t)\rangle\cos\Theta+\left[\langle\sigma_x(t)\rangle\cos\Phi+\langle\sigma_y(t)\rangle\sin\Phi\right]\sin\Theta\right\}. \tag{42}$$

It is worth noting that from the definition (39) the $SS_{AW}(t)$, cannot be negative as a result of the Q_A is a non-negative function. As it is generally difficult to find a closed form for the $SS_{AW}(t)$ numerical techniques have to be used. Nevertheless, at particular values of the interaction parameters the exact form can be obtained. The shifted (scaled) $S_{AW}(t)$ satisfies the following inequality

$$0\le SS_{AW}(t)\le \ln 2. \tag{43}$$

By integrating the atomic Wehrl density $S_q(\Theta,\Phi,t)$ over the atomic variable Φ, we obtain the marginal atomic Wehrl density as follows

$$S_\Phi=\int_0^{2\pi} S_q\sin\Theta d\Theta. \tag{44}$$

4. Some Statistical Aspects

In this section, we discuss and present some statistical aspects such as the atomic inversion $\rho_z(t)=\frac{1}{2}\langle\sigma_z(t)\rangle$, scaled atomic Wehrl entropy $SS_{AW}(t)$ and marginal atomic Wehrl density S_Φ. We have considered the time has been scaled "one the unit of time is given be the inverse of the coupling constant λ".

The atomic inversion of the atom is one of the important atomic dynamic variables of the system. This in fact would give us information about the behavior of the atom state during interaction time. In **Figure 1**, we have plotted the dynamical behavior for different values of the involved parameters. We concentrate on the variation of the initial atomic position θ from the excited state, *i.e.* $\theta=0$ to the superposition state, *i.e.* $\theta=\pi/2$ as well as on the excitation number m, which is in analogy with the usual Jaynes-Cummings model, corresponding to the number of photons. Firstly, we consider that the system is initially in the excited state $\theta=0$ and the absence of the detuning parameter $\delta=0$. It is observed that the atomic population inversion has a regular and periodic oscillation where the amplitude of oscillation is decreased when $\delta=5$ (see **Figure 1(c)**). **Figures 1(b)**, **(d)** depict the effect of the superposition state (*i.e.* $\theta=\pi/2$), where the amplitude of oscillations is very small when the detuning parameter is taken into consideration (see **Figure 1(d)**).

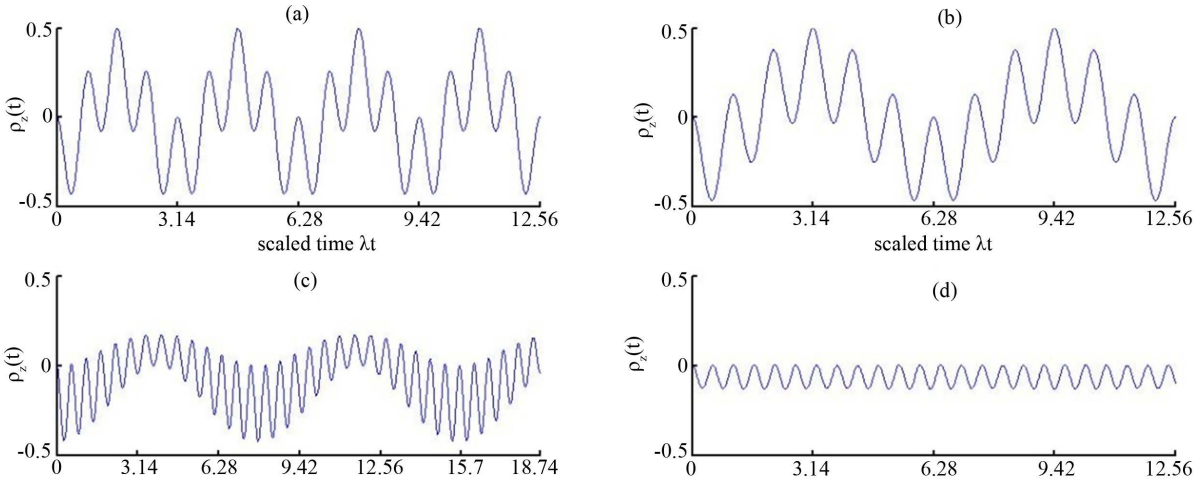

Figure 1. Time evolution of the atomic inversion $\rho_z(t)$ for $\phi=0$, $k_1=k_2=\frac{1}{4}$ and the excitation number $m_1=m_2=1$ and with different values of the initial atomic position θ and detuning parameter δ, where (a) $(\delta,\theta)=(0,0)$, (b) $(\delta,\theta)=\left(0,\frac{\pi}{2}\right)$, (c) $(\delta,\theta)=(5,0)$ and (d) $(\delta,\theta)=\left(5,\frac{\pi}{2}\right)$.

