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A Systematization for One-Loop 4D Feynman Integrals-Different Species of Massive Fields

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ABSTRACT

A systematization for the manipulations and calculations involving divergent (or not) Feynman integrals, typical of the one loop perturbative solutions of Quantum Field Theory, is proposed. A previous work on the same issue is generalized to treat theories and models having different species of massive fields. An improvement on the strategy is adopted so that no regularization needs to be used. The final results produced, however, can be converted into the ones of reasonable regularizations, especially those belonging to the dimensional regularization (in situations where the method applies). Through an adequate interpretation of the Feynman rules and a convenient representation for involved propagators, the finite and divergent parts are separated before the introduction of the integration in the loop momentum. Only the finite integrals obtained are in fact integrated. The divergent content of the amplitudes are written as a combination of standard mathematical object which are never really integrated. Only very general scale properties of such objects are used. The finite parts, on the other hand, are written in terms of basic functions conveniently introduced. The scale properties of such functions relate them to a well defined way to the basic divergent objects providing simple and transparent connection between both parts in the assintotic regime. All the arbitrariness involved in this type of calculations are preserved in the intermediary steps allowing the identification of universal properties for the divergent integrals, which are required for the maintenance of fundamental symmetries like translational invariance and scale independence in the perturbative amplitudes. Once these consistency relations are imposed no other symmetry is violated in perturbative calculations neither ambiguous terms survive at any theory or model formulated at any space-time dimension including nonrenormalizable cases. Representative examples of perturbative amplitudes involving different species of massive fermions are considered as examples. The referred amplitudes are calculated in detail within the context of the presented strategy (and systematization) and their relations among other Green functions are explicitly verified. At the end a generalization for the finite functions is presented.

Keywords: Feynman Integrals; Perturbative Amplitudes

1. Introduction

Given the fact that exact solutions for Quantum Field Theories (QFT) are rarely possible, almost all knowledge constructed through this formalism about the phenomenology of fundamental interacting particles has been obtained within the context of perturbative techniques. In order to get the predictions in such framework, many nontrivial mathematical difficulties must be circumvented due to the presence of infinities or divergences in the perturbative series for the elementary process. We have to find a consistent prescription to handle the mathematical indefiniteness involved, which means to avoid the breaking of global and local symmetries as well as simultaneously to avoid ambiguities in the produced results. By ambiguities we understand any dependence on the final results on possible arbitrary choices involved in intermediary steps of the calculations. If they exist, undoubtedly, the predictive power of the formalism it is destroyed. The first and most immediate of such ambiguities are those associated with the choices of the labels for the momenta carried by the internal lines of loop perturbative amplitudes. They naturally appear when the divergence degree is higher than the logarithmic one. The result for such amplitudes may be dependent on the particular choices for the routings due to the fact that in this case the amplitudes are not invariant under shifts in the loop momentum. A second and important type of choice is the regularization prescription. Two different choices for the regularization can lead to different results for the calculated amplitudes. These two kinds of ambiguities are very well-known in the corresponding literature. A third and more general one has been recently considered in the context of perturbative calculations, which is the denominated scale ambiguities [1]. They are related to the choice for a common scale for the finite and divergent parts when they are separated in a Feynman integral. There is an arbitrariness involved in the separation of these terms in a summation when they have different divergence degrees. The scale properties of the perturbative amplitudes are the most general guides for the consistency of the procedures. There are situations in which a symmetry violating is non-ambiguous relative to the choice for the labels of the internal lines momentum but it is ambiguous relative to the choice for the common scale. In addition to the difficulties coming from the divergences we frequently have also those coming from the extension of the mathematical expressions involved. Apart from a few number of simple amplitudes, the mathematical complexity of the obtained expression, not rarely, makes prohibitive any analysis of the obtained results.

Considering these aspects of the perturbative calculations in QFT it would be desirable to get a procedure to manipulate and calculate divergent physical amplitudes without compromising the results with a particular regularization scheme. In addition to this, we would like to make the calculations preserving all the possible choices for the arbitrariness involved like those related to the choice of routings for the internal momenta and for the common scale for the finite and divergent parts. To complete such adequate calculational strategy it would be desirable to get also a systematization for the finite parts of the amplitudes in a way that the mathematical expressions become simple allowing the required analysis and algebraic operations related to the renormalization procedures, among others.

If one agrees with this line of reasoning the present work may constitute a contribution on this direction. We present in this paper a calculational strategy which fulfills the requirements stated above. We start by formulating the steps involved in the calculation of perturbative amplitudes, through the corresponding Feynman rules, in such a way that no regularization needs to be specified. The calculations are made by using arbitrary choices for the internal lines of loop amplitudes and an arbitrary scale parameter is introduced in the separation of terms associated with different degrees of divergences. Through the procedure no divergent integral is really calculated. They are reduced to standard forms which are then untouched. The finite parts are not contaminated with any type of modification and a systematization through structure functions is introduced. The result is a completely algebraic procedure where no limits or expansions are taken. All the procedures like Ward identities verifications, renormalization procedures and so on, are made by using properties of the finite functions and basic divergent objects. In addition to this, the important aspect of the procedure is its general character; all the amplitudes in all theories and models are treated in an absolutely identical way. We treat amplitudes in renormalizable and non renormalizable theories formulated in even and odd space-time dimension within the same strategy. Symmetry violating terms as well as ambiguous ones may be simultaneously eliminated in a consistent way. Anomalous amplitudes are consistently described without the presence of ambiguities in any (even) space-time dimension.

The material we present in this work may be considered as an extension of that presented in [2]. The questions considered here are not new. In the literature there are many works about this issue and certainly many others continue to be done nowadays. In particular, the reduction of tensor integrals to scalar ones, made in the present work through the properties of the introduced finite functions, has been studied by Passarino and Veltman [3] as well as other authors [4-12]. The scalar integrals has been considered by G.'t Hooft and Veltman [13]. Recently, new works have been produced specially involving massless propagators like in [14-29] (and references therein). The present systematization for the perturbative calculations must be understood as a contribution to this type of investigation. The very general character of the procedure and the absence of restrictions of applicability may represent some advantages which can be useful for some users of the perturbative solutions of QFT's. With the material presented here any self-energy, decay amplitude and elastic scattering of two fields can be calculated in fundamental theories.

The work is organized as follows. In the Section 2 we define the set of basic one-loop 4D Feynman integrals which we will discuss in future sections. In the Section 3 we explain the strategy adopted to handle the divergences as well as we define the basic divergent objects used to write the divergent content of the perturbative amplitudes. The basic functions (and some of their useful properties) used to systematize the finite parts of the amplitudes are introduced in the Section 4. The solution of the basic one-loop integrals is considered in the Section 5 and the explicit calculation of perturbative amplitudes in the Section 6. In the Section 7 we consider the explicit verification of the relations among the Green functions for the calculated amplitudes and in the Section 8 the questions related to the ambiguities and symmetry relations are discussed. A generalization for the finite functions and their useful properties are presented in the Section 9 and, finally, in the Section 10 we present our final remarks and conclusions.

2. Basic One-Loop Feynman Integrals

First of all we call the attention to the fact that in perturbative calculations, independently of the specific theory or model, in loop amplitudes, we have to take the integration over the unrestricted momentum. We can consider such an operation as the last Feynman rule. Precisely at this step all the one-loop perturbative amplitudes will become combinations of a relatively small number of mathematical structures, the Feynman integrals. Some of such structures are undefined quantities because they are divergent integrals. Given this situation we have at our disposal two distinct but, in principle, equivalent attitudes to adopt. We can perform the calculation of the desired amplitudes one by one, within the context of a chosen regularization prescription or equivalent philosophy, ignoring any type of possible systematization of the procedures or identifying the set of operations we have to repeat in calculating different amplitudes considering such required operations in a separatedly way. In adopting the second option, the immediate systematization of the perturbative calculations is to consider the study of the set of Feynman integrals we need to solve in order to calculate all the one-loop amplitudes. Here we will restrict our attention to the fundamental theories but this attitude can always be followed.

In this linere of asoning we first separate the amplitudes by the number of internal lines or propagators. Thus the one propagator amplitudes in fundamental theories will be reduced, in some step of the calculations, to a combination of the integrals

$$\left(I_1;I_1^{\mu}\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(1;k^{\mu}\right)}{D_i}. \tag{1}$$

Here we introduced the definition $D_i=\left[\left(k+k_i\right)^2-m_i^2\right]$. Such structures are the most simple ones but are also those having the most severe degree of divergences: the cubic one $\left(I_1^{\mu}\right)$. The one-loop amplitudes having two internal propagators, on the other hand, will be written as a combination of the structures

$$\left(I_2;I_2^{\mu};I_2^{\mu\nu}\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(1;k^{\mu};k^{\mu}k^{\nu}\right)}{D_{ij}}. \tag{2}$$

Here $D_{ij}=D_iD_j$. The highest degree of divergence here is the quadratic one occurring in $I_2^{\mu\nu}$. In calculating amplitudes having three internal propagators we need to evaluate the integrals

$$\left(I_3;I_3^{\mu};I_3^{\mu\nu};I_3^{\mu\nu\lambda}\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(1;k^{\mu};k^{\mu}k^{\nu};k^{\mu}k^{\nu}k^{\lambda}\right)}{D_{ijl}}. \tag{3}$$

Here we have defined $D_{ijl}=D_{ij}D_l$. The higher degree of divergence involved in the above set of integrals is the linear one in $I_3^{\mu\nu\lambda}$. Two of them are finite structures. We can introduce also the ingredients required to calculate amplitudes having four internal lines, the four propagators Feynman integrals

$$\begin{aligned}&\left(I_4;I_4^{\mu};I_4^{\mu\nu};I_4^{\mu\nu\lambda};I_4^{\mu\nu\alpha\beta}\right)\\&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(1;k^{\mu};k^{\mu}k^{\nu};k^{\mu}k^{\nu}k^{\lambda};k^{\mu}k^{\nu}k^{\alpha}k^{\beta}\right)}{D_{ijlm}}.\end{aligned} \tag{4}$$

Now $D_{ijlm}=D_{ijl}D_m$. Only one of such structures is divergent which is the logarithmically divergent structure $I_4^{\mu\nu\alpha\beta}$.

In the above definitions $k+k_i$ and m_i are the arbitrary momentum carried by an internal propagator and its mass, respectively. The arbitrary internal momenta k_i are related to the external ones through the relations of energy-momentum conservation in vertices connecting the internal lines with the external ones. The adoption of arbitrary routing for the internal lines momenta is of crucial importance due to the divergent character of the Feynman integrals involved, in particular for those having degree of divergence higher than the logarithmic one just because in this case the result may be dependent on the chosen routing. In adopting such general arbitrary routing for the internal lines we can identify possible ambiguous terms arising in a certain calculation which are undefined combinations of the internal lines momenta (not related to the external ones). This aspect will become clear in a moment.

When we find a combination of divergent Feynman integrals in a certain step of the calculation of a perturbative amplitude, in order to give an additional step we have to specify the prescription we will adopt to handle the mathematical indefinitions involved. Usually this means adopting a regularization prescription or an equivalent philosophy. All the results, after this, will be compromised with the particular aspects of the chosen regularization. The so obtained results will represent only the consequences of the arbitrary choice made for the regularization. Even if there are elements of the calculations which are independent of the regularization scheme employed, certainly, there are parts of the result which will be specific of the particular regularization used.

In the present work we will follow an alternative procedure. We will not compromise the results with a particular choice in any step of the calculation. The choice for the regularization will be avoided. The routing of the internal lines momenta will be taken as arbitrary and the most important and new aspect specially for calculations involving different species of massive fields, the common scale for the finite and divergent parts, will be assumed also as being arbitrary. With this attitude all the possibilities for such choices will still remain in the final results. Thus, it will be possible to make a very general analysis of the results searching for the universal conditions which are necessary to be preserved in order to get consistent results in perturbative calculations. This means to obtain results which are simultaneously free from ambiguous and symmetry violating terms. In order to fulfill this program, in the next section, we will describe the strategy to be adopted in the manipulations and calculations of divergent Feynman integrals.

3. The Strategy to Handle Divergent Feynman Integrals and the Basic Divergent Structures

When we use the Feynman rules to construct the perturbative amplitudes there are two distinct steps. First, with propagators, vertex operators, combinatorial factors, traces over Dirac matrices, traces over internal symmetries operators and so on, we construct the amplitudes for one value of the loop momentum k. The next step is to take a summation over all values for such momentum, since it is not restricted by the energy momentum conservation at all vertices of the corresponding diagram. This means integrating over the loop momentum. It is possible to use these two distinct moments of the calculation to formulate a strategy to handle the divergences present in perturbative calculation of QFT which may avoid the use of a regularization [30]. The idea is very simple and does not involve any kind of magic. Only an adequate interpretation of the usual procedures is required. The first step is the same described above: to construct the amplitude corresponding to one value of the unrestricted momentum. Then before taking the integration, the last Feynman rule, we make a counting in the power of loop momentum in order to get the superficial degree of divergence of the amplitude in the space-time dimension we are working. Having this at hand we adopt the following representation for the involved propagators

$$\frac{1}{D_i}=\frac{1}{\left[\left(k+k_i\right)^2-m_i^2\right]}$$
$$=\sum_{j=0}^{N}\frac{(-1)^j\left(k_i^2+2k_i\cdot k+\lambda^2-m_i^2\right)^j}{\left(k^2-\lambda^2\right)^{j+1}}$$
$$+\frac{(-1)^{N+1}\left(k_i^2+2k_i\cdot k+\lambda^2-m_i^2\right)^{N+1}}{\left(k^2-\lambda^2\right)^{N+1}\left[\left(k+k_i\right)^2-m_i^2\right]}, \tag{5}$$

taking N in the summation as equal or major than the superficial degree of divergence. Here λ is an arbitrary parameter having dimension of mass which plays the role of a common scale to both finite and divergent parts of the corresponding Feynman integral. Through this parameter a precise connection between the finite and divergent parts is stated. Note that (as must be required) the expression above is an identity and in addition the right hand side is really independent of the arbitrary parameter λ^2. After the adoption of the adequate representation for the propagators and making all the convenient algebraic reorganizations, we take the integration over the loop momentum k. Then we note that the internal momenta dependent parts of the Feynman integrals are located only in finite integrals. On the other hand, the divergent parts will reside in standard forms of divergent integrals, after a convenient reorganization, where no physical parameter is present. Then we can perform the integration of the finite integrals obtained and in the divergent ones we need not to make any additional operation.

In order to allow a compactation of some expressions in future sections it is convenient to introduce the definition $A_i=k_i^2+2k_i\cdot k+\lambda^2-m_i^2$, so that we can write the above expression as

$$\frac{1}{D_i}=\sum_{j=0}^{N}\frac{(-1)^j\left(A_i\right)^j}{\left(k^2-\lambda^2\right)^{j+1}}+\frac{(-1)^{N+1}\left(A_i\right)^{N+1}}{\left(k^2-\lambda^2\right)^{N+1}D_i}. \tag{6}$$

The steps above described, required to implement the procedure, can be formulated within the context of the language of regularizations. In such formulation we take the integration over the loop momentum and then the divergences are stated. We adopt then a regularization in an implicit way in all Feynman integrals. It is required of such regularization distribution only very general properties. In addition to rendering the integral convergent we require that such distribution is even in the loop momentum in order to be consistent with the Lorentz symmetry and that a "connection limit" exists. Schematically

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}f(k)\to\int\frac{\mathrm{d}^4k}{(2\pi)^4}f(k)\left\{\lim_{\Lambda_i^2\to\infty}G_{\Lambda_i}\left(k^2,\Lambda_i^2\right)\right\}$$
$$=\int_\Lambda\frac{\mathrm{d}^4k}{(2\pi)^4}f(k).$$

where the $\Lambda_i's$ are parameters of the distribution $G\left(\Lambda_i^2,k^2\right)$, and the limits which allow to remove the distribution in the finite integrals

$$\lim_{\Lambda_i^2\to\infty}G_{\Lambda_i}\left(k^2,\Lambda_i^2\right)=1,$$

must be well-known. By assuming the presence of this very general regularization we can manipulate the integrand through algebraic identities just because the integrals are then finite. Next, the identity (5) is used to rewrite the propagators in the Feynman integrals. In the so obtained finite integrals we take the connection limit eliminating the regularization and performing then the integration. In the divergent integrals so obtained no additional modifications are made. Only a convenient reorganization in the form of standard objects is promoted.

There are no practical differences in both procedures described above. The only difference is the presence of the subscript Λ in the divergent integrals indicating that a regularization was assumed in an implicit way. The first formulation however represents the evolution of the second one proposed and developed by O. A. Battistel and denominated as implicit regularization, just because it allows us to perform all the necessary calculations without mentioning the word regularization in perturbative calculation for any purposes, as we shall see in what follows when representative examples of amplitudes calculations will be considered in detail.

The terms which will be converted in divergent integrals, when the integration over the loop momentum is taken, can be conveniently organized so that all the divergent content is present in the standard objects (at the one-loop level in fundamental theories)

$$\Box_{\alpha\beta\mu\nu}\left(\lambda^2\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{24k_\mu k_\nu k_\alpha k_\beta}{\left(k^2-\lambda^2\right)^4}-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4g_{\alpha\beta}k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4g_{\alpha\nu}k_\beta k_\mu}{\left(k^2-\lambda^2\right)^3}-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4g_{\alpha\mu}k_\beta k_\nu}{\left(k^2-\lambda^2\right)^3}, \tag{7}$$

$$\Delta_{\mu\nu}\left(\lambda^2\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{4k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{\left(k^2-\lambda^2\right)^2}, \tag{8}$$

$$\nabla_{\mu\nu}\left(\lambda^2\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2k_\nu k_\mu}{\left(k^2-\lambda^2\right)^2}-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{g_{\mu\nu}}{\left(k^2-\lambda^2\right)}, \tag{9}$$

$$I_{\log}\left(\lambda^2\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\left(k^2-\lambda^2\right)^2}, \tag{10}$$

$$I_{\text{quad}}\left(\lambda^2\right)=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{\left(k^2-\lambda^2\right)}. \tag{11}$$

In nonrenormalizable theories or in two or more loops calculations new objects analogous to these can be defined. Note that all the steps performed are perfectly valid within reasonable regularization prescriptions, including the dimensional regularization technique. This means that it is possible to make contact with the results corresponding to the ones belonging to such methods. To do this it is only necessary to evaluate the divergent structures obtained according to the specific chosen regularization prescription just because the finite parts must be the same due to the fact that, in all acceptable regularization the connection limit must exist. As a consequence, finite integrals must not be modified. More details about the procedure will be presented in a moment when examples of perturbative (divergent) amplitudes are considered.

4. Basic Structure Functions for the Finite Parts

Once the procedure described above is adopted, finite Feynman integrals must be solved. In general, to solve such integrals is not a problematic task. However, frequently, the obtained result is a very large mathematical expression making difficult any type of analysis. The experience, in realizing such type of calculations, revealed that it is possible to identify basic functions to systematize the results for the finite parts of the perturbative Green functions so that the results became very simplified and all the analysis required became simple and transparent. Such basic functions will emphasize, in a natural way, many important aspects typical of the perturbative physical amplitudes like, for example, unitarity. Further required manipulations, in renormalization procedures, in the verification of relations among Green functions or Ward identities, can be completely simplified in terms of simple properties of such basic functions. It is possible to show that the finite parts of amplitudes having a certain number of internal propagators can be reduced to a unique function written, in an integral form, in terms of Feynman parameters. Our next task will be to define the referred basic structures and to explicit their useful properties to be used in posterior sections where we will consider the evaluation of the divergent Feynman integrals defined in the first subsection above. The properties considered for such basic functions will be used in future sections, when we will consider explicit examples of amplitudes evaluation and in the verification of relations among Green functions.

4.1. Basic Two-Point Structure Functions

After the adoption of the procedure described in the Section 3 above, when we are considering a calculation involving amplitudes having two internal propagators the finite parts so obtained can be always written in terms of the following functions

$$Z_k\left(m_1^2;p^2,m_2^2;\lambda^2\right)=\int_0^1\mathrm{d}x\,x^k\ln\left(\frac{Q}{-\lambda^2}\right). \tag{12}$$

In the expression above, p is a momentum carried by an internal line or a combination of them, m_1 and m_2 are masses carried by the propagators, λ is a parameter with dimension of mass which plays the role of a common scale for all the involved physical quantities and $Q\left(m_1^2;p,m_2^2,x\right)=p^2x(1-x)+\left(m_1^2-m_2^2\right)x-m_1^2$. The role of the masses can be inverted through a simple change in the integration variable. In intermediary steps of perturbative calculations it is enough to maintain the integral representation but if one wants to solve the integration in the Feynman parameter this operation can be easily performed. For the first component of the above set of functions we will obtain

$$Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right) = -\left\{2-\ln\left(\frac{m_2^2}{\lambda^2}\right)+\frac{\left(p^2+m_1^2-m_2^2\right)}{2p^2}\ln\left(\frac{m_2^2}{m_1^2}\right) + \frac{h\left(m_1^2;p^2,m_2^2;\lambda^2\right)}{2p^2}\right\},$$

where $h\left(m_1^2;p^2,m_2^2;\lambda^2\right)$ possesses three representations:

1) for $p^2<\left(m_1-m_2\right)^2$. In this region of values for p^2 we have

$$h = 2\sqrt{\left(m_1-m_2\right)^2-p^2}\sqrt{\left(m_1+m_2\right)^2-p^2} \times\ln\left\{\frac{\sqrt{\left(m_1+m_2\right)^2-p^2}-\sqrt{\left(m_1-m_2\right)^2-p^2}}{\sqrt{\left(m_1+m_2\right)^2-p^2}+\sqrt{\left(m_1-m_2\right)^2-p^2}}\right\}.$$

2) for $\left(m_1-m\right)_2^2<p^2<\left(m_1+m_2\right)^2$. In this case we get

$$h = -4\sqrt{\left(m_1+m_2\right)^2-p^2}\sqrt{\left(m_1-m_2\right)^2+p^2} \times\arctan\left\{\frac{\sqrt{p^2-\left(m_1-m_2\right)^2}}{\sqrt{\left(m_1+m_2\right)^2-p^2}}\right\}$$

3) for $p^2>\left(m_1+m_2\right)^2$. In this region we write

$$h = 2\sqrt{p^2-\left(m_1+m_2\right)^2}\sqrt{p^2-\left(m_1-m_2\right)^2} \times\ln\left\{\frac{\sqrt{p^2-\left(m_1-m_2\right)^2}-\sqrt{p^2-\left(m_1+m_2\right)^2}}{\sqrt{p^2-\left(m_1-m_2\right)^2}+\sqrt{p^2-\left(m_1+m_2\right)^2}}\right\} + 2i\pi\sqrt{p^2-\left(m_1+m_2\right)^2}\sqrt{p^2-\left(m_1-m_2\right)^2}.$$

We can note then that the function $Z_k\left(m_1^2;p^2,m_2^2;\lambda^2\right)$ acquires an imaginary part in the region $p^2>\left(m_1+m_2\right)^2$, as required by unitarity. It is possible to state relations among the functions corresponding to different values for k. Examples of such relations are

$$Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right) = \frac{m_2^2}{2p^2}\ln\frac{m_2^2}{\lambda^2}-\frac{m_1^2}{2p^2}\ln\frac{m_1^2}{\lambda^2}+\frac{\left(m_1^2-m_2^2\right)}{2p^2} + \frac{\left(p^2+m_1^2-m_2^2\right)}{2p^2}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right], \quad (13)$$

$$Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right) = -\frac{1}{18}+\frac{\left(m_1^2-m_2^2\right)}{6p^2}+\frac{m_2^2}{3p^2}\ln\frac{m_2^2}{\lambda^2} + \frac{2}{3}\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] - \frac{m_1^2}{3p^2}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]. \quad (14)$$

Through such relations all components of the set can be reduced to that having the number of k reduced in one unity and successively to finally be reduced to only the $k=0$ function. These type of reduction is very useful in verifications of symmetry relations as we shall see in a moment.

4.2. Basic Three-Point Structure Functions

In evaluating the finite parts of Feynman integrals associated with amplitudes having three internal propagators, Equation (3), we can obtain considerable simplification if the results are written in terms of the following functions

$$\xi_{nm}\left(m_1^2;p,m_2^2;q,m_3^2\right)=\int_0^1\mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\,\frac{x_1^n x_2^m}{Q}, \quad (15)$$

where p and q are momenta of the internal lines or a combination of them and,

$$Q\equiv Q\left(m_1^2;p,m_2^2,x_1;q,m_3^2,x_2\right) = p^2x_1\left(1-x_1\right)+q^2x_2\left(1-x_2\right)-2\left(p\cdot q\right)x_1x_2 + \left(m_1^2-m_2^2\right)x_2+\left(m_1^2-m_3^2\right)x_1-m_1^2.$$

If the considered amplitude possesses two or more Lorentz indexes it is useful to define another set of auxiliary functions. They are defined as

$$\eta_{nm}\left(m_1^2;p,m_2^2;q,m_3^2;\lambda^2\right)=\int_0^1\mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\; x_1^n x_2^m\ln\left(\frac{Q}{-\lambda^2}\right). \quad (16)$$

The elements of the above set of functions can be reduced to ξ_{nm} and Z_k functions if useful or necessary. However, in intermediary steps of calculations it is frequently convenient to maintain the presence of η_{nm} function to give a compactation of the results and operations. Now we consider useful properties for the functions ξ_{nm} and η_{nm}.

The first aspect is relative to the reduction of all the elements of the set having a certain value for $n+m$ to that having $n+m-1$. We now show such reduction firstly considering those for $n+m=1$. We start by considering ξ_{01}. After some algebraic effort, which involves only basic mathematical operations like integration by parts, we can write the expression

$$\xi_{01} = \frac{C_1}{2}\left\{\left(\frac{q^2-(p\cdot q)}{p^2q^2}\right)\left[-Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{(p\cdot q)}{p^2q^2}\left[-Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]+\frac{1}{p^2}\left[Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]\right.$$
$$\left.+\left[\left(\frac{p^2+m_1^2-m_2^2}{p^2}\right)-\frac{(p\cdot q)}{p^2}\left(\frac{q^2+m_1^2-m_3^2}{q^2}\right)\right]\xi_{00}\right\},$$

where we have defined $C_1 = \dfrac{p^2q^2}{p^2q^2-(p\cdot q)^2}$.

Through the same type of manipulations the function ξ_{10} can be written as

$$\xi_{10} = \frac{C_1}{2}\left\{\left(\frac{p^2-(p\cdot q)}{p^2q^2}\right)\left[-Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{(p\cdot q)}{p^2q^2}Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)+\frac{1}{q^2}Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right.$$
$$\left.+\left[\left(\frac{q^2+m_1^2-m_3^2}{q^2}\right)-\frac{(p\cdot q)}{q^2}\left(\frac{p^2+m_1^2-m_2^2}{p^2}\right)\right]\xi_{00}\right\}.$$

In the last two equations above, we can note that both functions ξ_{10} and ξ_{01} may be related through a set of simultaneous transformations.

The reduction of the functions ξ_{20} and ξ_{02} can be written as

$$\xi_{02} = \frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-q^2}{p^2q^2}\right]\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{(p\cdot q)}{p^2q^2}Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right.$$
$$\left.+\frac{1}{p^2}\eta_{00}+\left[\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{01}\right\},$$

and

$$\xi_{20} = \frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-p^2}{p^2q^2}\right]\left[Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{(p\cdot q)}{p^2q^2}Z_1\left(m_1;q^2,m_3;\lambda^2\right)+\frac{1}{q^2}\eta_{00}\right.$$
$$\left.+\left[\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{10}\right\}.$$

For the component ξ_{11} on the other hand, it is interesting to obtain two alternative forms. First we write

$$\xi_{11} = \frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-q^2}{p^2q^2}\right]Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{p^2}Z_1\left(m_1;q^2,m_3;\lambda^2\right)-\frac{(p\cdot q)}{p^2q^2}\eta_{00}\right.$$
$$\left.+\left[\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{10}\right\}.$$

The second form is

$$\xi_{11} = \frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-p^2}{p^2q^2}\right]\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{q^2}Z_1\left(m_1;p^2,m_2;\lambda^2\right)\right.$$
$$\left.-\frac{(p\cdot q)}{p^2q^2}\eta_{00}+\left[\frac{\left(q^2+m_1^2-m_3^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(p^2+m_1^2-m_2^2\right)}{q^2}\right]\xi_{10}\right\}.$$

The explicit expressions for the ξ_{nm} functions, corresponding to $n+m=2$, can be completed if we develop the η_{00} in terms of ξ_{nm} and Z_k functions. Such function can be written as

$$\eta_{00} = \frac{1}{2}\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\left[\frac{1}{2}+m_1^2\xi_{00}\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{10}+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{01}.$$

The expressions corresponding to the first reduction of the ξ_{nm} functions having $n+m=3$ are

$$\xi_{30}=\frac{C_1}{2}\left\{\left[\frac{p\cdot q-p^2}{p^2q^2}\right]\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{p\cdot q}{p^2q^2}Z_2\left(m_1^2;q^2,m_3^2;\lambda^2\right)+\frac{1}{q^2}[2\eta_{10}]\right.$$
$$\left.+\left[\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}-\frac{(p\cdot q)}{q^2}\frac{\left(p^2+m_1^2-m_3^2\right)}{p^2}\right]\xi_{20}\right\}.$$

and

$$\xi_{03}=\frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-q^2}{p^2q^2}\right]\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-2Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\right.$$
$$\left.-\frac{p\cdot q}{p^2q^2}Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)+\frac{1}{p^2}[2\eta_{01}]+\left[\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{02}\right\}.$$

The two different forms for the function ξ_{12}. are written as

$$\xi_{21}=\frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-q^2}{p^2q^2}\right]Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{p^2}Z_2\left(m_1^2;q^2,m_3^2;\lambda^2\right)-\frac{p\cdot q}{p^2q^2}[2\eta_{10}]\right.$$
$$\left.+\left[\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{20}\right\},$$

and

$$\xi_{21}=\frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-p^2}{p^2q^2}\right]\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{q^2}[2\eta_{01}]-\frac{(p\cdot q)}{p^2q^2}[2\eta_{10}]\right.$$
$$\left.+\left[\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}-\frac{(p\cdot q)}{q^2}\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}\right]\xi_{11}\right\}.$$

Finally we consider the expressions for the function ξ_{12}. Firstly the form

$$\xi_{12}=\frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-p^2}{p^2q^2}\right]\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-2Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\right.$$
$$\left.+\frac{1}{q^2}Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)-\frac{(p\cdot q)}{p^2q^2}[2\eta_{01}]+\left[\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}-\frac{(p\cdot q)}{q^2}\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}\right]\xi_{02}\right\},$$

and then a second form can be obtained

$$\xi_{12}=\frac{C_1}{2}\left\{\left[\frac{(p\cdot q)-q^2}{p^2q^2}\right]\left[Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{p^2}\left[Z_2\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]\right.$$
$$\left.+\frac{1}{p^2}\left[Z_2\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]-\frac{(p\cdot q)}{p^2q^2}[2\eta_{01}]+\left[\frac{\left(p^2+m_1^2-m_2^2\right)}{p^2}-\frac{(p\cdot q)}{p^2}\frac{\left(q^2+m_1^2-m_3^2\right)}{q^2}\right]\xi_{11}\right\}.$$

For the η_{nm} used in the above expressions we have the following expressions

$$\eta_{10}=\frac{1}{3}\left\{Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-2\left[\frac{1}{6}+m_1^2\xi_{10}\right]+\left(q^2+m_1^2-m_3^2\right)\xi_{20}+\left(p^2+m_1^2-m_2^2\right)\xi_{11}\right\},$$

and

$$\eta_{01}=\frac{1}{3}\left\{\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-2\left[\frac{1}{6}+m_1^2\xi_{01}\right]+\left(p^2+m_1^2-m_2^2\right)\xi_{02}+\left(q^2+m_1^2-m_3^2\right)\xi_{11}\right\}.$$

With these expressions we can write the functions ξ_{nm} corresponding to $n+m\leq 3$ completely in terms of of functions Z_k and ξ_{nm} with $n+m\leq 2$.

The reductions present above are very useful in particular to allow the identification of important properties of the basic functions associated to amplitudes having three internal propagators. These referred properties are required when relations among Green functions or Ward identities are verified. They are particular combinations of a couple of elements of the set of functions which can be constructed directly from the reductions presented above. The usefulness of these properties will become very clear in future sections. They are

1) $n+m=1$:

$$q^2\xi_{01}+(p\cdot q)\xi_{10}=-\frac{1}{2}Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{00}, \tag{17}$$

$$p^2\xi_{10}+(p\cdot q)\xi_{01}=-\frac{1}{2}Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}Z_0\left(m_1^2;q^2,m_2^2;\lambda^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{00}. \tag{18}$$

2) $n+m=2$:

$$q^2\xi_{02}+(p\cdot q)\xi_{11}=-\frac{1}{4}Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}\eta_{00}+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{01}, \tag{19}$$

$$\begin{aligned}q^2\xi_{11}+(p\cdot q)\xi_{20}=&\frac{1}{2}\left[Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\\&+\frac{1}{2}Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{10},\end{aligned} \tag{20}$$

$$p^2\xi_{20}+(p\cdot q)\xi_{11}=-\frac{1}{4}Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}\eta_{00}+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{10}, \tag{21}$$

$$p^2\xi_{11}+(p\cdot q)\xi_{02}=-\frac{1}{2}Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}Z_1\left(m_1^2;q^2,m_3^2;\lambda^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{01}. \tag{22}$$

3) $n+m=3$:

$$\begin{aligned}q^2\xi_{21}+(p\cdot q)\xi_{30}=&-\frac{1}{2}\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-2Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\\&+\frac{1}{2}Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{20},\end{aligned} \tag{23}$$

$$q^2\xi_{03}+(p\cdot q)\xi_{12}=-\frac{1}{2}Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\eta_{01}+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{02}, \tag{24}$$

$$q^2\xi_{12}+(p\cdot q)\xi_{21}=\frac{1}{2}\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{2}\eta_{10}+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\xi_{11}, \tag{25}$$

$$p^2\xi_{12}+(p\cdot q)\xi_{03}=-\frac{1}{2}Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+\frac{1}{2}Z_2\left(m_1^2;q^2,m_3^2;\lambda^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{02}, \tag{26}$$

$$\begin{aligned}p^2\xi_{30}+(p\cdot q)\xi_{21}=&-\frac{1}{2}\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-2Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)+Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\\&+\eta_{10}+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{20},\end{aligned} \tag{27}$$

$$p^2\xi_{21}+(p\cdot q)\xi_{12}=\frac{1}{2}\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{2}\eta_{01}+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\xi_{11}. \tag{28}$$

It is also useful to note similar properties involving the η_{nm} functions,

$$\begin{aligned}q^2[\eta_{01}]+(p\cdot q)[\eta_{10}]=&\frac{1}{2}m_2^2\left[1-\ln\left(\frac{m_2^2}{\lambda^2}\right)\right]-\frac{1}{2}m_3^2\left[1-\ln\left(\frac{m_3^2}{\lambda^2}\right)\right]+p^2\left[Z_2\left(m_1^2;p^2,m_2^2\right)-Z_1\left(m_1^2;p^2,m_2^2\right)\right]\\&-(p-q)^2\left[Z_2\left(m_2^2;(p-q)^2,m_3^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2\right)\right]+\frac{1}{2}\left(p^2+m_2^2-m_1^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2\right)\right]\\&-\frac{1}{2}\left[(p-q)^2+m_3^2-m_2^2\right]\left[Z_1\left(m_2^2;(p-q)^2,m_3^2\right)\right]+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)[\eta_{00}].\end{aligned} \tag{29}$$

$$p^2[\eta_{10}]+(p\cdot q)[\eta_{01}]=\frac{1}{2}m_3^2\left[1-\ln\left(\frac{m_3^2}{\lambda^2}\right)\right]-\frac{1}{2}m_2^2\left[1-\ln\left(\frac{m_2^2}{\lambda^2}\right)\right]+q^2\left[Z_2\left(m_1^2;q^2,m_3^2\right)-Z_1\left(m_1^2;q^2,m_3^2\right)\right]$$
$$-(p-q)^2\left[Z_2\left(m_3^2;(p-q)^2,m_2^2\right)-Z_1\left(m_3^2;(p-q)^2,m_2^2\right)\right]+\frac{1}{2}\left(q^2+m_3^2-m_1^2\right)\left[Z_1\left(m_1^2;q^2,m_3^2\right)\right] \quad (30)$$
$$-\frac{1}{2}\left[(p-q)^2+m_2^2-m_3^2\right]\left[Z_1\left(m_3^2;(p-q)^2,m_2^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\eta_{00}].$$

Furthermore, note that when on the left hand side we have ξ_{nm} for what $n+m=3$, on the right hand side we will have only functions with $n+m=2$, and so on. Such type of structures are precisely the expected ones when the Ward identities are considered. It is clear that other functions corresponding to higher values of n and m, and analogous relations among them, can be obtained. In the final Section 9 we will show how to generalize all above functions and their relations to an arbitrary number of points. At the present purposes the ξ_{nm} given above will be enough.

4.3. Basic Four-Point Structure Functions

The finite parts of four-point functions calculations admit a systematization analogous to the three-point functions. The basic functions are defined as

$$\zeta_{ijk}=\int_0^1 dx_1\int_0^{1-x_1}dx_2\int_0^{1-x_1-x_2}dx_3\,\frac{x_1^i x_2^j x_3^k}{[Q]^2}, \quad (31)$$

where

$$Q\equiv Q\left(m_1^2;p,m_2^2,x_1;q,m_3^2,x_2;r,m_4^2,x_3\right)=p^2x_1(1-x_1)+q^2x_2(1-x_2)+r^2x_3(1-x_3)-2(p\cdot q)x_1x_2$$
$$-2(p\cdot r)x_1x_3-2(q\cdot r)x_2x_3+\left(m_1^2-m_2^2\right)x_1+\left(m_1^2-m_3^2\right)x_2+\left(m_1^2-m_4^2\right)x_3-m_1^2.$$

If the considered amplitude possesses at least two Lorentz indexes it is useful to define another set of auxiliary functions

$$\xi_{ijk}=\int_0^1 dx_1\int_0^{1-x_1}dx_2\int_0^{1-x_1-x_2}dx_3\,\frac{x_1^i x_2^j x_3^k}{Q}, \quad (32)$$

and if four or more Lorentz indexes are involved it is convenient to define also the functions

$$\eta_{ijk}=\int_0^1 dx_1\int_0^{1-x_1}dx_2\int_0^{1-x_1-x_2}dx_3\; x_1^i x_2^j x_3^k\ln\left(\frac{Q}{-\lambda^2}\right). \quad (33)$$

The elements of the set of functions ξ_{ijk} and η_{ijk} defined above can be reduced to functions ξ_{ijk} if useful or necessary. However, in order to give a compactation of the results and operations, in intermediary steps of calculations, frequently, it is convenient to maintain the ξ_{ijk} and η_{ijk} in the corresponding expressions. All the functions of the set ξ_{ijk} can be, at the final, reduced to the most simple ones ζ_{000}. As examples of such reductions let us consider those corresponding to $i+j+k=1$. They can be written as

1) Functions ξ_{ijk}:

$$\zeta_{100}=\frac{\left[q^2r^2-(q\cdot r)^2\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{000}]\right\}$$
$$+\frac{\left[(q\cdot r)(r\cdot p)-r^2(q\cdot p)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_3^2;p-q,m_2^2;r-q,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2;r,m_4^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)[\zeta_{000}]\right\}$$
$$+\frac{\left[(q\cdot p)(r\cdot q)-q^2(r\cdot p)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_4^2;p-r,m_2^2;q-r,m_3^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)+\frac{1}{2}\left(r^2+m_1^2-m_4^2\right)[\zeta_{000}]\right\},$$

$$\zeta_{010}=\frac{\left[(p\cdot r)(r\cdot q)-r^2(p\cdot q)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{000}]\right\}$$
$$+\frac{\left[p^2r^2-(p\cdot r)^2\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_3^2;p-q,m_2^2;r-q,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2;r,m_4^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)[\zeta_{000}]\right\}$$
$$+\frac{\left[(p\cdot q)(r\cdot p)-p^2(r\cdot q)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_4^2;p-r,m_2^2;q-r,m_3^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)+\frac{1}{2}\left(r^2+m_1^2-m_4^2\right)[\zeta_{000}]\right\},$$

$$\zeta_{001}=\frac{\left[(p\cdot q)(q\cdot r)-q^2(p\cdot r)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_2^2,p-q,m_3^2;p-r,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\zeta_{000}\right]\right\}$$
$$+\frac{\left[(p\cdot r)(q\cdot p)-p^2(q\cdot r)\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_3^2;p-q,m_2^2;r-q,m_4^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2,;r,m_4^2\right)+\frac{1}{2}\left(q^2+m_1^2-m_3^2\right)\left[\zeta_{000}\right]\right\}$$
$$+\frac{\left[p^2q^2-(p\cdot q)^2\right]}{C_2}\left\{\frac{1}{2}\xi_{00}\left(m_4^2;p-r,m_3^2;q-r,m_2^2\right)-\frac{1}{2}\xi_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)+\frac{1}{2}\left(r^2+m_1^2-m_4^2\right)\left[\zeta_{000}\right]\right\},$$

where we have defined

$$C_2=p^2q^2r^2+2(p\cdot q)(p\cdot r)(q\cdot r)-p^2(q\cdot r)^2-q^2(p\cdot r)^2-r^2(p\cdot q)^2.$$

Note that $\zeta_{010}=\zeta_{100}\left(p\leftrightarrow q,m_2\leftrightarrow m_3\right)$ and $\zeta_{001}=\zeta_{100}\left(p\leftrightarrow r,m_2\leftrightarrow m_4\right)$.