Figure 2 depicts the dynamical behavior of the scaled atomic Wehrl entropy SS_{AW} for different values of detuning parameter and initial atomic position when the phase shift between the two levels is neglected and the excitation number is taken to be the unity. From **Figure 1(a)**, one can infer that before the interaction the scaled atomic Wehrl entropy is equal to zero and information about energy levels is not available. This implies that entanglement cannot be performed before the interaction is switched on. As the scaled time goes on, one see that SS_{AW} is growing and reaches a local maximum value but after a sometime interaction the difference between local maximum and local minimum becomes bigger. The system returns on its separable state (*i.e.* $SS_{AW}=0$) at $gt=m\pi$, where $m=0,1,2,\cdots$. The high amount of entanglement is achieved around the half of periodic time at $gt=(2m+1)\pi$. The amplitude of SS_{AW} is decrees which means that the case of weak entanglement between the two-level atom and input field when the effect of the detuning parameter is taken into account (see **Figure 2(c)**). It is observed that the dynamical behavior of SS_{AW} is completely changed when the two-level atom starts the interaction from the superposition state $(\theta=\pi/4)$. In general one can see that the high amount of entanglement is achieved during the time evolution in the comparison with the upper state case $(\theta=0)$. On the other hand $SS_{AW}=0$ is dropped to zero at $gt=(2m+1)\pi$. Finally, **Figure 2(d)** presents the influence of the detuning parameter on the evolution of SS_{AW} when the initial atomic position $\theta=\pi/4$.

Now we are in a position to discuss the evolution of the marginal atomic Wehrl density S_{Φ} as a function of the time and atomic phase space parameter Φ for different values of initial state setting and detuning parameter. It is interesting to mention here that the behavior of $S_{\Phi}(t)$ for different values of the non-fluctuating components of Rabi frequency. It is observed that $S_{\Phi}(t)$ oscillates between minimum and maximum peaks during the time evolution. The distribution of the marginal atomic Wehrl density peaks in depending the initial state setting of the two-level atom when the effect detuning parameter is neglected. The behavior of $S_{\Phi}(t)$ peaks becomes regular and periodic when the effect of the detuning parameter is considered. In this case there the initial state setting has weak effect on the dynamical behavior of S_{Φ} (**Figure 3**).

5. Conclusion

Quantum entanglement is a key resource which distinguishes quantum information theory from classical one. It plays a central role in quantum information and computation. In this paper, we have discussed the problem of the interaction between two-level atom and SU(1,1) quantum system. The model was considered when the two-level atom is initially in superposition state and the expectation values of the atomic variable are obtained