2) Functions ξ_{ijk}:

$$\xi_{100}=\frac{\left[q^2r^2-(q\cdot r)^2\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{\left(p^2+m_1^2-m_2^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[(q\cdot r)(r\cdot p)-r^2(q\cdot p)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_3^2;q-p,m_2^2;q-r,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;r,m_4^2\right)\right]+\frac{\left(q^2+m_1^2-m_3^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[(q\cdot p)(r\cdot q)-q^2(r\cdot p)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;r-p,m_3^2;r-q,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)\right]+\frac{\left(r^2+m_1^2-m_4^2\right)}{2}\left[\xi_{000}\right]\right\},$$

$$\xi_{010}=\frac{\left[(p\cdot r)(r\cdot q)-r^2(p\cdot q)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{\left(p^2+m_1^2-m_2^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[p^2r^2-(p\cdot r)^2\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;q-r,m_3^2;q-p,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;r,m_4^2\right)\right]+\frac{\left(q^2+m_1^2-m_3^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[(p\cdot q)(r\cdot p)-p^2(r\cdot q)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;r-p,m_3^2;r-q,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)\right]+\frac{\left(r^2+m_1^2-m_4^2\right)}{2}\left[\xi_{000}\right]\right\},$$

$$\xi_{001}=\frac{\left[(p\cdot q)(q\cdot r)-q^2(p\cdot r)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{\left(p^2+m_1^2-m_2^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[(p\cdot r)(q\cdot p)-p^2(q\cdot r)\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;q-r,m_3^2;q-p,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;r,m_4^2\right)\right]+\frac{\left(q^2+m_1^2-m_3^2\right)}{2}\left[\xi_{000}\right]\right\}$$
$$+\frac{\left[p^2q^2-(p\cdot q)^2\right]}{C_2}\left\{-\frac{1}{2}\left[\eta_{00}\left(m_2^2;r-p,m_3^2;r-q,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{00}\left(m_1^2;p,m_2^2;q,m_3^2\right)\right]+\frac{\left(r^2+m_1^2-m_4^2\right)}{2}\left[\xi_{000}\right]\right\}.$$

3) Functions η_{ijk}:

$$\eta_{000}=\frac{1}{3}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{3}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{100}\right]+\frac{1}{3}\left(q^2+m_1^2-m_3^2\right)\left[\xi_{010}\right]$$
$$+\frac{1}{3}\left(r^2+m_1^2-m_4^2\right)\left[\xi_{001}\right]-\frac{2}{3}\left\{\frac{1}{6}+m_1^2\left[\xi_{000}\right]\right\},$$

$$\eta_{100}=\frac{1}{4}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{4}\left[\eta_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{4}\left[\eta_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$+\frac{1}{4}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{200}\right]+\frac{1}{4}\left(q^2+m_1^2-m_3^2\right)\left[\xi_{110}\right]+\frac{1}{4}\left(r^2+m_1^2-m_4^2\right)\left[\xi_{101}\right]-\frac{1}{2}\left\{\frac{1}{24}+m_1^2\left[\xi_{100}\right]\right\},$$

$$\eta_{010}=\frac{1}{4}\left[\eta_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{4}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{110}\right]+\frac{1}{4}\left(q^2+m_1^2-m_3^2\right)\left[\xi_{020}\right]$$
$$+\frac{1}{4}\left(r^2+m_1^2-m_4^2\right)\left[\xi_{011}\right]-\frac{1}{2}\left\{\frac{1}{24}+m_1^2\left[\xi_{010}\right]\right\},$$

$$\eta_{001}=\frac{1}{4}\left[\eta_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{4}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{101}\right]+\frac{1}{4}\left(q^2+m_1^2-m_3^2\right)\left[\xi_{011}\right]$$
$$+\frac{1}{4}\left(r^2+m_1^2-m_4^2\right)\left[\xi_{002}\right]-\frac{1}{2}\left\{\frac{1}{24}+m_1^2\left[\xi_{001}\right]\right\}.$$

The systematization obtained through the functions ζ_{ijk}, ξ_{ijk} and η_{ijk} is enough to write all four-point amplitude. In order to verify relations among Green functions or Ward identities some properties of those functions are useful too. In our case it is sufficient the following properties:

i) $i+j+k=1$:

$$p^2\left[\zeta_{100}\right]+\left(p\cdot q\right)\left[\zeta_{010}\right]+\left(p\cdot r\right)\left[\zeta_{001}\right]$$
$$=\frac{1}{2}\left[\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\zeta_{000}\right],$$

$$p^2\left[\xi_{100}\right]+\left(p\cdot q\right)\left[\xi_{010}\right]+\left(p\cdot r\right)\left[\xi_{001}\right]$$
$$=-\frac{1}{2}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\eta_{00}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{000}\right].$$

2) $i+j+k=2$:

$$p^2\left[\zeta_{200}\right]+\left(p\cdot q\right)\left[\zeta_{110}\right]+\left(p\cdot r\right)\left[\zeta_{101}\right]$$
$$=\frac{1}{2}\left[\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)+\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$-\frac{1}{2}\left[\xi_{000}\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\zeta_{100}\right],$$

$$p^2\left[\zeta_{110}\right]+\left(p\cdot q\right)\left[\zeta_{020}\right]+\left(p\cdot r\right)\left[\zeta_{011}\right]$$
$$=\frac{1}{2}\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{10}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\zeta_{010}\right],$$

$$p^2\left[\zeta_{101}\right]+\left(p\cdot q\right)\left[\zeta_{011}\right]+\left(p\cdot q\right)\left[\zeta_{002}\right]$$
$$=\frac{1}{2}\left[\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{01}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\zeta_{001}\right],$$

$$p^2\left[\xi_{200}\right]+\left(p\cdot q\right)\left[\xi_{110}\right]+\left(p\cdot r\right)\left[\xi_{101}\right]$$
$$=-\frac{1}{2}\left[\eta_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{2}\left[\eta_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)+\eta_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$+\frac{1}{2}\left[\eta_{000}\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{100}\right],$$

$$p^2\left[\xi_{110}\right]+\left(p\cdot q\right)\left[\xi_{020}\right]+\left(p\cdot r\right)\left[\xi_{011}\right]$$
$$=-\frac{1}{2}\left[\eta_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\eta_{10}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{010}\right],$$

$$p^2\left[\xi_{101}\right]+\left(p\cdot q\right)\left[\xi_{011}\right]+\left(p\cdot r\right)\left[\xi_{002}\right]$$
$$=-\frac{1}{2}\left[\eta_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\eta_{01}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)\left[\xi_{001}\right].$$

3) $i+j+k=3$:

$$p^2[\zeta_{300}]+(p\cdot q)[\zeta_{210}]+(p\cdot r)[\zeta_{201}]$$
$$=\frac{1}{2}\left[\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$+\frac{1}{2}\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r;m_4^2\right)\right]+\frac{1}{2}\left[\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$-[\xi_{100}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{200}],$$

$$p^2[\zeta_{120}]+(p\cdot q)[\zeta_{030}]+(p\cdot r)[\zeta_{021}]$$
$$=\frac{1}{2}\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r;m_4^2\right)-\xi_{20}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{020}],$$

$$p^2[\zeta_{102}]+(p\cdot q)[\zeta_{012}]+(p\cdot r)[\zeta_{003}]$$
$$=\frac{1}{2}\left[\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{02}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{002}],$$

$$p^2[\zeta_{210}]+(p\cdot q)[\zeta_{120}]+(p\cdot r)[\zeta_{111}]=\frac{1}{2}\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$-\frac{1}{2}\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)+\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}[\xi_{010}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{110}],$$

$$p^2[\zeta_{201}]+(p\cdot q)[\zeta_{111}]+(p\cdot r)[\zeta_{102}]$$
$$=\frac{1}{2}\left[\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)+\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$-\frac{1}{2}[\xi_{001}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{101}],$$

$$p^2[\zeta_{111}]+(p\cdot q)[\zeta_{021}]+(p\cdot r)[\zeta_{012}]$$
$$=\frac{1}{2}\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{11}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{011}].$$

4) $i+j+k=4$:

$$p^2[\zeta_{400}]+(p\cdot q)[\zeta_{310}]+(p\cdot r)[\zeta_{301}]$$
$$=\frac{1}{2}\left[\xi_{00}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{3}{2}\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{3}{2}\left[\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$+\frac{3}{2}\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+3\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{3}{2}\left[\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right] \quad (34)$$
$$-\frac{1}{2}\left[\xi_{30}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{3}{2}\left[\xi_{21}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{3}{2}\left[\xi_{12}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]$$
$$-\frac{1}{2}\left[\xi_{03}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{3}{2}[\xi_{200}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{300}],$$

$$p^2[\zeta_{130}]+(p\cdot q)[\zeta_{040}]+(p\cdot r)[\zeta_{031}]$$
$$=\frac{1}{2}\left[\xi_{30}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{30}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{030}], \quad (35)$$

$$p^2[\zeta_{103}]+(p\cdot q)[\zeta_{013}]+(p\cdot r)[\zeta_{004}]$$
$$=\frac{1}{2}\left[\xi_{03}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{03}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{003}], \quad (36)$$

$$
\begin{aligned}
&p^2[\zeta_{310}]+(p\cdot q)[\zeta_{220}]+(p\cdot r)[\zeta_{211}]\\
&=\frac{1}{2}\left[\xi_{10}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad+\frac{1}{2}\left[\xi_{30}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\left[\xi_{21}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad+\frac{1}{2}\left[\xi_{12}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-[\xi_{110}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{210}],
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&p^2[\zeta_{301}]+(p\cdot q)[\zeta_{211}]+(p\cdot r)[\zeta_{202}]\\
&=\frac{1}{2}\left[\xi_{01}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\left[\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad+\frac{1}{2}\left[\xi_{21}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\left[\xi_{12}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]+\frac{1}{2}\left[\xi_{03}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad-[\xi_{101}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{201}],
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&p^2[\zeta_{211}]+(p\cdot q)[\zeta_{121}]+(p\cdot r)[\zeta_{112}]\\
&=\frac{1}{2}\left[\xi_{11}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}\left[\xi_{21}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad-\frac{1}{2}\left[\xi_{12}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}[\xi_{011}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{111}],
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&p^2[\zeta_{220}]+(p\cdot q)[\zeta_{130}]+(p\cdot r)[\zeta_{121}]\\
&=\frac{1}{2}\left[\xi_{20}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}\left[\xi_{30}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad-\frac{1}{2}\left[\xi_{21}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}[\xi_{020}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{120}],
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&p^2[\zeta_{202}]+(p\cdot q)[\zeta_{112}]+(p\cdot r)[\zeta_{103}]\\
&=\frac{1}{2}\left[\xi_{02}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}\left[\xi_{12}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]\\
&\quad-\frac{1}{2}\left[\xi_{03}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)\right]-\frac{1}{2}[\xi_{002}]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{102}],
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
&p^2[\zeta_{130}]+(p\cdot q)[\zeta_{040}]+(p\cdot r)[\zeta_{031}]\\
&=\frac{1}{2}\left[\xi_{30}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{30}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{030}],
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&p^2[\zeta_{103}]+(p\cdot q)[\zeta_{013}]+(p\cdot r)[\zeta_{004}]\\
&=\frac{1}{2}\left[\xi_{03}\left(m_2^2;p-q,m_3^2;p-r,m_4^2\right)-\xi_{03}\left(m_1^2;q,m_3^2;r,m_4^2\right)\right]+\frac{1}{2}\left(p^2+m_1^2-m_2^2\right)[\zeta_{003}].
\end{aligned}
\tag{43}
$$

Similar relations can be obtained for others components of the set by exploring the properties relating these functions which are the interchanges $p\leftrightarrow q$, $p\leftrightarrow r$, $m_2\leftrightarrow m_3$, and $m_2\leftrightarrow m_4$ (analogously to the ξ_{ij} functions). The systematization allows us to treat the perturbative four-point amplitudes in an exact way. By successive reductions all the content of finite parts of a four-point function will be written in terms of only ξ_{000} (more ξ_{00} and Z_0). Let us now consider the evaluation of the integrals (1)-(4) in terms of the systematization introduced.

5. Manipulations and Calculations of the One-Loop Feynman Integrals

After introducing the strategy to be adopted to handle with the divergences in perturbative calculations of QFT, as well as to state the standard divergent structures in terms of which the divergent parts will be written and to define the set of basic functions in terms of which the finite parts will be written, we can consider the solution of the divergent Feynman integrals presented in (1)-(4).

5.1. One-point Feynman Integrals

If we want to solve the Feynman integral $(I_1)_\mu$ defined in (1), by using the procedure described in previous sections, first we identify the divergence degree $D=3$. After this we have to adopt the adequate representation for the propagator. This means taking $N=3$ in the expression (5) to get

$$\frac{k_\mu}{D_1}=k_\mu\left\{\frac{1}{(k^2-\lambda^2)}-\frac{A_1}{(k^2-\lambda^2)^2}+\frac{(A_1)^2}{(k^2-\lambda^2)^3}-\frac{(A_1)^3}{(k^2-\lambda^2)^4}+\frac{(A_1)^4}{(k^2-\lambda^2)^4 D_1}\right\}. \tag{44}$$

Next we reorganize in a convenient way in order to get the basic divergent structures defined in Section 3. Then we write the above expression in the form

$$\left(\frac{k_\mu}{D_1}\right)_{\text{even}}=-k_1^\alpha\left\{\frac{2k_\alpha k_\mu}{(k^2-\lambda^2)^2}\right\}+\left(k_1^2+\lambda^2-m_1^2\right)k_1^\alpha\left\{\frac{4k_\alpha k_\mu}{(k^2-\lambda^2)^3}\right\}-\frac{k_1^\beta k_1^\alpha k_1^\nu}{3}\left\{\frac{24k_\alpha k_\beta k_\nu k_\mu}{(k^2-\lambda^2)^4}\right\}$$
$$-3\left(k_1^2+\lambda^2-m_1^2\right)^2\frac{(2k_1\cdot k)k_\mu}{(k^2-\lambda^2)^4}+\frac{(A_1)^4 k_\mu}{(k^2-\lambda^2)^4 D_1},$$

where we have written only the terms which are even in the loop momentum k by simplicity just because the odd ones will be ruled out after the introduction of the integration sign. Convenient reorganizations are made to get the divergent terms written completely in combinations of the five objects (7)-(11) and then we get

$$I_{1\mu}=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu}{D_1}=-k_1^\xi\left[\nabla_{\xi\mu}(\lambda^2)\right]-\frac{1}{3}k_1^\xi k_1^\chi k_1^\tau\left[\Box_{\xi\chi\tau\mu}(\lambda^2)\right]-\frac{1}{2}k_{1\mu}k_1^\xi k_1^\chi\left[\Delta_{\xi\chi}(\lambda^2)\right]-\frac{1}{2}\left(k_1^2\right)k_1^\xi\left[\Delta_{\mu\xi}(\lambda^2)\right]$$
$$+\left(k_1^2+\lambda^2-m_1^2\right)k_1^\xi\left[\Delta_{\xi\mu}(\lambda^2)\right]-k_{1\mu}\left\{\left[I_{\text{quad}}(\lambda^2)\right]+\left(m^2-\lambda^2\right)\left[I_{\log}(\lambda^2)\right]\right\}$$
$$-3\left(k_1^2+\lambda^2-m_1^2\right)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{2(k\cdot k_1)k_\mu}{(k^2-\lambda^2)^4}+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(A_1)^4 k_\mu}{(k^2-\lambda^2)^4 D_1}.$$

Only finite terms will be integrated in the next step and no additional modification will be made. The result is the expression

$$I_{1\mu}=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu}{D_1}=-k_1^\xi\left[\nabla_{\xi\mu}(\lambda^2)\right]-\frac{1}{3}k_1^\xi k_1^\chi k_1^\tau\left[\Box_{\xi\chi\tau\mu}(\lambda^2)\right]-\frac{1}{2}k_{1\mu}k_1^\xi k_1^\chi\left[\Delta_{\xi\chi}(\lambda^2)\right]-\frac{1}{2}\left(k_1^2\right)k_1^\xi\left[\Delta_{\mu\xi}(\lambda^2)\right]$$
$$+\left(k_1^2+\lambda^2-m_1^2\right)k_1^\xi\left[\Delta_{\xi\mu}(\lambda^2)\right]-k_{1\mu}\left\{\left[I_{\text{quad}}(\lambda^2)\right]+\left(m_1^2-\lambda^2\right)\left[I_{\log}(\lambda^2)\right]+\frac{\mathrm{i}}{(4\pi)^2}\left[\left(m_1^2-\lambda^2\right)-m_1^2\ln\left(\frac{m_1^2}{\lambda^2}\right)\right]\right\}.$$

The reasons for the definition of the divergent objects precisely on this form will become clear in future sections. It is possible to show that for any value of N in the expression (44) major than 3 the result can be put in the above form. Note that, following our strategy, no mention needs to be made to regularization techniques until this step. On the other hand, the above result can be converted to any regularization prescription since all the steps performed are perfectly valid in the presence of all regularization distribution. Such eventually adopted regularization, in this case, will be present only in the basic divergent objects just because it can be removed from the finite integrals by taking the connection limit. If, on the other hand, we want to attribute a definite value for the involved divergent objects a regularization must be assumed and the integration made. However, as we shall see in a moment, this is not necessary in any situation.

Now we can consider the quadratically divergent integral defined in (1). For this purpose we follow the same procedure applied above. Strictly speaking, the same representation for the propagator used in (44) can be adopted. However, algebraic effort can be avoided by taking the value $N = 2$ in the expression (5) just because the obtained expression may be put in the same form for any superior value. Having this in mind in all situations where we have to calculate the integral I_1 we will have to integrate the expression (omitting an odd term in the k loop momentum)

$$\left[\frac{1}{D_1}\right]_{\text{even}}=\frac{1}{\left(k^2-\lambda^2\right)}-\left(k_1^2+\lambda^2-m_1^2\right)\frac{1}{\left(k^2-\lambda^2\right)^2}+k_1^\alpha k_1^\beta\frac{4k_\alpha k_\beta}{\left(k^2-\lambda^2\right)^3}+\frac{\left(k_1^2+\lambda^2-m_1^2\right)^2}{\left(k^2-\lambda^2\right)^3}-\frac{\left(A_1\right)^3}{\left(k^2-\lambda^2\right)^3\left[D_1\right]}.$$

So, taking the integration after some convenient reorganization, we will get

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{D_1}=\left[I_{\text{quad}}\left(\lambda^2\right)\right]+\left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right]+k_1^\xi k_1^\chi\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right]+\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(k_1^2+\lambda^2-m_1^2\right)^2}{\left(k^2-\lambda^2\right)^3}$$
$$-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\left(A_1\right)^3}{\left(k^2-\lambda^2\right)^3\left[\left(k+k_1\right)^2-m_1^2\right]}.$$

Solving the finite terms we obtain

$$I_1=\left[I_{\text{quad}}\left(\lambda^2\right)\right]+\left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right]+\frac{i}{(4\pi)^2}\left[m_1^2-\lambda^2-m_1^2\ln\left(\frac{m_1^2}{\lambda^2}\right)\right]+k_1^\xi k_1^\chi\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right]. \tag{45}$$

Again note the general character of the expression. Only mathematical operations free from choices have been made.

5.2. Two-Point Feynman Integrals

Now we consider the integrals having two propagators. First we take the simplest one: the I_2 integral. When this integral needs to be solved, as a consequence of the application of Feynman rules, we first adopt the representation (5) for the propagators. If one wants to use an unique representation for the propagators the expression may be that used in (44). However, given the divergence degree involved, some algebraic simplification can be obtained assuming the value $N=1$ for both propagators. We have to integrate the summation of terms

$$\frac{1}{D_{12}}=\frac{1}{\left(k^2-\lambda^2\right)^2}-\sum_{i=1}^{2}\frac{A_i}{\left(k^2-\lambda^2\right)^2D_i}+\frac{A_1A_2}{\left(k^2-\lambda^2\right)^2D_{12}},$$

where we have used the definition (6) in order to write the expressions in a more compact way. Now we introduce the integration sign to get

$$I_2\equiv\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{D_{12}}=\left[I_{\log}\left(\lambda^2\right)\right]-\sum_{i=1}^{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{A_i}{\left(k^2-\lambda^2\right)^2D_i}$$
$$-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{A_1A_2}{\left(k^2-\lambda^2\right)^2D_{12}}.$$

The finite ones can be integrated by using usual tools to yield

$$I_2=\left[I_{\log}\left(\lambda^2\right)\right]-\frac{i}{(4\pi)^2}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right], \tag{46}$$

where we have introduced the definition $k_2-k_1=p$.

The same procedure can be adopted when the integral $I_{2\mu}$ needs to be solved. In our procedure, before taking the integration, we first write

$$\left[\frac{k_\mu}{D_{12}}\right]_{\text{even}}=-\frac{1}{2}\left(k_2+k_1\right)^\xi\left\{\frac{4k_\xi k_\mu}{\left(k^2-\lambda^2\right)^3}\right\}+\left(k_2^2+\lambda^2-m_2^2\right)\frac{\left(2k\cdot k_1\right)k_\mu}{\left(k^2-\lambda^2\right)^4}+\left(k_1^2+\lambda^2-m_1^2\right)\frac{\left(2k\cdot k_2\right)k_\mu}{\left(k^2-\lambda^2\right)^4}$$
$$-\sum_{i\neq j=1}^{2}\frac{A_i\left(A_j\right)^2k_\mu}{\left(k^2-\lambda^2\right)^4D_j}+\sum_{i=1}^{2}\frac{\left(A_i\right)^2k_\mu}{\left(k^2-\lambda^2\right)^3D_i}+\frac{\left(A_1\right)^2\left(A_2\right)^2k_\mu}{\left(k^2-\lambda^2\right)^4D_{12}}.$$

Note that odd terms have been omitted. After some reorganization, we take the integration solving the finite integrals obtained to get

$$\begin{aligned}I_{2\mu}=&-\frac{P_\mu}{2}\left[I_{\log}\left(\lambda^2\right)\right]-\frac{P^\xi}{2}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\\&-\frac{\mathrm{i}}{(4\pi)^2}\left\{p_\mu\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)-\frac{Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)}{2}\right]-\frac{P_\mu}{2}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right\}.\end{aligned} \tag{47}$$

Here we have defined $P=k_1+k_2$.

Next, we can follow strictly the same procedure to get the expression for the integral $I_{2\mu\nu}$ in our procedure. The first step is to write

$$\left[\frac{k_\mu k_\nu}{D_{12}}\right]_{\text{even}} = \frac{1}{2}\left\{\frac{2k_\mu k_\nu}{\left(k^2-\lambda^2\right)^2}\right\} - \frac{1}{4}\left[\left(k_2^2+\lambda^2-m_2^2\right)+\left(k_1^2+\lambda^2-m_1^2\right)\right]\left\{\frac{4k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}\right\}$$
$$+\frac{1}{6}\left[k_1^\alpha k_1^\beta + k_2^\alpha k_2^\beta + k_1^\alpha k_2^\beta\right]\left\{\frac{24 k_\alpha k_\beta k_\mu k_\nu}{\left(k^2-\lambda^2\right)^4}\right\}$$
$$+\left\{\left(k_2^2+\lambda^2-m_2^2\right)^2+\left(k_1^2+\lambda^2-m_1^2\right)^2+\left(k_2^2+\lambda^2-m_2^2\right)\left(k_1^2+\lambda^2-m_1^2\right)\right\}\left\{\frac{k_\mu k_\nu}{\left(k^2-\lambda^2\right)^4}\right\}$$
$$-\sum_{i\neq j=1}^{2}\frac{A_i\left(A_i\right)^2 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^5}+\frac{\left(A_1\right)^2\left(A_2\right)^2 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^6}-\sum_{i=1}^{2}\frac{\left(A_i\right)^3 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^4 D_i}+\sum_{i=1}^{2}\frac{A_i\left(A_j\right)^3 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^5 D_j}$$
$$-\sum_{i=1}^{2}\frac{\left(A_i\right)^2\left(A_j\right)^3 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^6 D_j}+\frac{\left(A_1\right)^3\left(A_2\right)^3 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^6 D_{12}}.$$

Now we take the integration, after a convenient reorganization of the terms to write the divergent terms as a combination of the basic divergent structures, and perform the integration in the finite terms by using standard techniques, to get

$$\begin{aligned}
I_{2\mu\nu} ={}& \frac{1}{2}\left[\nabla_{\mu\nu}\left(\lambda^2\right)\right]+\frac{1}{6}\left(k_2^\xi k_2^\chi + k_2^\xi k_1^\chi + k_1^\xi k_1^\chi\right)\left[\Box_{\xi\chi\mu\nu}\left(\lambda^2\right)\right]-\frac{1}{4}\left(2\lambda^2-m_1^2-m_2^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]\\
&+\frac{1}{12}\left(-2k_2^2+k_2\cdot k_1-2k_1^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+\frac{1}{12}g_{\mu\nu}\left(k_2^\xi k_2^\chi + k_2^\xi k_1^\chi + k_1^\xi k_1^\chi\right)\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right]\\
&+\frac{1}{12}\left(2k_{2\mu}k_2^\xi + k_{2\mu}k_1^\xi + k_{1\mu}k_2^\xi + 2k_{1\mu}k_1^\xi\right)\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]\\
&+\frac{1}{12}\left(2k_{2\nu}k_2^\xi + k_{2\nu}k_1^\xi + k_{1\nu}k_2^\xi + 2k_{1\nu}k_1^\xi\right)\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]+\frac{1}{2}g_{\mu\nu}\left[I_{\text{quad}}\left(\lambda^2\right)\right]\\
&-\frac{1}{4}g_{\mu\nu}\left(2\lambda^2-m_1^2-m_2^2\right)\left[I_{\log}\left(\lambda^2\right)\right]-\frac{1}{12}g_{\mu\nu}\left(k_2-k_1\right)^2\left[I_{\log}\left(\lambda^2\right)\right]\\
&+\frac{1}{6}\left(2k_{2\mu}k_{2\nu}+k_{2\mu}k_{1\nu}+k_{2\nu}k_{1\mu}+2k_{1\mu}k_{1\nu}\right)\left[I_{\log}\left(\lambda^2\right)\right]+\frac{1}{2}g_{\mu\nu}\left[m_2^2-\lambda^2-m_2^2\ln\left(\frac{m_2^2}{\lambda^2}\right)\right]\\
&+g_{\mu\nu}p^2\left[Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)-\frac{1}{2}Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-p_\mu p_\nu\left[Z_2\left(p_1^2,m_1^2,m_2^2;\lambda^2\right)\right]\\
&-\frac{1}{2}g_{\mu\nu}\left(m_1^2-m_2^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-\left(k_{1\mu}p_\nu+k_{1\nu}p_\mu\right)\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\
&-k_{1\mu}k_{1\nu}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right],
\end{aligned}\tag{48}$$

which completes the calculation of the Feynman integrals having two internal propagators.

5.3. Three-Point Feynman Integrals

Now we evaluate the integrals having three propagators. The first element of the set (3) is finite and may be calculated by taking any value for N in the expression (5). We write the result as

$$I_3=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{D_{123}}=\frac{i}{(4\pi)^2}\left[\xi_{00}\left(m_1^2;p^2,m_2^2,q^2,m_3^2\right)\right],\tag{49}$$

where we adopted the definitions $k_3-k_1=q$ and $k_2-k_1=p$. The definition (15) for the ξ_{nm} functions has been used. The same comment applies to the second element of the set (3). The result can be written as

$$\begin{aligned}
I_{3\mu}&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu}{D_{123}}\\
&=-\frac{\mathrm{i}}{(4\pi)^2}\left[p_\mu\xi_{01}+q_\mu\xi_{10}\right]-\frac{i}{(4\pi)^2}k_{1\mu}\left[\xi_{00}\right].
\end{aligned}\tag{50}$$

By simplicity, we will omit the arguments of three-point functions ξ_{nm} and η_{nm} whenever it is not involved four-point structures. The next integral of the set (3), which is $I_{3\mu\nu}$, is logarithmically divergent. Then after taking the integration we have to adopt the adequate representation for the propagators. In this case we can first write

$$\left[\frac{k_\mu k_\nu}{D_{123}}\right]_{\text{even}}=\frac{1}{4}\left\{\frac{4k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}\right\}-\sum_{i=1}^{3}\frac{A_i k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3 D_i}+\sum_{i\neq j=1}^{3}\frac{A_i A_j k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3 D_{ij}}-\frac{A_1A_2A_3 k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3 D_{123}}.$$

Only the first term will be converted in a divergent object after we take the integration. Solving the finite integrals we can put the results in the form

$$\begin{aligned}I_{3\mu\nu}=&\frac{1}{4}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+\frac{1}{4}g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]+\frac{i}{(4\pi)^2}\left[q_\nu q_\mu\xi_{20}+p_\nu p_\mu\xi_{02}+p_\nu q_\mu\xi_{11}+q_\nu p_\mu\xi_{11}-\frac{1}{2}g_{\nu\mu}\eta_{00}\right]\\&+\frac{i}{(4\pi)^2}k_{1\mu}\left[p_\nu\xi_{01}+q_\nu\xi_{10}\right]+\frac{i}{(4\pi)^2}k_{1\nu}\left[p_\mu\xi_{01}+q_\mu\xi_{10}\right]+\frac{i}{(4\pi)^2}k_{1\mu}k_{1\nu}\left[\xi_{00}\right].\end{aligned}\tag{51}$$

Now let us consider the linearly divergent structure, the integral $I_{3\mu\nu}$. The first step is to rewrite it using (5), as we did above, and next we solve the finite integrals to write the result as

$$\begin{aligned}\left[\frac{k_\mu k_\nu k_\lambda}{D_{123}}\right]_{\text{even}}=&-\frac{\left(k_3+k_2+k_1\right)^\alpha}{12}\left\{\frac{24k_\alpha k_\mu k_\nu k_\lambda}{\left(k^2-\lambda^2\right)^4}\right\}+\sum_{i\neq j=1}^{3}\frac{A_iA_j}{\left(k^2-\lambda^2\right)^5}-\frac{A_1A_2A_3}{\left(k^2-\lambda^2\right)^6}+\sum_{i\neq j=1}^{3}\frac{\left(A_i\right)^2}{\left(k^2-\lambda^2\right)^4 D_i}-\sum_{i\neq j=1}^{3}\frac{A_i\left(A_j\right)^2}{\left(k^2-\lambda^2\right)^5 D_j}\\&+\sum_{i,j,l=1}^{3}\frac{A_iA_j\left(A_l\right)^2}{\left(k^2-\lambda^2\right)^6 D_l}+\sum_{i\neq j=1}^{3}\frac{\left(A_i\right)^2\left(A_j\right)^2}{\left(k^2-\lambda^2\right)^5 D_{ij}}-\sum_{i,j,l=1}^{3}\frac{A_i\left(A_j\right)^2\left(A_l\right)^2}{\left(k^2-\lambda^2\right)^6 D_{jl}}+\frac{\left(A_1\right)^2\left(A_2\right)^2\left(A_3\right)^2}{\left(k^2-\lambda^2\right)^6 D_{123}}.\end{aligned}$$

By reorganizing in a convenient way the first term so that it is written as a combination of the basic divergent objects (7)-(11), and after this taking the integration and performing the operations in the finite terms the result can be put into the form

$$\begin{aligned}I_{3\mu\nu\lambda}=&-\frac{1}{12}\left(k_1+k_2+k_3\right)^\xi\left[\square_{\xi\lambda\mu\nu}\left(\lambda^2\right)\right]-\frac{1}{24}\left(k_1+k_2+k_3\right)^\xi\left\{g_{\mu\lambda}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]+g_{\mu\nu}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right]+g_{\lambda\nu}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\right\}\\&-\frac{1}{12}\left(k_1+k_2+k_3\right)_\mu\left\{g_{\lambda\nu}\left[I_{\log}\left(\lambda^2\right)\right]+\frac{1}{2}\left[\Delta_{\lambda\nu}\left(\lambda^2\right)\right]\right\}-\frac{1}{12}\left(k_1+k_2+k_3\right)_\lambda\left\{g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]+\frac{1}{2}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]\right\}\\&-\frac{1}{12}\left(k_1+k_2+k_3\right)_\nu\left\{g_{\mu\lambda}\left[I_{\log}\left(\lambda^2\right)\right]+\frac{1}{2}\left[\Delta_{\mu\lambda}\left(\lambda^2\right)\right]\right\}-\frac{i}{(4\pi)^2}\Big\{q_\mu q_\lambda q_\nu\xi_{30}-p_\mu p_\lambda p_\nu\xi_{03}-q_\mu q_\lambda p_\nu\xi_{21}\\&-q_\mu p_\lambda q_\nu\xi_{21}-p_\mu q_\lambda q_\nu\xi_{21}-p_\mu q_\lambda p_\nu\xi_{12}-q_\mu p_\lambda p_\nu\xi_{12}-p_\mu p_\lambda q_\nu\xi_{12}\\&+\frac{1}{2}\left[g_{\nu\mu}q_\lambda\eta_{10}+g_{\lambda\mu}q_\nu\eta_{10}+g_{\nu\lambda}q_\mu\eta_{10}\right]+\frac{1}{2}\left[g_{\nu\mu}p_\lambda\eta_{01}+g_{\lambda\mu}p_\nu\eta_{01}+g_{\nu\lambda}p_\mu\eta_{01}\right]\Big\}\\&-\frac{i}{(4\pi)^2}\Big\{k_{1\mu}k_{1\lambda}k_{1\nu}\left[\xi_{00}\right]+k_{1\mu}k_{1\lambda}\left[p_\nu\xi_{01}+q_\nu\xi_{10}\right]+k_{1\mu}k_{1\nu}\left[p_\lambda\xi_{01}+q_\lambda\xi_{10}\right]\\&+k_{1\lambda}k_{1\nu}\left[p_\mu\xi_{01}+q_\mu\xi_{10}\right]+k_{1\mu}\left[p_\lambda p_\nu\xi_{02}+q_\lambda q_\nu\xi_{20}+q_\lambda p_\nu\xi_{11}+p_\lambda q_\nu\xi_{11}-\frac{1}{2}g_{\nu\lambda}\eta_{00}\right]\\&+k_{1\lambda}\left[p_\mu p_\nu\xi_{02}+q_\mu q_\nu\xi_{20}+p_\mu q_\nu\xi_{11}+q_\mu p_\nu\xi_{11}-\frac{1}{2}g_{\nu\mu}\eta_{00}\right]\\&+k_{1\nu}\left[p_\mu q_\lambda\xi_{11}+q_\mu p_\lambda\xi_{11}+p_\mu p_\lambda\xi_{02}+q_\mu q_\lambda\xi_{20}-\frac{1}{2}g_{\lambda\mu}\eta_{00}\right]\Big\}.\end{aligned}\tag{52}$$

In fundamental theories the considered integrals are enough to evaluate the one-loop amplitudes having three internal propagators.

5.4. Four-Point Feynman Integrals

Finally, we consider the four-point function integrals. Only one of them is a divergent structure which makes the job easy. The first, the scalar one, can be written as

$$I_4 = i(4\pi)^{-2}\left[\zeta_{000}\right], \tag{53}$$

where we have identified the four-point structure functions previously defined in the Equation (31) and also the external momentum $r = k_4 - k_1$. Next, one can immediatly see that, for the vector integral, we can write

$$I_{4\mu} = J_{4\mu} - k_{1\mu}\left[I_4\right], \tag{54}$$

$$J_{4\mu} = -i(4\pi)^{-2}\left\{p_\mu\left[\zeta_{100}\right] + q_\mu\left[\zeta_{010}\right] + r_\mu\left[\zeta_{001}\right]\right\}, \tag{55}$$

and that for the one having two Lorentz indexes, we have

$$I_{4\mu\nu} = J_{4\mu\nu} - k_{1\mu}\left[I_{4\nu}\right] - k_{1\nu}\left[I_{4\mu}\right] + k_{1\mu}k_{1\nu}\left[I_4\right], \tag{56}$$

where

$$J_{4\mu\nu} = \left[J'_{\mu\nu}\left(m_1^2;p,m_2^2;q,m_3^2;r,m_4^2\right)\right] + \left[J'_{\mu\nu}\left(p\leftrightarrow q;m_2^2\leftrightarrow m_3^2\right)\right] + \left[J'_{\mu\nu}\left(p\leftrightarrow r;m_2^2\leftrightarrow m_4^2\right)\right], \tag{57}$$

$$J'_{\mu\nu}\left(m_1^2;p,m_2^2;q,m_3^2;r,m_4^2\right) = -i(4\pi)^{-2}\left\{\frac{1}{6}g_{\mu\nu}\left[\xi_{000}\right] + p_\mu p_\nu\left[\zeta_{200}\right] + p_\mu q_\nu\left[\zeta_{110}\right] + p_\mu r_\nu\left[\zeta_{101}\right]\right\} \tag{58}$$

On the other hand,

$$I_{4\mu\nu\lambda} = J_{4\mu\nu\lambda} - k_{1\lambda}\left[I_{4\mu\nu}\right] - k_{1\nu}\left[I_{4\mu\lambda}\right] - k_{1\mu}\left[I_{4\nu\lambda}\right] + k_{1\mu}k_{1\nu}\left[I_{4\lambda}\right] + k_{1\mu}k_{1\lambda}\left[I_{4\nu}\right] + k_{1\nu}k_{1\lambda}\left[I_{4\mu}\right] - k_{1\mu}k_{1\nu}k_{1\lambda}\left[I_4\right], \tag{59}$$

where

$$J_{4\mu\nu\lambda} = \left[J'_{\mu\nu\lambda}\left(m_1^2;p,m_2^2;q,m_3^2;r,m_4^2\right)\right] + \left[J'_{\mu\nu\lambda}\left(p\leftrightarrow q;m_2^2\leftrightarrow m_3^2\right)\right] + \left[J'_{\mu\nu\lambda}\left(p\leftrightarrow r;m_2^2\leftrightarrow m_4^2\right)\right] \tag{60}$$

$$\begin{aligned} J'_{\mu\nu\lambda} = -i(4\pi)^{-2}\Big\{&-\frac{1}{2}\left(g_{\mu\nu}p_\lambda + g_{\mu\lambda}p_\nu + g_{\nu\lambda}p_\mu\right)\left[\xi_{100}\right] - p_\mu p_\nu p_\lambda\left[\zeta_{300}\right] \\ &-\left(p_\mu p_\nu q_\lambda + p_\mu q_\nu p_\lambda + q_\mu p_\nu p_\lambda\right)\left[\zeta_{210}\right] - \left(p_\mu p_\nu r_\lambda + p_\mu r_\nu p_\lambda + r_\mu p_\nu p_\lambda\right)\left[\zeta_{201}\right]\Big\}. \end{aligned} \tag{61}$$

The last one we consider is the logarithmically divergent one, which we write as

$$\begin{aligned} I_{4\mu\nu\alpha\beta} = J_{4\mu\nu\alpha\beta} &+ \frac{1}{24}\left[\Box_{\alpha\beta\mu\nu}\left(\lambda^2\right)\right] + \frac{1}{48}\left\{g_{\mu\alpha}\left[\Delta_{\beta\nu}\left(\lambda^2\right)\right] + g_{\alpha\beta}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] + g_{\alpha\nu}\left[\Delta_{\mu\beta}\left(\lambda^2\right)\right]\right\} \\ &+ \frac{1}{48}\left\{g_{\mu\beta}\left[\Delta_{\alpha\nu}\left(\lambda^2\right)\right] + g_{\mu\nu}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right] + g_{\beta\nu}\left[\Delta_{\mu\alpha}\left(\lambda^2\right)\right]\right\} + \frac{1}{24}\left(g_{\mu\alpha}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta} + g_{\alpha\beta}g_{\mu\nu}\right)\left[I_{\log}\left(\lambda^2\right)\right] \\ &-\left(g_{\alpha\rho}g_{\beta\gamma}g_{\mu\tau}g_{\nu\lambda} + g_{\beta\rho}g_{\nu\gamma}g_{\alpha\tau}g_{\mu\lambda} + g_{\mu\rho}g_{\beta\gamma}g_{\nu\tau}g_{\alpha\lambda} + g_{\nu\rho}g_{\alpha\gamma}g_{\mu\tau}g_{\beta\lambda}\right) \\ &\times\left\{k_1^\rho\left[I_4^{\gamma\tau\lambda}\right] - \frac{1}{2}k_1^\rho k_1^\gamma\left[I_4^{\tau\lambda}\right] - \frac{1}{2}k_1^\rho k_1^\tau\left[I_4^{\gamma\lambda}\right] - \frac{1}{2}k_1^\rho k_1^\lambda\left[I_4^{\tau\gamma}\right] + k_1^\rho k_1^\gamma k_1^\tau\left[I_4^\lambda\right] - k_1^\rho k_1^\gamma k_1^\tau k_1^\lambda\left[I_4\right]\right\}, \end{aligned} \tag{62}$$

where

$$J_{4\mu\nu\alpha\beta} = \left[J'_{\mu\nu\alpha\beta}\left(m_1^2;p,m_2^2;q,m_3^2;r,m_4^2\right)\right] + \left[J'_{\mu\nu\alpha\beta}\left(p\leftrightarrow q;m_2^2\leftrightarrow m_3^2\right)\right] + \left[J'_{\mu\nu\alpha\beta}\left(p\leftrightarrow r;m_2^2\leftrightarrow m_4^2\right)\right], \tag{63}$$

$$\begin{aligned} J'_{\mu\nu\alpha\beta} &= J''_{\mu\nu\alpha\beta} + J''_{\nu\mu\alpha\beta} + J''_{\beta\nu\alpha\mu} \\ J''_{\mu\nu\alpha\beta} &= i(4\pi)^{-2}\Big\{-\frac{1}{12}g_{\alpha\mu}g_{\beta\nu}\left[\eta_{000}\right] + \frac{1}{2}\left(g_{\mu\alpha}p_\nu p_\beta + g_{\nu\beta}p_\mu p_\alpha\right)\left[\xi_{200}\right] \\ &+\frac{1}{2}\left[g_{\mu\alpha}\left(q_\nu r_\beta + r_\nu q_\beta\right) + g_{\nu\beta}\left(q_\mu r_\alpha + r_\mu q_\alpha\right)\right]\left[\xi_{011}\right] + \frac{1}{3}p_\mu p_\nu p_\alpha p_\beta\left[\zeta_{400}\right] \\ &+\left(\frac{1}{3}r_\alpha q_\mu q_\nu q_\beta + r_\mu q_\nu q_\alpha q_\beta\right)\left[\zeta_{031}\right] + \left(\frac{1}{3}q_\alpha r_\mu r_\nu r_\beta + q_\mu r_\nu r_\alpha r_\beta\right)\left[\zeta_{013}\right] + \left(q_\mu r_\nu q_\alpha r_\beta + r_\mu q_\nu r_\alpha q_\beta\right)\left[\zeta_{022}\right] \\ &+\left(p_\mu q_\nu p_\alpha r_\beta + p_\mu r_\nu p_\alpha q_\beta + q_\mu p_\nu r_\alpha p_\beta + r_\mu p_\nu q_\alpha p_\beta\right)\left[\zeta_{211}\right]\Big\}. \end{aligned} \tag{64}$$

With the above results for the Feynman integrals at hand we can perform all the one-loop amplitudes for one, two, three and four fermionic propagators in the context of fundamental gauge theories. In the next section we evaluate some representative amplitudes involving vector vertexes.