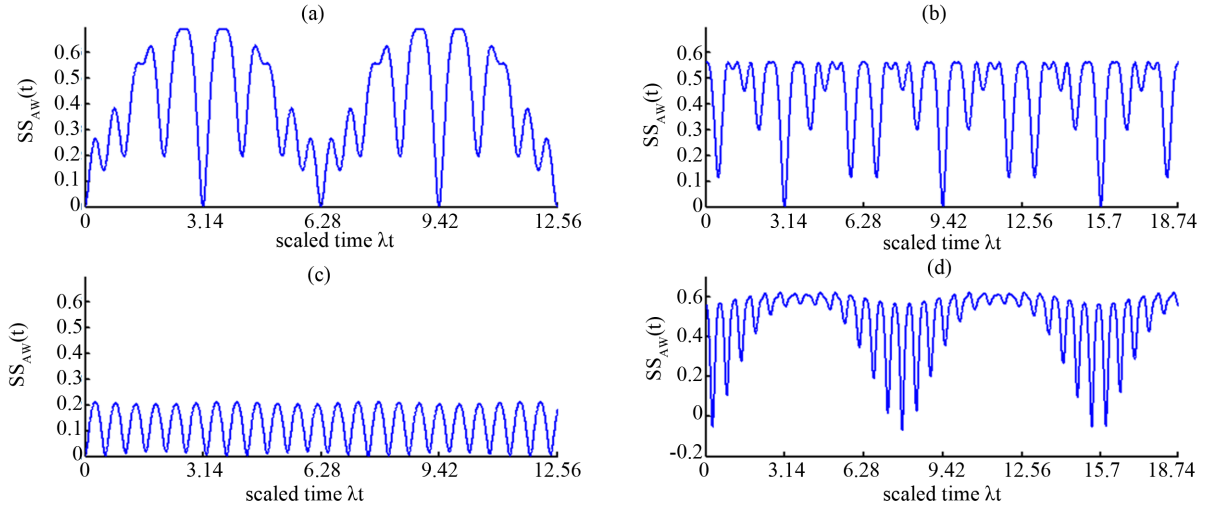

Figure 2. Time evolution of the scaled atomic Wehrl entropy $SS_{AW}(t)$, for $\phi=0$, $k_1=k_2=\frac{1}{4}$ and the excitation number $m_1=m_2=1$ and with different values of the initial atomic position θ and detuning parameter δ, where (a) $(\delta,\theta)=(0,0)$, (b) $(\delta,\theta)=\left(0,\frac{\pi}{2}\right)$, (c) $(\delta,\theta)=(5,0)$ and (d) $(\delta,\theta)=\left(5,\frac{\pi}{2}\right)$.

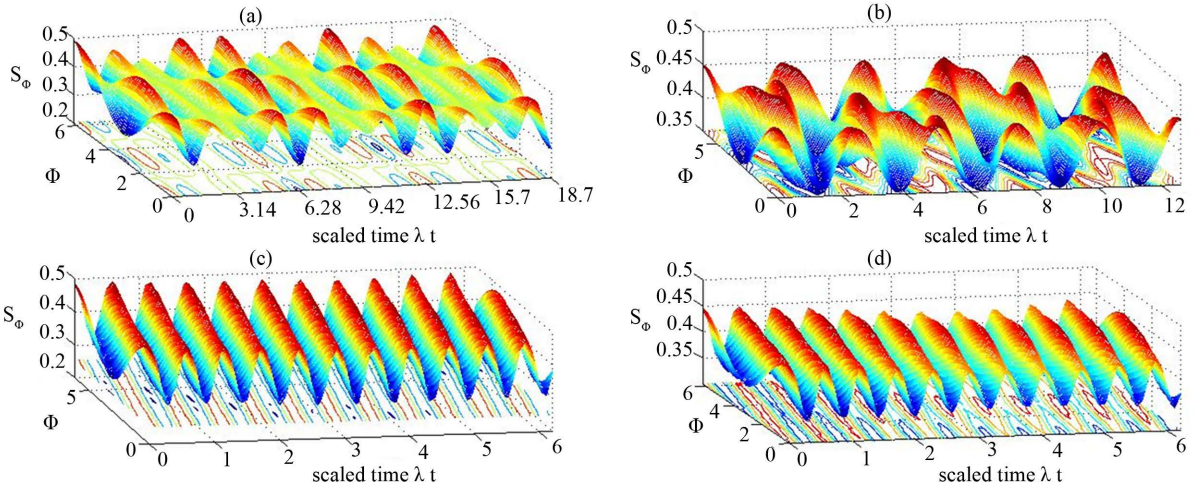

Figure 3. The surface plot of the marginal atomic Wehrl entropy $S_\Phi(t)$ versus the scaled time λt and the phase space parameter Φ for $\phi = 0$, $k_1 = k_2 = \frac{1}{4}$ and the excitation number $m_1 = m_2 = 1$ and with different values of the initial atomic position θ and detuning parameter δ, where (a) $(\delta,\theta) = (0,0)$, (b) $(\delta,\theta) = \left(0,\frac{\pi}{2}\right)$, (c) $(\delta,\theta) = (5,0)$ and (d) $(\delta,\theta) = \left(5,\frac{\pi}{2}\right)$.

analytically. Using the scaled atomic Wehrl entropy the system entanglement has been investigated. The analysis herein has been carried out at two distinct considerations of the detuning parameter and initial atomic state setting. Our results show that the SU(1,1) quantum field-atom interaction considering the effect of the initial state setting and detuning parameter has much richer structure. The initial atomic state position and detuning parameter has an important role on the dynamics of the atomic inversion, scaled atomic Wehrl entropy and marginal atomic Wehrl density.

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