6. Physical Amplitudes

In the preceding sections we have considered the evaluation of the Feynman integrals introduced in the Section 2, which are crucial for the one-loop calculation in the context of fundamental gauge theories like QED. All the integrals have been written in terms of the set of divergent objects; $\square_{\alpha\beta\mu\nu}$, $\Delta_{\mu\nu}$, $\nabla_{\mu\nu}$, $I_{\log}$ and I_{quad}, defined in the Equations (7)-(11) and in terms of the functions Z_k, ξ_{nm} and, ζ_{nml} defined in the Equations (12), (15) and, (31) for two, three and four-point functions, respectively. By using properties relating the above cited functions, all one-loop amplitudes can be reduced to a combination of only three basic pieces: Z_0, ξ_{00} and, ζ_{000}.

In the present section we will evaluate some representative amplitudes of the perturbative calculations by using the systematization introduced in the preceding sections. We will consider an example for each number of points taking the amplitude corresponding to the higher degree of divergence. With this attitude we will have an opportunity to use all the ingredients we have introduced in our proposed systematization. In next sections we will consider the relations among Green functions, ambiguities and Ward identities. We choose for this purpose simple but representative Green functions of the Standard model; the one-loop Green functions having only fermionic internal lines. It is simple to state relations among these structures as well as to state Ward identities to be obeyed by them.

In the construction of such Green functions through the Feynman rules, apart from coupling constants, internal symmetry operators and so on, we have to state the amplitudes for one value of the loop momentum k, which are the quantities

$$t^{\Gamma_i\Gamma_j\cdots\Gamma_l} = Tr\left\{\Gamma_i S_F\left(k+k_a;m_a\right)\Gamma_j S_F\left(k+k_b;m_b\right)\cdots \Gamma_l S_F\left(k+k_d;m_d\right)\right\}. \tag{65}$$

The Γ quantities are vertice operators belonging to the set

$$\Gamma_i = 1,\gamma_5,\gamma_\alpha,\gamma_\alpha\gamma_5,$$

appearing in the coupling of fermionic currents to the bosonic fields in the Lagrangian. After defining the operators corresponding Lorentz indexes are attached to $t^{\Gamma_i\Gamma_j\cdots\Gamma_l}$. The quantities S_F are fermionic propagators carrying momentum $k+k_a$ and mass m_a which we will write as

$$S_F = \frac{\left(k+k_a\right)+m_a}{D_a},$$

where through the quantity $D_a = \left[\left(k+k_a\right)^2 - m_a^2\right]$ we state a connection with the procedure described in the proceeding sections. The corresponding one-loop amplitudes are obtained by taking the integration of the t structures in the loop momentum k;

$$T^{\Gamma_i\Gamma_j\cdots\Gamma_l} = \int\frac{\mathrm{d}^4k}{\left(2\pi\right)^4}t^{\Gamma_i\Gamma_j\cdots\Gamma_l}.$$

In the present work we will consider the cases where the structures above correspond to divergent amplitudes for one, two, three and four-point functions. They are all connected due to relations among Green functions and Ward identities as we will see.

6.1. One-Point Functions

We start by taking the cases having the highest divergence degrees; the one-point functions. First, we write for the one value of the k momentum, the quantities

$$t^{\Gamma_i} = Tr\left\{\Gamma_i S_F\left(k+k_a;m_a\right)\right\},$$

or

$$t^{\Gamma_1} = Tr\left\{\Gamma_1\gamma_\alpha\right\}\frac{\left(k+k_1\right)^\alpha}{D_1} + m_1 Tr\left\{\Gamma_1\right\}\frac{1}{D_1}. \tag{66}$$

The corresponding one-loop amplitudes, obtained by integrating the above structures in the loop momentum,

$$T^{\Gamma_1} = \int\frac{\mathrm{d}^4k}{\left(2\pi\right)^4}t^{\Gamma_1},$$

are divergent quantities. The superficial degree of divergence is cubic. Now, taking two different possibilities to the vertice operators we can construct the one-point functions which will be useful in future developments. First we take the scalar one-point function which means to assume $\Gamma_1 = 1$. We get then

$$t^S = Tr\left\{\gamma_\alpha\right\}\frac{\left(k+k_1\right)^\alpha}{D_1} + m_1 Tr\left\{1\right\}\frac{1}{D_1},$$

or, solving the Dirac traces,

$$t^S = 4m_1\left[\frac{1}{D_1}\right].$$

At this point we adopt the adequate representation for the propagator, as we have made when we discussed the solution of the I_1 integral. Then we get

$$T^S \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} t^S = 4m_1 k_1^\alpha k_1^\beta \left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right]$$
$$+4m_1\left\{\left[I_{\text{quad}}\left(\lambda^2\right)\right]+\left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right]\right.$$
$$\left.+\frac{i}{(4\pi)^2}\left[m_1^2-\lambda^2+m_1^2\ln\left(\frac{\lambda^2}{m_1^2}\right)\right]\right\}.$$

Note the presence of the basic divergent objects as well as the presence of a potentially ambiguous term, the last one, since here $k_{1\alpha}$ is arbitrary.

Now taking $\Gamma_1 = \gamma_\mu$ in the expression (66) we get the vector one-point function

$$t_\mu^V = Tr\left\{\gamma_\mu \gamma_\alpha\right\}\frac{(k+k_1)^\alpha}{D_1} + m_1 Tr\left\{\gamma_\mu\right\}\frac{1}{D_1}.$$

Using the results for the Dirac traces involved we get

$$t_\mu^V = 4\left[\frac{k_\mu}{D_1} + k_{1\mu}\frac{1}{D_1}\right].$$

Adopting the adequate representation for the propagator as we have made in the calculation of the integrals $I_{1\mu}$ and I_1 we get

$$T_\mu^V(k_1) \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} t_\mu^V = -4k_1^\xi\left[\nabla_{\xi\mu}\left(\lambda^2\right)\right]$$
$$+2\left(k_1^2+2\lambda^2-2m_1^2\right)k_1^\xi\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]$$
$$-\frac{4}{3}k_1^\xi k_1^\chi k_1^\tau\left[\Box_{\xi\chi\tau\mu}\left(\lambda^2\right)\right]$$
$$+2k_{1\mu}k_1^\xi k_1^\chi\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right].$$

Note that the result is completely potentially ambiguous since all the quantities involved are arbitrary (the momentum k_1 and the scale λ^2). Let us now consider an example of two-point functions.

6.2. Two-Point Function

If one wants to consider a representative Green function of the perturbative calculation, concerning the consistency in the manipulations and calculations involving divergent Feynman integrals, certainly there is no better one than the fermionic two-point functions. We will consider three of such amplitudes related among them through Ward identities. We write them from the definition (65) as

$$t^{\Gamma_1\Gamma_2} = Tr\left\{\Gamma_1\gamma_\alpha\Gamma_2\gamma_\beta\right\}\frac{(k+k_1)^\alpha (k+k_2)^\beta}{D_{12}}$$
$$+m_2 Tr\left\{\Gamma_1\gamma_\alpha\Gamma_2\right\}\frac{(k+k_1)^\alpha}{D_{12}}$$
$$+m_1 Tr\left\{\Gamma_1\Gamma_2\gamma_\alpha\right\}\frac{(k+k_2)^\alpha}{D_{12}} + m_1 m_2 Tr\left\{\Gamma_1\Gamma_2\right\}\frac{1}{D_{12}}.$$

Firstly we consider the scalar-scalar where $\Gamma_1 = \Gamma_2 = 1$ (SS). For this case we get first (after taking the Dirac traces)

$$t^{SS} = \frac{1}{D_1} + \frac{1}{D_2} - \left[(k_1-k_2)^2 - (m_1+m_2)^2\right]\frac{1}{D_{12}}.$$

Now when the integration is taken the problems we have to solve are the integrals (45) and (46). Following the procedure we have adopted we get

$$\frac{T^{SS}}{2} \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{t^{SS}}{2} = \left[I_{\text{quad}}\left(\lambda^2\right)\right] + \left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}\left[m_1^2-\lambda^2+m_1^2\ln\left(\frac{\lambda^2}{m_1^2}\right)\right]$$
$$+\left[I_{\text{quad}}\left(\lambda^2\right)\right] + \left(m_2^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}\left[m_2^2-\lambda^2+m_2^2\ln\left(\frac{\lambda^2}{m_2^2}\right)\right] - \left[p^2-(m_1+m_2)^2\right]\left[I_{\log}\left(\lambda^2\right)\right]$$
$$+\frac{i}{(4\pi)^2}\left[p^2-(m_1+m_2)^2\right]\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] + \left(\frac{p^\alpha p^\beta}{2}+\frac{P^\alpha P^\beta}{2}\right)\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right],$$

Next, we consider the amplitude scalar-vector (*SV*) by taking $\Gamma_1 = 1$ and $\Gamma_2 = \gamma_\mu$, we get

$$t_\mu^{SV} = 2(m_2+m_1)\frac{k_\mu}{D_{12}} + 2\left(m_1 k_{2\mu} + m_2 k_{1\mu}\right)\frac{1}{D_{12}}.$$

To calculate the corresponding amplitude we have to solve the integrals (46) and (47). We get then

$$\frac{T_\mu^{SV}}{4} \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{t_\mu^{SV}}{4} = -\frac{1}{2}(m_2+m_1)P^\alpha\left[\Delta_{\alpha\mu}\left(\lambda^2\right)\right] + \frac{1}{2}p_\mu(m_2-m_1)\left[I_{\log}\left(\lambda^2\right)\right]$$
$$-\frac{i}{(4\pi)^2}p_\mu\left\{(m_2+m_1)\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] - m_1\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right\}. \tag{67}$$

Now we consider the most complex and interesting case; the vector-vector (*VV*) amplitude. It is obtained from the general definition (65) by assuming $\Gamma_1 = \gamma_\mu$, $\Gamma_2 = \gamma_\nu$. We get the the expression

$$t_{\mu\nu}^{VV} = 2\left[t_{2\mu\nu}^{(+)}\left(k_1,k_2\right)\right] + g_{\mu\nu}\left[t^{PP}\right],$$

where we have adopted the definitions

$$t_{2\mu\nu}^{(s)}\left(k_i,k_j\right) = \left[\left(k+k_i\right)_\mu\left(k+k_j\right)_\nu + s\left(k+k_i\right)_\nu\left(k+k_j\right)_\mu\right]\frac{1}{D_{12}}, \tag{68}$$

and

$$t^{PP} = -\frac{1}{D_1} - \frac{1}{D_2} + \left[p^2 - \left(m_1 - m_2\right)^2\right]\frac{1}{D_{12}},$$

which is precisely the pseudoscalar scalar (*PP*) two-point function. In the definition (68) above s assumes the values ± 1. After taking the integration in these expressions we have to solve the integrals (45), (46), (47) and, (48). Substituting the obtained results we get

$$\begin{aligned}
T_{\mu\nu} \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} t_{2\mu\nu}^{(+)}\left(k_1,k_2\right) &= 4\left[\nabla_{\mu\nu}\left(\lambda^2\right)\right] - 2\left(\lambda^2 - m_1^2 + \lambda^2 - m_2^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] \\
&+ \frac{2}{3}\left(-2k_2^2 + k_1\cdot k_2 - 2k_1^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] + \frac{4}{3}\left(k_2^\xi k_2^\chi + k_1^\xi k_2^\chi + k_1^\xi k_1^\chi\right)\left[\Box_{\xi\chi\mu\nu}\left(\lambda^2\right)\right] \\
&+ \frac{2}{3}\left(-k_{2\mu}k_2^\xi - 2k_{1\mu}k_2^\xi - 2k_1^\xi k_{2\mu} - k_{1\mu}k_1^\xi\right)\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right] + \frac{2}{3}\left(-k_{2\nu}k_2^\xi - 2k_{1\nu}k_2^\xi - 2k_{2\nu}k_1^\xi - k_{1\nu}k_1^\xi\right)\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right] \\
&+ \frac{2}{3}g_{\mu\nu}\left(k_2^\xi k_2^\chi + k_1^\xi k_2^\chi + k_1^\xi k_1^\chi\right)\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right] + 2g_{\mu\nu}\left\{\left[I_{\text{quad}}\left(\lambda^2\right)\right] - \left(\lambda^2 - m_1^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + m_1^2 - \lambda^2 - m_1^2\ln\left(\frac{m_1^2}{\lambda^2}\right)\right\} \\
&+ 2g_{\mu\nu}\left\{\left[I_{\text{quad}}\left(\lambda^2\right)\right] - \left(\lambda^2 - m_2^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + m_2^2 - \lambda^2 - m_2^2\ln\left(\frac{m_2^2}{\lambda^2}\right)\right\} \\
&- \frac{2}{3}\left(g_{\mu\nu}p^2 + 2p_\nu p_\mu\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}8\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right) - Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] \\
&+ \frac{i}{(4\pi)^2}2g_{\mu\nu}\left(p^2 + m_1^2 - m_2^2\right)\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] - \frac{i}{(4\pi)^2}4g_{\mu\nu}\left(m_1^2 - m_2^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right],
\end{aligned}$$

and

$$\begin{aligned}
T^{PP} \equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4} t^{PP} &= -2\left\{\left[I_{\text{quad}}\left(\lambda^2\right)\right] + \left(m_1^2 - \lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}\left[m_1^2 - \lambda^2 + m_1^2\ln\left(\frac{\lambda^2}{m_1^2}\right)\right]\right\} \\
&- 2\left\{\left[I_{\text{quad}}\left(\lambda^2\right)\right] + \left(m_2^2 - \lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}\left[m_2^2 - \lambda^2 + m_2^2\ln\left(\frac{\lambda^2}{m_2^2}\right)\right]\right\} \\
&+ 2\left[p^2 - \left(m_1 - m_2\right)^2\right]\left\{\left[I_{\log}\left(\lambda^2\right)\right] - \frac{i}{(4\pi)^2}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right\} - \left(p^\alpha p^\beta + P^\alpha P^\beta\right)\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right].
\end{aligned}$$

Then

$$\begin{aligned}
T_{\mu\nu}^{VV} &= \frac{4}{3}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[I_{\log}\left(\lambda^2\right)\right] - 2g_{\mu\nu}\left(m_1 - m_2\right)^2\left[I_{\log}\left(\lambda^2\right)\right] \\
&+ \frac{i}{(4\pi)^2}2\left\{4\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right) - Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right. \\
&+ g_{\mu\nu}\left(m_1^2 - m_2^2\right)\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right) - 2Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] \\
&\left.+ g_{\mu\nu}\left(m_1 - m_2\right)^2\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right\} + A_{\mu\nu},
\end{aligned} \tag{69}$$

where we have defined the quantity

$$A_{\mu\nu}=4\left[\nabla_{\mu\nu}\left(\lambda^2\right)\right]-2\left(2\lambda^2-m_1^2-m_2^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]-\frac{1}{6}\left(3P^2+5p^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]$$
$$+\frac{1}{3}\left[3P^{\xi}P^{\chi}-p^{\xi}P^{\chi}+p^{\chi}P^{\xi}+p^{\xi}p^{\chi}\right]\left[\square_{\xi\chi\mu\nu}\left(\lambda^2\right)\right]+\frac{1}{3}\left[-3P_{\mu}P^{\xi}+p_{\mu}p^{\xi}\right]\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]$$
$$+\frac{1}{3}\left[-3P_{\nu}P^{\xi}+p_{\nu}p^{\xi}\right]\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]+\frac{1}{6}g_{\mu\nu}\left[-3P^{\xi}P^{\chi}-p^{\xi}P^{\chi}+p^{\chi}P^{\xi}-5p^{\xi}p^{\chi}\right]\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right].$$

Note the presence in the above expression of potentially ambiguous terms since the quantity
$$P=k_1+k_2$$
is dependent on choices for arbitrary quantities as well as the presence of terms dependent on physical combination of the arbitrary internal momentum $p=k_2-k_1$ which are not dependent on the choices for the routing of the internal lines momenta of the loop amplitude but are dependent on the arbitrary choice for the common scale.

6.3. Three-Point Functions

Now we consider the case of three-point functions. In this case the higher degree of divergence involved is the linear one. We will take three related amplitudes in order to exploit the potentiality of the proposed systematization. From the definition (65) we get first the expression

$$t^{\Gamma_1\Gamma_2\Gamma_3}=Tr\left\{\Gamma_1\gamma_{\alpha}\Gamma_2\gamma_{\beta}\Gamma_3\gamma_{\xi}\right\}\frac{\left(k+k_1\right)^{\alpha}\left(k+k_2\right)^{\beta}\left(k+k_3\right)^{\xi}}{D_{123}}+m_1Tr\left\{\Gamma_1\Gamma_2\gamma_{\alpha}\Gamma_3\gamma_{\beta}\right\}\frac{\left(k+k_2\right)^{\alpha}\left(k+k_3\right)^{\beta}}{D_{123}}$$
$$+m_2Tr\left\{\Gamma_1\gamma_{\alpha}\Gamma_2\Gamma_3\gamma_{\beta}\right\}\frac{\left(k+k_1\right)^{\alpha}\left(k+k_3\right)^{\beta}}{D_{123}}+m_3Tr\left\{\Gamma_1\gamma_{\alpha}\Gamma_2\gamma_{\beta}\Gamma_3\right\}\frac{\left(k+k_1\right)^{\alpha}\left(k+k_2\right)^{\beta}}{D_{123}}$$
$$+m_1m_2Tr\left\{\Gamma_1\Gamma_2\Gamma_3\gamma_{\alpha}\right\}\frac{\left(k+k_3\right)^{\alpha}}{D_{123}}+m_1m_3Tr\left\{\Gamma_1\Gamma_2\gamma_{\alpha}\Gamma_3\right\}\frac{\left(k+k_2\right)^{\alpha}}{D_{123}} \tag{70}$$
$$+m_2m_3Tr\left\{\Gamma_1\gamma_{\alpha}\Gamma_2\Gamma_3\right\}\frac{\left(k+k_1\right)^{\alpha}}{D_{123}}+m_1m_2m_3Tr\left\{\Gamma_1\Gamma_2\Gamma_3\right\}\frac{1}{D_{123}}.$$

So if we take in all vertice scalar operators $\left(\Gamma_1=\Gamma_2=\Gamma_3=\hat{1}\right)$ we get

$$t^{SSS}=2\left(m_1+m_3\right)\frac{1}{D_{13}}+2\left(m_1+m_2\right)\frac{1}{D_{12}}+2\left(m_2+m_3\right)\frac{1}{D_{23}}-2\left\{m_1\left[\left(k_3-k_2\right)^2-\left(m_3+m_2\right)^2\right]\right.$$
$$\left.+m_2\left[\left(k_3-k_1\right)^2-\left(m_3+m_1\right)^2\right]+m_3\left[\left(k_2-k_1\right)^2-\left(m_2-m_1\right)^2\right]\right\}\frac{1}{D_{123}}.$$

By using the developments made in solving the integrals (46) and (49) we get the expression

$$T^{SSS}\equiv\int\frac{\mathrm{d}^4k}{\left(2\pi\right)^4}t^{SSS}=4\left(m_1+m_2+m_3\right)\left[I_{\log}\left(\lambda^2\right)\right]$$
$$+\frac{i}{\left(4\pi\right)^2}2m_1\left\{-\left[Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-\left[\left(p-q\right)^2-\left(m_3+m_2\right)^2\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{\left(4\pi\right)^2}2m_2\left\{-\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-\left[q^2-\left(m_3+m_1\right)^2\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{\left(4\pi\right)^2}2m_3\left\{-\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]-\left[p^2-\left(m_2-m_1\right)^2\right]\left[\xi_{00}\right]\right\}$$

On the other hand, taking $\Gamma_1=\gamma_{\lambda}$, $\Gamma_2=1$ and, $\Gamma_3=1$ in Equation (70) and by using the results obtaining in the study of integrals (46), (47) and, (49) we get

$$T_{\lambda}^{VSS} = -2\left(k_3+k_1\right)^{\xi}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right]+2\left(q-2p\right)_{\lambda}\left[I_{\log}\left(\lambda^2\right)\right]+\frac{i}{\left(4\pi\right)^2}2p_{\lambda}\left\{\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\right.$$
$$+2\left[p^2-\left(p\cdot q\right)-\left(m_3+m_2\right)\left(m_2+m_1\right)\right]\left[\xi_{10}\right]+\left.\left[q^2-\left(m_3-m_1\right)^2\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{\left(4\pi\right)^2}2q_{\lambda}\left\{\left[2Z_1\left(m_1^2;q^2,m_3^2;\lambda^2\right)-Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]-\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]\right.$$
$$+2\left[p^2-\left(p\cdot q\right)-\left(m_3+m_2\right)\left(m_2+m_1\right)\right]\left[\xi_{01}\right]-\left.\left[p^2-\left(m_2+m_1\right)^2\right]\left[\xi_{00}\right]\right\}.$$

Having two vector indexes we get the SVV amplitude

$$t_{\mu\nu}^{SVV} = m_2\left[t_{3\mu\nu}^{(+)}\left(k_1,k_3\right)\right]+m_1\left[t_{3\mu\nu}^{(-)}\left(k_3,k_2\right)\right]+m_3\left[t_{3\mu\nu}^{(+)}\left(k_1,k_2\right)\right]+g_{\mu\nu}\left[t^{SPP}\right],$$

where we have defined

$$t_{3\mu\nu}^{(s)}\left(k_i,k_j\right)=\left[\left(k+k_i\right)_{\mu}\left(k+k_j\right)_{\nu}+s\left(k+k_i\right)_{\nu}\left(k+k_j\right)_{\mu}\right]\frac{1}{D_{123}},$$

with $s=\pm1$, and

$$t^{SPP} = -2\left(m_1+m_3\right)\frac{1}{D_{13}}-2\left(m_1-m_2\right)\frac{1}{D_{12}}+2\left(m_2-m_3\right)\frac{1}{D_{23}}$$
$$+2\left\{m_1\left[\left(k_3-k_2\right)^2-\left(m_3-m_2\right)^2\right]-m_2\left[\left(k_3-k_1\right)^2-\left(m_3+m_1\right)^2\right]+m_3\left[\left(k_2-k_1\right)^2-\left(m_2+m_1\right)^2\right]\right\}\frac{1}{D_{123}}.$$

We get then

$$T_{\mu\nu}^{SVV} = 2\left(m_1+m_3\right)\left\{\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]\right\}+\frac{i}{\left(4\pi\right)^2}4\left\{-g_{\mu\nu}\left(m_1+m_3\right)\left[\eta_{00}\right]+2q_{\mu}q_{\nu}\left[\left(m_1+m_3\right)\xi_{02}-m_1\xi_{01}\right]\right.$$
$$+2p_{\mu}p_{\nu}\left(m_1+m_3\right)\left(\xi_{20}-\xi_{10}\right)+p_{\mu}q_{\nu}\left[\left(m_1+m_3\right)\left(2\xi_{11}-\xi_{01}\right)-\left(m_1+m_2\right)\xi_{10}+m_1\xi_{00}\right] \tag{71}$$
$$\left.+q_{\mu}p_{\nu}\left[\left(m_1+m_3\right)\left(2\xi_{11}-\xi_{01}\right)-\left(m_1-m_2\right)\xi_{10}+m_1\xi_{00}\right]\right\}+g_{\mu\nu}\left[T^{SPP}\right],$$

$$T^{SPP} = -4\left(m_1-m_2+m_3\right)\left[I_{\log}\left(\lambda^2\right)\right]$$
$$-\frac{i}{\left(4\pi\right)^2}2m_1\left\{-\left[Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-\left[\left(k_3-k_2\right)^2-\left(m_3-m_2\right)^2\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{\left(4\pi\right)^2}2m_2\left\{-\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]-\left[\left(k_3-k_1\right)^2-\left(m_3+m_1\right)^2\right]\left[\xi_{00}\right]\right\} \tag{72}$$
$$-\frac{i}{\left(4\pi\right)^2}2m_3\left\{-\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)\right]-\left[\left(k_2-k_1\right)^2-\left(m_2+m_1\right)^2\right]\left[\xi_{00}\right]\right\}.$$

Finally, let us consider the case of triple vector operators. First we get

$$t_{\lambda\mu\nu}^{VVV} = 4\left[t_{3\lambda\mu\nu}^{(+)}\left(k_1,k_2,k_3\right)\right]+4\left[t_{3\lambda\mu\nu}^{(-)}\left(k_2,k_1,k_3\right)\right]+4\left[t_{3\lambda\mu\nu}^{(+)}\left(k_3,k_1,k_2\right)\right]+g_{\mu\nu}\left[t_{\lambda}^{VPP}\right]+g_{\lambda\nu}\left[t_{\mu}^{PVP}\right]+g_{\lambda\mu}\left[t_{\nu}^{PPV}\right],$$

where the following definitions have been introduced

$$t_{\lambda}^{VPP} = 4\left\{-\left(k+k_1\right)_{\lambda}\left[\left(k+k_2\right)\cdot\left(k+k_3\right)-m_2m_3\right]+\left(k+k_2\right)_{\lambda}\left[\left(k+k_1\right)\cdot\left(k+k_3\right)-m_1m_3\right]\right.$$
$$\left.-\left(k+k_3\right)_{\lambda}\left[\left(k+k_1\right)\cdot\left(k+k_2\right)-m_1m_2\right]\right\}\frac{1}{D_{123}},$$

$$t_{\mu}^{PVP} = 4\left\{-\left(k+k_1\right)_{\mu}\left[\left(k+k_2\right)\cdot\left(k+k_3\right)-m_2m_3\right]-\left(k+k_2\right)_{\mu}\left[\left(k+k_1\right)\cdot\left(k+k_3\right)-m_1m_3\right]\right.$$
$$\left.+\left(k+k_3\right)_{\mu}\left[\left(k+k_1\right)\cdot\left(k+k_2\right)-m_1m_2\right]\right\}\frac{1}{D_{123}},$$

$$t_{\nu}^{PPV}=4\left\{\left(k+k_{1}\right)_{\nu}\left[\left(k+k_{2}\right)\cdot\left(k+k_{3}\right)-m_{2}m_{3}\right]-\left(k+k_{2}\right)_{\nu}\left[\left(k+k_{1}\right)\cdot\left(k+k_{3}\right)-m_{1}m_{3}\right]\right.$$
$$\left.-\left(k+k_{3}\right)_{\nu}\left[\left(k+k_{1}\right)\cdot\left(k+k_{2}\right)-m_{1}m_{2}\right]\right\}\frac{1}{D_{123}},$$

and

$$t_{3\lambda\mu\nu}^{(s)}\left(k_{i},k_{j},k_{l}\right)=\left(k+k_{i}\right)_{\lambda}\left[\left(k+k_{j}\right)_{\mu}\left(k+k_{l}\right)_{\nu}+s\left(k+k_{j}\right)_{\nu}\left(k+k_{l}\right)_{\mu}\right]\frac{1}{D_{123}},$$

with $s=\pm1$. With the aid of the integrals (49), (50), (51) and, (52) the tensors $t_{3\lambda\mu\nu}^{(s)}\left(k_{i},k_{j},k_{l}\right)$ may be written explicitly by

$$T_{\lambda\nu\mu}^{(s)}\left(k_{1},k_{2},k_{3}\right)=-\frac{1}{12}\left(k_{1\xi}+k_{2\xi}+k_{3\xi}\right)(1+s)\left[\Box_{\xi\nu\mu\lambda}\left(\lambda^{2}\right)\right]-\frac{1}{24}g_{\nu\lambda}\left(k_{1\xi}+k_{2\xi}+k_{3\xi}\right)(1+s)\left[\Delta_{\xi\mu}\left(\lambda^{2}\right)\right]$$
$$-\frac{1}{24}g_{\lambda\mu}\left(k_{1\xi}+k_{2\xi}+k_{3\xi}\right)(1+s)\left[\Delta_{\xi\nu}\left(\lambda^{2}\right)\right]-\frac{1}{24}g_{\nu\mu}\left(k_{1\xi}+k_{2\xi}+k_{3\xi}\right)(1+s)\left[\Delta_{\xi\lambda}\left(\lambda^{2}\right)\right]$$
$$+\frac{1}{24}\left[-(1+s)k_{1\nu}+(5-s)k_{2\nu}-(1-5s)k_{3\nu}\right]\left[\Delta_{\lambda\mu}\left(\lambda^{2}\right)\right]$$
$$+\frac{1}{24}\left[-(1+s)k_{1\mu}+(5-s)k_{3\mu}-(1-5s)k_{2\mu}\right]\left[\Delta_{\lambda\nu}\left(\lambda^{2}\right)\right]-\frac{1}{24}\left(-5k_{1\lambda}+k_{2\lambda}+k_{3\lambda}\right)(1+s)\left[\Delta_{\nu\mu}\left(\lambda^{2}\right)\right]$$
$$+\frac{1}{12}g_{\lambda\nu}\left[q_{\mu}+sp_{\mu}+(1-s)(q-p)_{\mu}\right]\left[I_{\log}\left(\lambda^{2}\right)\right]+\frac{1}{12}g_{\lambda\mu}\left[p_{\nu}+sq_{\nu}-(1-s)(q-p)_{\nu}\right]\left[I_{\log}\left(\lambda^{2}\right)\right]$$
$$-\frac{1}{12}g_{\nu\mu}\left[p_{\lambda}+q_{\lambda}\right](1+s)\left[I_{\log}\left(\lambda^{2}\right)\right]+\frac{1}{2}g_{\lambda\nu}\left\{p_{\mu}\left[(1+s)\eta_{10}-s\eta_{00}\right]+q_{\mu}\left[(1+s)\eta_{01}-\eta_{00}\right]\right\}$$
$$+\frac{1}{2}g_{\lambda\mu}\left\{p_{\nu}\left[(1+s)\eta_{10}-\eta_{00}\right]+q_{\nu}\left[(1+s)\eta_{01}-s\eta_{00}\right]\right\}+\frac{1}{2}g_{\nu\mu}\left\{p_{\lambda}\left[(1+s)\eta_{10}\right]+q_{\lambda}\left[(1+s)\eta_{01}\right]\right\}$$
$$+p_{\lambda}p_{\nu}p_{\mu}(1+s)\left[-\xi_{30}+\xi_{20}\right]+q_{\lambda}q_{\nu}q_{\mu}(1+s)\left[-\xi_{03}+\xi_{02}\right]+p_{\lambda}p_{\nu}q_{\mu}\left[-(1+s)\xi_{21}+\xi_{20}+\xi_{11}-\xi_{10}\right]$$
$$+p_{\lambda}q_{\nu}p_{\mu}\left[-(1+s)\xi_{21}+s\left(\xi_{20}+\xi_{11}-\xi_{10}\right)\right]+q_{\lambda}p_{\nu}p_{\mu}\left[-(1+s)\xi_{21}+(1+s)\xi_{11}\right]$$
$$+p_{\lambda}q_{\nu}q_{\mu}\left[-(1+s)\xi_{12}+(1+s)\xi_{11}\right]+q_{\lambda}p_{\nu}q_{\mu}\left[-(1+s)\xi_{12}+\xi_{02}+\xi_{11}-\xi_{01}\right]$$
$$+q_{\lambda}q_{\nu}p_{\mu}\left[-(1+s)\xi_{12}+s\left(\xi_{02}+\xi_{11}-\xi_{01}\right)\right].$$

On the other hand, the expressions for T_{λ}^{VPP}, T_{μ}^{PVP} and, T_{ν}^{PPV} may be written as

$$T_{\lambda}^{VPP}=2\left(k_{3}+k_{1}\right)^{\xi}\left[\Delta_{\lambda\xi}\left(\lambda^{2}\right)\right]+2(2p-q)_{\mu}\left[I_{\log}\left(\lambda^{2}\right)\right]+\frac{i}{(4\pi)^{2}}2q_{\lambda}\left\{-2Z_{1}\left(m_{1}^{2};q^{2},m_{3}^{2};\lambda^{2}\right)+Z_{0}\left(m_{1}^{2};q^{2},m_{3}^{2};\lambda^{2}\right)\right.$$
$$\left.+Z_{0}\left(m_{2}^{2};(p-q)^{2},m_{3}^{2};\lambda^{2}\right)-2\left[p^{2}-(p\cdot q)+\left(m_{2}-m_{1}\right)\left(m_{3}-m_{2}\right)\right]\left[\xi_{01}\right]+\left[p^{2}-\left(m_{2}-m_{1}\right)^{2}\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{(4\pi)^{2}}2p_{\lambda}\left\{-\left[Z_{0}\left(m_{1}^{2};p^{2},m_{2}^{2};\lambda^{2}\right)+Z_{0}\left(m_{2}^{2};(p-q)^{2},m_{3}^{2};\lambda^{2}\right)\right]\right.$$
$$\left.-2\left[p^{2}-(p\cdot q)+\left(m_{2}-m_{1}\right)\left(m_{3}-m_{2}\right)\right]\left[\xi_{10}\right]-\left[q^{2}-\left(m_{3}-m_{1}\right)^{2}\right]\left[\xi_{00}\right]\right\},$$

$$T_{\mu}^{PVP}=2\left(k_{2}+k_{1}\right)^{\xi}\left[\Delta_{\mu\xi}\left(\lambda^{2}\right)\right]+2(2q-p)_{\mu}\left[I_{\log}\left(\lambda^{2}\right)\right]+\frac{i}{(4\pi)^{2}}2p_{\mu}\left\{\left[Z_{0}\left(m_{1}^{2};p^{2},m_{2}^{2};\lambda^{2}\right)+Z_{0}\left(m_{2}^{2};(p-q)^{2},m_{3}^{2};\lambda^{2}\right)\right]\right.$$
$$-2\left[Z_{1}\left(m_{1}^{2};p^{2},m_{2}^{2};\lambda^{2}\right)\right]-2\left[q^{2}-(p\cdot q)-\left(m_{3}-m_{1}\right)\left(m_{3}-m_{2}\right)\right]\left[\xi_{10}\right]$$
$$\left.+\left[q^{2}-\left(m_{3}-m_{1}\right)^{2}\right]\left[\xi_{00}\right]\right\}+\frac{i}{(4\pi)^{2}}2q_{\mu}\left\{-\left[Z_{0}\left(m_{1}^{2};q^{2},m_{3}^{2};\lambda^{2}\right)+Z_{0}\left(m_{2}^{2};(p-q)^{2},m_{3}^{2};\lambda^{2}\right)\right]\right.$$
$$\left.-2\left[q^{2}-(p\cdot q)-\left(m_{3}-m_{1}\right)\left(m_{3}-m_{2}\right)\right]\left[\xi_{01}\right]-\left[p^{2}-\left(m_{2}-m_{1}\right)^{2}\right]\left[\xi_{00}\right]\right\},$$

$$T_{\nu}^{PPV}=2\left\{\left(k_3+k_2\right)^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]-\left(p+q\right)_{\nu}\left[I_{\log}\left(\lambda^2\right)\right]\right\}+\frac{i}{\left(4\pi\right)^2}2p_{\nu}\left\{\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)-Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]\right.$$
$$+2\left[Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]-2\left[\left(p\cdot q\right)-\left(m_3-m_1\right)\left(m_2-m_1\right)\right]\left[\xi_{10}\right]+\left[q^2-\left(m_3-m_1\right)^2\right]\left[\xi_{00}\right]\right\}$$
$$+\frac{i}{\left(4\pi\right)^2}2q_{\nu}\left\{\left[Z_0\left(m_1^2;q^2,m_3^2;\lambda^2\right)+Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]-2\left[Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]\right.$$
$$\left.-2\left[\left(p\cdot q\right)-\left(m_3-m_1\right)\left(m_2-m_1\right)\right]\left[\xi_{01}\right]+\left[p^2-\left(m_2-m_1\right)^2\right]\left[\xi_{00}\right]\right\}.$$

Finally, in the next section we perform the calculation of four-point functions.

6.4. The Four-Vector Four-Point Function

As an example of calculation of a Green function of the perturbative calculations having four fermionic propagators, we consider the four-vector four-point function, given by

$$t_{\mu\nu\alpha\beta}^{VVVV}=Tr\left\{\gamma_{\mu}\frac{1}{k+k_1-m_1}\gamma_{\nu}\frac{1}{k+k_2-m_2}\gamma_{\alpha}\frac{1}{k+k_3-m_3}\gamma_{\beta}\frac{1}{k+k_4-m_4}\right\},$$

or

$$t_{\mu\nu\alpha\beta}^{VVVV}=Tr\left\{\gamma_{\mu}\gamma_{\tau_1}\gamma_{\nu}\gamma_{\tau_2}\gamma_{\alpha}\gamma_{\tau_3}\gamma_{\beta}\gamma_{\tau_4}\right\}\frac{\left(k+k_1\right)^{\tau_1}\left(k+k_2\right)^{\tau_2}\left(k+k_3\right)^{\tau_3}\left(k+k_4\right)^{\tau_4}}{D_{1234}}$$
$$+m_1m_2Tr\left\{\gamma_{\mu}\gamma_{\nu}\gamma_{\alpha}\gamma_{\tau_1}\gamma_{\beta}\gamma_{\tau_2}\right\}\frac{\left(k+k_3\right)^{\tau_1}\left(k+k_4\right)^{\tau_2}}{D_{1234}}+m_1m_3Tr\left\{\gamma_{\mu}\gamma_{\nu}\gamma_{\tau_1}\gamma_{\alpha}\gamma_{\beta}\gamma_{\tau_2}\right\}\frac{\left(k+k_2\right)^{\tau_1}\left(k+k_4\right)^{\tau_2}}{D_{1234}}$$
$$+m_1m_4Tr\left\{\gamma_{\mu}\gamma_{\nu}\gamma_{\tau_1}\gamma_{\alpha}\gamma_{\tau_2}\gamma_{\beta}\right\}\frac{\left(k+k_2\right)^{\tau_1}\left(k+k_3\right)^{\tau_2}}{D_{1234}}+m_2m_3Tr\left\{\gamma_{\mu}\gamma_{\tau_1}\gamma_{\nu}\gamma_{\alpha}\gamma_{\beta}\gamma_{\tau_2}\right\}\frac{\left(k+k_1\right)^{\tau_1}\left(k+k_4\right)^{\tau_2}}{D_{1234}}$$
$$+m_2m_4Tr\left\{\gamma_{\mu}\gamma_{\tau_1}\gamma_{\nu}\gamma_{\alpha}\gamma_{\tau_2}\gamma_{\beta}\right\}\frac{\left(k+k_1\right)^{\tau_1}\left(k+k_3\right)^{\tau_2}}{D_{1234}}+m_3m_4Tr\left\{\gamma_{\mu}\gamma_{\tau_1}\gamma_{\nu}\gamma_{\tau_2}\gamma_{\alpha}\gamma_{\beta}\right\}\frac{\left(k+k_1\right)^{\tau_1}\left(k+k_2\right)^{\tau_2}}{D_{1234}}$$
$$+m_1m_2m_3m_4Tr\left\{\gamma_{\mu}\gamma_{\nu}\gamma_{\alpha}\gamma_{\beta}\right\}\frac{1}{D_{1234}}.$$

After performing the Dirac traces we identify the following structure

$$t_{\mu\nu\alpha\beta}^{VVVV}=4t_{\mu\nu\alpha\beta}+g_{\mu\nu}\left[t_{\alpha\beta}^{PPVV}\right]+g_{\mu\alpha}\left[t_{\nu\beta}^{PVPV}\right]+g_{\mu\beta}\left[t_{\nu\alpha}^{PVVP}\right]+g_{\nu\alpha}\left[t_{\mu\beta}^{VPPV}\right]+g_{\nu\beta}\left[t_{\mu\alpha}^{VPVP}\right]+g_{\alpha\beta}\left[t_{\mu\nu}^{VVPP}\right]$$
$$-\left(g_{\mu\beta}g_{\nu\alpha}-g_{\mu\alpha}g_{\nu\beta}+g_{\mu\nu}g_{\alpha\beta}\right)\left[t^{PPPP}\right]. \quad (73)$$

In the above expression a convenient and useful tensorial systematization was introduced

$$t_{\mu\nu\alpha\beta}=\left[t_{\mu\nu;\alpha\beta}^{(+;+)}\left(k_1,k_2;k_3,k_4\right)\right]+\left[t_{\mu\alpha;\nu\beta}^{(-;+)}\left(k_1,k_2;k_3,k_4\right)\right]-\left[t_{\mu\beta;\nu\alpha}^{(-;-)}\left(k_1,k_2;k_3,k_4\right)\right]+\left[t_{\nu\alpha;\mu\beta}^{(+;+)}\left(k_1,k_2;k_3,k_4\right)\right]$$
$$-\left[t_{\nu\beta;\mu\alpha}^{(+;-)}\left(k_1,k_2;k_3,k_4\right)\right]+\left[t_{\alpha\beta;\mu\nu}^{(-;-)}\left(k_1,k_2;k_3,k_4\right)\right], \quad (74)$$

where

$$t_{4\mu\nu;\alpha\beta}^{(\sigma_1;\sigma_2)}\left(k_i,k_j;k_l,k_n\right)=\left[\left(k+k_i\right)_{\mu}\left(k+k_j\right)_{\nu}+\sigma_1\left(k+k_i\right)_{\nu}\left(k+k_j\right)_{\mu}\right]$$
$$\times\left[\left(k+k_l\right)_{\alpha}\left(k+k_n\right)_{\beta}+\sigma_2\left(k+k_l\right)_{\beta}\left(k+k_n\right)_{\alpha}\right]\frac{1}{D_{1234}}. \quad (75)$$

Here σ_1 and σ_2 assume the values ±1. We also see that the coefficients of the metric tensor are four-point amplitudes with vector and pseudoscalar vertices defined as

$$t_{\mu\nu}^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4}=Tr\left\{\Gamma_1\frac{1}{k+k_1-m_1}\Gamma_2\frac{1}{k+k_2-m_2}\Gamma_3\frac{1}{k+k_3-m_3}\Gamma_4\frac{1}{k+k_4-m_4}\right\}.$$

After performing the Dirac traces, the four-point amplitudes with vector and pseudoscalar vertices acquire the form

$$\begin{aligned} t_{\mu\nu}^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4} &= 4s_1\left[(k+k_3)\cdot(k+k_4)-m_3m_4\right]\left[(k+k_1)_\mu(k+k_2)_\nu+s_2(k+k_1)_\nu(k+k_2)_\mu\right] \\ &+4s_3\left[(k+k_2)\cdot(k+k_4)-m_2m_4\right]\left[(k+k_1)_\mu(k+k_3)_\nu+s_4(k+k_1)_\nu(k+k_3)_\mu\right] \\ &+4s_5\left[(k+k_2)\cdot(k+k_3)-m_2m_3\right]\left[(k+k_1)_\mu(k+k_4)_\nu+s_6(k+k_1)_\nu(k+k_4)_\mu\right] \\ &+4s_7\left[(k+k_1)\cdot(k+k_4)-m_1m_4\right]\left[(k+k_2)_\mu(k+k_3)_\nu+s_8(k+k_2)_\nu(k+k_3)_\mu\right] \\ &+4s_9\left[(k+k_1)\cdot(k+k_3)-m_1m_3\right]\left[(k+k_2)_\mu(k+k_4)_\nu+s_{10}(k+k_2)_\nu(k+k_4)_\mu\right] \\ &+4s_{11}\left[(k+k_1)\cdot(k+k_2)-m_1m_2\right]\left[(k+k_3)_\mu(k+k_4)_\nu+s_{12}(k+k_3)_\nu(k+k_4)_\mu\right] \\ &+4s_{13}g_{\mu\nu}\left[t^{PPPP}\right], \end{aligned} \tag{76}$$

where $\Gamma_i=\{\gamma_5,\gamma_\mu\}$ and $s_i=\pm 1$ and

$$t^{PPPP}=Tr\left\{\gamma_5\frac{1}{k+k_1-m_1}\gamma_5\frac{1}{k+k_2-m_2}\gamma_5\frac{1}{k+k_3-m_3}\gamma_5\frac{1}{k+k_4-m_4}\right\}.$$

Below we identify the values of s_i according to the corresponding amplitude

$$\begin{bmatrix} \Gamma_1\Gamma_2\Gamma_3\Gamma_4 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ s_{10} & s_{11} & s_{12} & s_{13} & & & & & & \\ PPVV & - & - & + & + & + & - & - & + & - \\ - & - & + & + & & & & & & \\ PVPV & + & + & - & + & - & - & - & - & - \\ + & + & + & - & & & & & & \\ PVVP & - & + & - & - & + & - & - & + & + \\ + & - & - & + & & & & & & \\ VPPV & + & - & - & - & - & + & + & - & + \\ + & - & + & + & & & & & & \\ VPVP & - & - & - & + & + & + & + & + & - \\ + & + & - & - & & & & & & \\ VVPP & - & + & + & + & - & + & - & - & + \\ - & - & - & + & & & & & & \end{bmatrix}$$

Some algebraic effort is necessary in order to obtain an expression for the above amplitudes. This is a tedious task, although easy, because the number of external momenta and Lorentz indexes involved produce very large mathematical expressions. Consider first the tensor (74) for $i=1$, $j=2$, $k=3$ and $l=4$. From the results (53), (54), (56), (59) and, (62) we get

$$\begin{aligned} T_{4\mu\nu;\alpha\beta}^{(s_1;s_2)} &= \frac{1}{6}(1+s_1)(1+s_2)\left\{\left[\Box_{\mu\nu\alpha\beta}(\lambda^2)\right]+\frac{1}{2}g_{\mu\alpha}\left[\Delta_{\beta\nu}(\lambda^2)\right]+\frac{1}{2}g_{\alpha\beta}\left[\Delta_{\mu\nu}(\lambda^2)\right]+\frac{1}{2}g_{\alpha\nu}\left[\Delta_{\mu\beta}(\lambda^2)\right]\right. \\ &\left.+\frac{1}{2}g_{\mu\beta}\left[\Delta_{\alpha\nu}(\lambda^2)\right]+\frac{1}{2}g_{\mu\nu}\left[\Delta_{\alpha\beta}(\lambda^2)\right]+\frac{1}{2}g_{\beta\nu}\left[\Delta_{\mu\alpha}(\lambda^2)\right]\right\} \\ &+\frac{1}{6}\left(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}\right)\left[I_{\log}(\lambda^2)\right]+4(1+s_1)(1+s_2)\left[J_{\mu\nu\alpha\beta}\right] \\ &+4(1+s_1)\left\{(r_\beta+s_2q_\beta)\left[J_{\mu\nu\alpha}\right]+(q_\alpha+s_2r_\alpha)\left[J_{\mu\nu\beta}\right]\right\}+4(1+s_2)\left\{p_\nu\left[J_{\mu\alpha\beta}\right]+s_1p_\mu\left[J_{\nu\alpha\beta}\right]\right\} \\ &+4(1+s_1)(q_\alpha r_\beta+s_2q_\beta r_\alpha)\left[J_{\mu\nu}\right]+4(r_\beta+s_2q_\beta)\left\{p_\nu\left[J_{\mu\alpha}\right]+s_1p_\mu\left[J_{\nu\alpha}\right]\right\} \\ &+4(q_\alpha+s_2r_\alpha)\left\{p_\nu\left[J_{\mu\beta}\right]+s_1p_\mu\left[J_{\nu\beta}\right]\right\}+4(q_\alpha r_\beta+s_2q_\beta r_\alpha)\left\{p_\nu\left[J_\mu\right]+s_1p_\mu\left[J_\nu\right]\right\}, \end{aligned}$$

where $J_{\mu\nu\alpha\beta}$, $J_{\mu\nu\alpha}$, $J_{\mu\nu}$ and J_{μ} are given by in Equations (55), (57), (60) and (63). Replacing the above result (with appropriate values for the symbols σ_1 and σ_2) in Equation (74) gives

$$
\begin{aligned}
T_{\mu\nu\alpha\beta} =& \frac{4}{3}\Big\{\left[\Box_{\alpha\beta\mu\nu}\left(\lambda^2\right)\right] + \frac{1}{2}g_{\mu\alpha}\left[\Delta_{\beta\nu}\left(\lambda^2\right)\right] + \frac{1}{2}g_{\alpha\beta}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] + \frac{1}{2}g_{\alpha\nu}\left[\Delta_{\mu\beta}\left(\lambda^2\right)\right] \\
&+ \frac{1}{2}g_{\mu\beta}\left[\Delta_{\alpha\nu}\left(\lambda^2\right)\right] + \frac{1}{2}g_{\mu\nu}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right] + \frac{1}{2}g_{\beta\nu}\left[\Delta_{\mu\alpha}\left(\lambda^2\right)\right]\Big\} + \frac{4}{3}\left(g_{\alpha\beta}g_{\mu\nu} + g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\mu\beta}\right)\left[I_{\log}\left(\lambda^2\right)\right] \\
&+ 32\left[J_{\mu\nu\alpha\beta}\right] + 16\left\{r_\mu\left[J_{\nu\alpha\beta}\right] + p_\nu\left[J_{\mu\alpha\beta}\right] + \left(q_\beta + r_\beta\right)\left[J_{\mu\nu\alpha}\right] + \left(q_\alpha + p_\alpha\right)\left[J_{\mu\nu\beta}\right]\right\} + 8\left(p_\nu r_\mu - r_\nu p_\mu\right)\left[J_{\alpha\beta}\right] \\
&+ 8\left(r_\beta q_\mu + q_\beta r_\mu\right)\left[J_{\nu\alpha}\right] + 8\left(p_\nu q_\alpha + q_\nu p_\alpha\right)\left[J_{\mu\beta}\right] + 8\left(r_\nu p_\beta + p_\nu r_\beta + p_\nu q_\beta - q_\nu p_\beta\right)\left[J_{\mu\alpha}\right] \\
&+ 8\left(q_\alpha r_\mu - r_\alpha q_\mu + r_\alpha p_\mu + p_\alpha r_\mu\right)\left[J_{\nu\beta}\right] + 8\left(q_\alpha r_\beta + q_\beta r_\alpha + p_\alpha q_\beta + q_\alpha p_\beta + p_\alpha r_\beta - r_\alpha p_\beta\right)\left[J_{\mu\nu}\right] \\
&+ 4\left\{p_\nu\left(q_\alpha r_\beta + q_\beta r_\alpha\right) + p_\alpha\left(q_\nu r_\beta + q_\beta r_\nu\right) - p_\beta\left(q_\nu r_\alpha - q_\alpha r_\nu\right)\right\}\left[J_\mu\right] \\
&+ 4\left\{p_\mu\left(q_\alpha r_\beta + q_\beta r_\alpha\right) + p_\alpha\left(q_\mu r_\beta + q_\beta r_\mu\right) - p_\beta\left(q_\mu r_\alpha - q_\alpha r_\mu\right)\right\}\left[J_\nu\right] \\
&+ 4\left\{p_\nu\left(q_\mu r_\beta + q_\beta r_\mu\right) - p_\mu\left(q_\nu r_\beta + q_\beta r_\nu\right) + p_\beta\left(q_\mu r_\nu - q_\nu r_\mu\right)\right\}\left[J_\alpha\right] \\
&+ 4\left\{p_\mu\left(q_\nu r_\alpha - q_\alpha r_\nu\right) - p_\nu\left(q_\mu r_\alpha - q_\alpha r_\mu\right) - p_\alpha\left(q_\mu r_\nu - q_\nu r_\mu\right)\right\}\left[J_\beta\right].
\end{aligned}
$$

For the amplitudes listed in the table above we may write

$$
\begin{aligned}
T_{\mu\nu}^{\Gamma_1\Gamma_2\Gamma_3\Gamma_4} =& s_1'\left\{\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] + g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]\right\} + g_{\mu\nu}\left[F_0\right] + p_\mu p_\nu\left[F_1\right] + q_\mu q_\nu\left[F_2\right] + r_\mu r_\nu\left[F_3\right] + p_\mu r_\nu\left[F_4\right] + r_\mu p_\nu\left[F_5\right] \\
&+ p_\mu q_\nu\left[F_6\right] + q_\mu p_\nu\left[F_7\right] + q_\mu r_\nu\left[F_8\right] + r_\mu q_\nu\left[F_9\right] + s_{13}g_{\mu\nu}\left[T^{PPPP}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
F_0 =& -s_2'\left[\eta_{00}''\right] - s_3'\left[\eta_{00}'\right] - s_4'\left[\eta_{00}'''\right] - s_5'\left[\eta_{00}''''\right] + \left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-s_1\left(1+s_2\right)\eta_{000}\right] \\
&+ \left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-s_3\left(1+s_4\right)\eta_{000}\right] + \left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-s_5\left(1+s_6\right)\eta_{000}\right] \\
&+ \left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-s_7\left(1+s_8\right)\eta_{000}\right] + \left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-s_9\left(1+s_{10}\right)\eta_{000}\right] \\
&+ \left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-s_{11}\left(1+s_{12}\right)\eta_{000}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_1 =& 2s_2'\left[\xi_{20}''\right] + 2s_3'\left[\xi_{20}'\right] + 2s_5'\left[\xi_{20}'''' + \xi_{02}''''\right] + 4s_5'\left[\xi_{11}''''\right] - 2\left(s_1 + s_9 + s_1 s_2 + s_9 s_{10}\right)\left[\xi_{10}''\right] \\
&- 2\left(s_1 + s_7 + s_7 s_8 + s_1 s_2\right)\left[\xi_{10}'\right] - 2\left(2s_{11} + 2s_{11}s_{12} + s_7 + s_9 + s_7 s_8 + s_9 s_{10}\right)\left[\xi_{10}''''\right] \\
&- 2\left(2s_{11} + 2s_{11}s_{12} + s_7 + s_9 + s_7 s_8 + s_9 s_{10}\right)\left[\xi_{01}''''\right] + 2s_{11}\left(1+s_{12}\right)\left[\xi_{00}''''\right] \\
&+ 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{200} + \left(1+s_2\right)\zeta_{100}\right] + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{200}\right] \\
&+ 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{200}\right] + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{200} + \left(1+s_8\right)\zeta_{100}\right] \\
&+ 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{200} + \left(1+s_{10}\right)\zeta_{100}\right] + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{200}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_2 =& 2s_3'\left[\xi_{02}'\right] + 2s_4'\left[\xi_{20}'''\right] + 2s_5'\left[\xi_{20}''''\right] - 2\left(s_7 + s_3\right)\left[\xi_{01}'\right] - 2\left(s_7 s_8 + s_3 s_4\right)\left[\xi_{01}'\right] - 2\left(s_{11}s_{12} + s_3\right)\left[\xi_{10}'''\right] \\
&- 2\left(s_{11} + s_3 s_4\right)\left[\xi_{10}'''\right] - 2\left(s_{11}s_{12} + s_7 + s_{11} + s_7 s_8\right)\left[\xi_{10}''''\right] + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{020}\right] \\
&+ 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{020} + \left(1+s_4\right)\zeta_{010}\right] \\
&+ 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{020}\right] + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{020} + \left(1+s_8\right)\zeta_{010}\right], \\
&+ 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{020}\right] + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{020} + \left(1+s_{12}\right)\zeta_{010}\right]
\end{aligned}
$$

$$
\begin{aligned}
F_3 = {} & 2s_2'\left[\xi_{02}''\right] + 2s_4'\left[\xi_{02}'''\right] + 2s_5'\left[\xi_{02}''''\right] - 2\left(s_9 + s_5\right)\left[\xi_{01}''\right] - 2\left(s_9 s_{10} + s_5 s_6\right)\left[\xi_{01}''\right] - 2\left(s_{11} + s_5\right)\left[\xi_{01}'''\right] \\
& - 2\left(s_{11}s_{12} + s_5 s_6\right)\left[\xi_{01}'''\right] - 2\left(s_{11} + s_9 + s_{11}s_{12} + s_9 s_{10}\right)\left[\xi_{01}''''\right] \\
& + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{002}\right] + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{002}\right] \\
& + 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{002} + \left(1+s_6\right)\zeta_{001}\right] \\
& + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{002}\right] + 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{002} + \left(1+s_{10}\right)\zeta_{001}\right] \\
& + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{002} + \left(1+s_{12}\right)\zeta_{001}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_4 = {} & 2s_2'\left[\xi_{11}''\right] - 2s_5'\left[\xi_{02}'''' + \xi_{11}''''\right] - 2\left(s_9 + s_5\right)\left[\xi_{10}''\right] - 2\left(s_9 + s_1 s_2\right)\left[\xi_{01}''\right] + 2\left(s_{11} + s_9 + s_{11}s_{12} + s_9 s_{10} + s_{11} + s_7 s_8\right)\left[\xi_{01}''''\right] \\
& + 2\left(s_{11} + s_9\right)\left[\xi_{10}''''\right] + 2s_9\left[\xi_{00}''\right] - 2s_{11}\left[\xi_{00}''''\right] + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{101} + s_2\zeta_{001}\right] \\
& + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{101}\right] + 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{101} + \zeta_{100}\right] \\
& + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{101} + \zeta_{001}\right] + 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{101} + \zeta_{100} + \zeta_{001} - \zeta_{000}\right] \\
& + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{101} + \zeta_{100}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_5 = {} & 2s_2'\left[\xi_{11}''\right] - 2s_5'\left[\xi_{02}'''' + \xi_{11}''''\right] - 2\left(s_9 s_{10} + s_1\right)\left[\xi_{01}''\right] - 2\left(s_9 s_{10} + s_5 s_6\right)\left[\xi_{10}''\right] \\
& + 2\left(s_{11} + s_9 + s_7 + s_9 s_{10} + 2s_{11}s_{12}\right)\left[\xi_{01}''''\right] + 2\left(s_{11}s_{12} + s_9 s_{10}\right)\left[\xi_{10}''''\right] + 2s_9 s_{10}\left[\xi_{00}''\right] - 2s_{11}s_{12}\left[\xi_{00}''''\right] \\
& + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{101} + \zeta_{001}\right] + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{101}\right] \\
& + 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{101} + s_6\zeta_{100}\right] + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{101} + s_8\zeta_{001}\right] \\
& + 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{101} + s_{10}\left(\zeta_{001} + \zeta_{100} - \zeta_{000}\right)\right] \\
& + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{101} + s_{12}\zeta_{100}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_6 = {} & 2s_3'\left[\xi_{11}'\right] - 2s_5'\left[\xi_{20}'''' + \xi_{11}''''\right] - 2\left(s_7 + s_3\right)\left[\xi_{10}'\right] - 2\left(s_7 + s_1 s_2\right)\left[\xi_{01}'\right] + 2\left(2s_{11}s_{12} + s_7 + s_{11} + s_7 s_8 + s_9 s_{10}\right)\left[\xi_{10}''''\right] \\
& + 2\left(s_{11}s_{12} + s_7\right)\left[\xi_{01}''''\right] + 2s_7\left[\xi_{00}'\right] - 2s_{11}s_{12}\left[\xi_{00}''''\right] + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{110} + s_2\zeta_{010}\right] \\
& + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{110} + \zeta_{100}\right] + 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{110}\right] \\
& + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{110} + \zeta_{100} + \zeta_{010} - \zeta_{000}\right] + 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{110} + \zeta_{010}\right] \\
& + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{110} + s_{12}\zeta_{100}\right],
\end{aligned}
$$

$$
\begin{aligned}
F_7 = {} & 2s_3'\left[\xi_{11}'\right] - 2s_5'\left[\xi_{20}'''' + \xi_{11}''''\right] - 2\left(s_7 s_8 + s_1\right)\left[\xi_{01}'\right] - 2\left(s_7 s_8 + s_3 s_4\right)\left[\xi_{10}'\right] + 2\left(s_{11}s_{12} + s_7 + s_{11} + s_7 s_8 + s_{11} + s_9\right)\left[\xi_{10}''''\right] \\
& + 2\left(s_{11} + s_7 s_8\right)\left[\xi_{01}''''\right] + 2s_7 s_8\left[\xi_{00}'\right] - 2s_{11}\left[\xi_{00}''''\right] + 2s_1\left[\left(r-q\right)^2 - \left(m_4 - m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{110} + \zeta_{010}\right] \\
& + 2s_3\left[\left(r-p\right)^2 - \left(m_4 - m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{110} + s_4\zeta_{100}\right] + 2s_5\left[\left(q-p\right)^2 - \left(m_3 - m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{110}\right] \\
& + 2s_7\left[r^2 - \left(m_4 - m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{110} + s_8\left(\zeta_{010} + \zeta_{100} - \zeta_{000}\right)\right] \\
& + 2s_9\left[q^2 - \left(m_3 - m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{110} + s_{10}\zeta_{010}\right] \\
& + 2s_{11}\left[p^2 - \left(m_2 - m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{110} + \zeta_{100}\right],
\end{aligned}
$$

$$F_8 = 2s_4'\left[\xi_{11}'''\right]+2s_5'\left[\xi_{11}''''\right]-2\left(s_{11}+s_5\right)\left[\xi_{10}'''\right]-2\left(s_{11}+s_9\right)\left[\xi_{10}''''\right]-2\left(s_{11}+s_3s_4\right)\left[\xi_{01}'''\right]-2\left(s_{11}+s_7s_8\right)\left[\xi_{01}''''\right]+2s_{11}\left[\xi_{00}'''+\xi_{00}''''\right]$$
$$+2s_1\left[\left(r-q\right)^2-\left(m_4-m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{011}\right]+2s_3\left[\left(r-p\right)^2-\left(m_4-m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{011}+s_4\zeta_{001}\right]$$
$$+2s_5\left[\left(q-p\right)^2-\left(m_3-m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{011}+\zeta_{010}\right]+2s_7\left[r^2-\left(m_4-m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{011}+s_8\zeta_{001}\right]$$
$$+2s_9\left[q^2-\left(m_3-m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{011}+\zeta_{010}\right]+2s_{11}\left[p^2-\left(m_2-m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{011}+\zeta_{010}+\zeta_{001}-\zeta_{000}\right],$$

$$F_9 = 2s_4'\left[\xi_{11}'''\right]+2s_5'\left[\xi_{11}''''\right]-2\left(s_{11}s_{12}+s_3\right)\left[\xi_{01}'''\right]-2\left(s_{11}s_{12}+s_7\right)\left[\xi_{01}''''\right]-2\left(s_{11}s_{12}+s_5s_6\right)\left[\xi_{10}'''\right]$$
$$-2\left(s_{11}s_{12}+s_9s_{10}\right)\left[\xi_{10}''''\right]+2s_{11}s_{12}\left[\xi_{00}'''+\xi_{00}''''\right]+2s_1\left[\left(r-q\right)^2-\left(m_4-m_3\right)^2\right]\left[-\left(1+s_2\right)\zeta_{011}\right]$$
$$+2s_3\left[\left(r-p\right)^2-\left(m_4-m_2\right)^2\right]\left[-\left(1+s_4\right)\zeta_{011}+\zeta_{001}\right]+2s_5\left[\left(q-p\right)^2-\left(m_3-m_2\right)^2\right]\left[-\left(1+s_6\right)\zeta_{011}+s_6\zeta_{010}\right]$$
$$+2s_7\left[r^2-\left(m_4-m_1\right)^2\right]\left[-\left(1+s_8\right)\zeta_{011}+\zeta_{001}\right]+2s_9\left[q^2-\left(m_3-m_1\right)^2\right]\left[-\left(1+s_{10}\right)\zeta_{011}+s_{10}\zeta_{010}\right]$$
$$+2s_{11}\left[p^2-\left(m_2-m_1\right)^2\right]\left[-\left(1+s_{12}\right)\zeta_{011}+s_{12}\left(\zeta_{001}+\zeta_{010}-\zeta_{000}\right)\right],$$

$$T^{PPPP} = 4\left[I_{\log}\left(\lambda^2\right)\right]-2\left[Z_0\left(m_2^2;\left(r-p\right)^2,m_4^2\right)+Z_0\left(m_1^2;q^2,m_2^2\right)\right]$$
$$-2\left[q^2-\left(r\cdot q\right)+\left(r\cdot p\right)-\left(q\cdot p\right)\right]\left[\xi_{00}''''\right]-2\left[r^2-\left(r\cdot q\right)\right]\left[\xi_{00}'''\right]-2\left(r\cdot p\right)\left[\xi_{00}''\right]$$
$$-\left[2p^2-2\left(q\cdot p\right)\right]\left[\xi_{00}'\right]+\left[p^2\left(r-q\right)^2-q^2\left(r-p\right)^2+r^2\left(q-p\right)^2\right]\left[\zeta_{000}\right].$$

Above, the following compact definitions were also used

$$\xi_{nm}' = \xi_{nm}\left(m_1^2;p,m_2^2;q,m_3^2\right),$$
$$\xi_{nm}'' = \xi_{nm}\left(m_1^2;p,m_2^2;r,m_4^2\right),$$
$$\xi_{nm}''' = \xi_{nm}\left(m_1^2;q,m_3^2;r,m_4^2\right),$$
$$\xi_{nm}'''' = \xi_{nm}\left(m_2^2;q-p,m_3^2;r-p,m_4^2\right),$$

and

$$s_1' = s_1\left(1+s_2\right)+s_3\left(1+s_4\right)+s_5\left(1+s_6\right)+s_7\left(1+s_8\right)+s_9\left(1+s_{10}\right)+s_{11}\left(1+s_{12}\right),$$
$$s_2' = s_1\left(1+s_2\right)+s_5\left(1+s_6\right)+s_9\left(1+s_{10}\right),$$
$$s_3' = s_1\left(1+s_2\right)+s_3\left(1+s_4\right)+s_7\left(1+s_8\right),$$
$$s_4' = s_3\left(1+s_4\right)+s_5\left(1+s_6\right)+s_{11}\left(1+s_{12}\right),$$
$$s_5' = s_7\left(1+s_8\right)+s_9\left(1+s_{10}\right)+s_{11}\left(1+s_{12}\right).$$

Our main purpose has been, at this point, fulfilled which is to show how the proposed systematization works in the calculation of physical amplitudes. However, another important aspect involved in perturbative calculations can be also considered which, within the context of our procedure, became very simple and transparent, that is the verification of relations among the Green functions and, consequently, of the associated Ward identities. We perform such task in the next section.

7. Relations among Green Functions

In the preceding sections we have described in details a procedure to handle the divergences typical of the perturbative calculations in QFT. The procedure is very general since all the choices involved have been preserved; the internal momenta were taken as arbitrary so that all possible choices can be made in the final results, the choice of regularization is avoided since all the steps performed are allowed in the context of all reasonable regularization prescription and an arbitrary scale was adopted in the separation of terms having different degrees of divergent and finite ones. We can ask ourselves at this point about the consistency of the performed operations as usual in such type of manipulations and calculations. In order to verify this aspect we can make a minimal test of consistency by verifying if the relations among the calculated Green functions remain preserved after the realized operations. The required consistency is to verify such identities without assuming particular choices for the involved arbitrariness, which means that the relations need to be satisfied in the presence of potentially ambiguous and symmetry violating terms. Essentially, what we want to know is if the performed operations have preserved the property of linearity of the integration which seems to be a trivial task but, given the mathematical indefinitions involved, it is not. Only if the operations realized until this point possess the desired consistency we can give an additional step which is to verify if the potentially ambiguous and symmetry violating terms can be eliminated in a consistent way. Let us consider this aspect in detail now.

We start by considering the *VV* two-point function whose calculation we have considered in detail in the Sec. (VI). In order to state a relation with other calculated amplitudes it is enough to note the identity bellow

$$\begin{aligned}&(k_2-k_1)^{\mu}\left\{\gamma_\nu\frac{1}{(k+k_1)-m_1}\gamma_\mu\frac{1}{(k+k_2)-m_2}\right\}\\&=\left\{\gamma_\nu\frac{1}{[k+k_1-m_1]}\right\}-\left\{\gamma_\nu\frac{1}{[k+k_2-m_2]}\right\}\\&\quad+(m_2-m_1)\left\{\gamma_\nu\frac{1}{(k+k_1)-m_1}\frac{1}{(k+k_2)-m_2}\right\}.\end{aligned}$$

After taking the Dirac traces in both sides we can identify that

$$\begin{aligned}&(k_2-k_1)^{\mu}t_{\mu\nu}^{VV}\\&=t_\nu^V(k_1,m_1)-t_\nu^V(k_2,m_2)+(m_2-m_1)t_\nu^{VS},\end{aligned}\tag{77}$$

The above relation means that it is expected that if we integrate both sides in the loop momentum k the corresponding relation among the loop amplitudes remain valid, *i.e.*,

$$\begin{aligned}&(k_2-k_1)^{\mu}T_{\mu\nu}^{VV}\\&=T_\nu^V(k_1,m_1)-T_\nu^V(k_2,m_2)+(m_2-m_1)T_\nu^{VS}.\end{aligned}\tag{78}$$

This means that by calculating all the involved amplitudes in a separated way and after this contracting the *VV* amplitude the reorganization of the terms must allow the identification of the amplitudes in the specific combination of the right hand side. This type of identity is highly nontrivial to be preserved in traditional regularization prescriptions. A similar procedure allows us to state that

$$\begin{aligned}&(k_2-k_1)^{\mu}T_{\mu}^{VS}\\&=T^S(k_1,m_1)-T^S(k_2,m_2)+(m_2-m_1)T^{SS},\end{aligned}\tag{79}$$

which implies that

$$\begin{aligned}&(k_2-k_1)^{\mu}(k_2-k_1)^{\nu}T_{\mu\nu}^{VV}\\&=(k_2-k_1)^{\mu}\left[T_\mu^V(k_1,m_1)-T_\mu^V(k_2,m_2)\right]\\&\quad+(m_2-m_1)\left[T^S(k_1,m_1)-T^S(k_2,m_2)\right]\\&\quad+(m_2-m_1)^2T^{SS}.\end{aligned}\tag{80}$$

We can note from the above expressions that all amplitudes of the perturbative calculations are related among them. In particular, the above considered relations involve the amplitudes: *VV*, *VS*, *SS*, *PP*, *V* and, *S*.

For the calculated three-point function structures we can verify the relations

$$\begin{aligned}&(k_3-k_1)^{\lambda}T_{\lambda\mu\nu}^{VVV}\\&=\left[T_{\mu\nu}^{VV}(k_2,k_1)\right]-\left[T_{\mu\nu}^{VV}(k_2,k_3)\right]+(m_3-m_1)\left[T_{\mu\nu}^{SVV}\right],\end{aligned}\tag{81}$$

$$\begin{aligned}&(k_2-k_1)^{\mu}T_{\mu\nu}^{SVV}\\&=\left[T_\nu^{SV}(k_1,k_3)\right]-\left[T_\nu^{SV}(k_2,k_3)\right]+(m_2-m_1)\left[T_\nu^{SSV}\right],\end{aligned}\tag{82}$$

$$\begin{aligned}&(k_3-k_2)^{\nu}T_{\nu}^{SSV}\\&=\left[T^{SS}(k_1,k_2)\right]-\left[T^{SS}(k_1,k_3)\right]+(m_3-m_2)\left[T^{SSS}\right].\end{aligned}\tag{83}$$

Now we can note that all the three, two and one-point calculated functions are in fact related among them through precise relations. In the above considered relations the following structures are involved: *VVV*, *VVS*, *VSS*, *VV*, *VS* and, *SS* plus the ones which appear as substructures: *VPP*, *SPP*, *PP* and *S*.

If we consider four-point functions, the same will occur. To evaluate the *VVVV* function all the above mentioned structures will appear as well as other four-point structures. This is a very crucial point. We can start from a finite amplitude and by successive contractions we can relate such amplitude with the cubically divergent one-point function. The challenge is then to evaluate all the perturbative amplitudes within a certain prescription maintaining all the relations among them preserved in a simultaneous way. Within the context of our procedure we will show that all the relations presented above can be verified in the presence of all remaining arbitrariness. We emphasize that such type of verifications are very nontrivial for all traditional techniques.

Let us start by the property (78). Taking the expression for the *VV* amplitude, Equation (69), and contracting with $k_2-k_1=p$ we get

$$\begin{aligned}&p^{\nu}T_{\mu\nu}^{VV}\left(k_1,k_2\right)\\&=\left(m_2-m_1\right)\Big\{-2\left(m_2-m_1\right)p_\mu\left[I_{\log}\left(\lambda^2\right)\right]\\&\quad+4\left(m_2+m_1\right)p_\mu\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\&\quad-4m_1p_\mu\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\Big\}\\&\quad+p^{\nu}\left[A_{\mu\nu}\left(k_1,k_2\right)\right].\end{aligned}$$

By comparing to the result (67) for the VS amplitude we can identify

$$\begin{aligned}p^{\nu}T_{\mu\nu}^{VV}\left(k_1,k_2\right)=\;&p^{\nu}\left[A_{\mu\nu}\left(k_1,k_2\right)\right]\\&+2\left(m_2^2-m_1^2\right)\left(k_2+k_1\right)^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\\&+\left(m_2-m_1\right)\left[T_{\nu}^{VS}\left(k_1,k_2\right)\right].\end{aligned}$$

In order to complete the verification of the property (78), the last term in the above equation must be identified with the one-point vector functions. It is simple to note that if an $\nabla_{\mu\nu}$ is added and subtracted in the expression for $A_{\mu\nu}$, a reorganization allows us to identify

$$\begin{aligned}p^{\nu}A_{\mu\nu}=\;&\left[T_{\mu}^{V}\left(k_1\right)-T_{\mu}^{V}\left(k_2\right)\right]\\&-2\left(m_2^2-m_1^2\right)\left(k_2+k_1\right)^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right].\end{aligned}$$

So, the relation (78) is obtained preserved by our calculation.

The relation (80) is, on the other hand, emblematic to explain many aspects of our procedure and we will make the discussion in details. First we note that by contracting the expression (67) for the VS amplitude it is obtained

$$\begin{aligned}&p^{\nu}T_{\nu}^{VS}\\&=2\left(m_1+m_2\right)\left[\left(k_1\right)^{\chi}\left(k_1\right)^{\xi}-\left(k_2\right)^{\chi}\left(k_2\right)^{\xi}\right]\left[\Delta_{\chi\xi}\left(\lambda^2\right)\right]\\&\quad-2\left(m_2-m_1\right)p^2\left[I_{\log}\left(\lambda^2\right)\right]\\&\quad+4\left(m_1+m_2\right)p^2\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\&\quad-4m_1p^2\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right].\end{aligned}\tag{84}$$

We know that this result needs to be related to the SS amplitude as well as with S amplitudes having different masses. This means that quadratic divergences need to appear from the right hand side in a non-cancelling way. At first sight it seems that it is not possible to satisfy the relation. However, we note that on the left hand side of the identity (84) we have the function Z_1 and in the right hand side only Z_0 must appear. Let us consider the reduction of Z_1 to Z_0 through the property (13) in order to adequate the right hand side of the Equation (84). The referred reduction is the property (13) which allows us to write

$$\begin{aligned}&2p^2Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\\&+\left(p^2+m_1^2-m_2^2\right)\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\&=m_1^2-m_1^2\ln\left(\frac{m_1^2}{\lambda^2}\right)-m_2^2+m_2^2\ln\left(\frac{m_2^2}{\lambda^2}\right).\end{aligned}\tag{85}$$

Now consider the result obtained for the I_1 integral at the value $k_i=0,$ which is nothing more than a scale property of the basic quadratic divergent object $I_{\text{quad}}\left(\lambda^2\right)$,

$$\begin{aligned}I_{\text{quad}}\left(m_1^2\right)=\;&I_{\text{quad}}\left(\lambda^2\right)+\left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right]\\&+\frac{i}{\left(4\pi\right)^2}\left[m_1^2-\lambda^2+m_1^2\ln\left(\frac{\lambda^2}{m_1^2}\right)\right].\end{aligned}\tag{86}$$

We get then

$$\begin{aligned}&I_{\text{quad}}\left(m_1^2\right)-I_{\text{quad}}\left(m_2^2\right)=m_1^2\left[I_{\log}\left(\lambda^2\right)-\frac{i}{\left(4\pi\right)^2}\ln\left(\frac{m_1^2}{\lambda^2}\right)\right]\\&-m_2^2\left[I_{\log}\left(\lambda^2\right)-\frac{i}{\left(4\pi\right)^2}\ln\left(\frac{m_2^2}{\lambda^2}\right)\right]\\&+\frac{i}{\left(4\pi\right)^2}\left\{m_1^2-m_1^2\ln\left(\frac{m_1^2}{\lambda^2}\right)-m_2^2+m_2^2\ln\left(\frac{m_2^2}{\lambda^2}\right)\right\}.\end{aligned}$$

Now note that we can relate the reduction of the finite functions to the scale properties of the divergent objects

$$\begin{aligned}&2p^2\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\&-\left(p^2+m_1^2-m_2^2\right)\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\\&=\left[I_{\text{quad}}\left(m_1^2\right)\right]-\left[I_{\text{quad}}\left(m_2^2\right)\right]\\&-m_1^2\left[I_{\log}\left(\lambda^2\right)-\frac{i}{\left(4\pi\right)^2}\ln\left(\frac{m_1^2}{\lambda^2}\right)\right]\\&+m_2^2\left[I_{\log}\left(\lambda^2\right)-\frac{i}{\left(4\pi\right)^2}\ln\left(\frac{m_2^2}{\lambda^2}\right)\right],\end{aligned}$$

Substituting in the expression for VS amplitude we will identify the relation (79) among the Green functions VS, SS, and, S. Note that the precise connection between the finite functions and the basic divergent object allows us to verify in an exact way the considered relation among Green functions. It is not necessary to emphasize that the same procedure is nontrivial within the context of traditional regularization methods.

Let us now consider the relations among the three-point functions calculated in the previous section. Contracting the VVV amplitude, calculated in last section, with q^{λ} and using the properties (23), (24), (25) and (29) in order to eliminate the ξ_{nm} functions having $n+m=3$ in favor of those having $n+m=2$ we get

$$
\begin{aligned}
q^{\lambda}T^{VVV}_{\lambda\mu\nu} =& \left[A_{\mu\nu}\left(k_1,k_2\right)\right]-\left[A_{\mu\nu}\left(k_2,k_3\right)\right]+\frac{4}{3}\left(g_{\mu\nu}p^2-p_\mu p_\nu\right)\left[I_{\log}\left(\lambda^2\right)\right]-\frac{4}{3}\left[g_{\mu\nu}\left(p-q\right)^2-\left(p-q\right)_\mu\left(p-q\right)_\nu\right]\left[I_{\log}\left(\lambda^2\right)\right] \\
&-2g_{\mu\nu}\left[\left(m_1-m_2\right)^2-\left(m_2-m_3\right)^2\right]\left[I_{\log}\left(\lambda^2\right)\right]+\frac{i}{\left(4\pi\right)^2}4g_{\mu\nu}\Big\{2p^2\left[Z_2\left(m_1^2;p^2,m_2^2\right)-Z_1\left(m_1^2;p^2,m_2^2\right)\right] \\
&+\left(m_2^2-m_1^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2\right)\right]+\left(m_1-m_2\right)m_1\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] \\
&-2\left(p-q\right)^2\left[Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2\right)-Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2\right)\right]-\left(m_3^2-m_2^2\right)\left[Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2\right)\right] \\
&-\left(m_2-m_3\right)m_2\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]\Big\}+\frac{i}{\left(4\pi\right)^2}8p_\mu p_\nu\Big\{Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right) \\
&-2Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)-Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)+Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+2\left[q^2\xi_{11}+\left(p\cdot q\right)\xi_{20}\right]-q^2\left[\xi_{10}\right]\Big\} \\
&+\frac{i}{\left(4\pi\right)^2}8q_\mu q_\nu\Big\{Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)-\frac{1}{2}Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)+\left[q^2\xi_{02}+\left(p\cdot q\right)\xi_{11}\right] \\
&-\frac{1}{2}\left[q^2-\left(m_3-m_1\right)^2\right]\left[\xi_{01}\right]-\frac{1}{2}\left[\eta_{00}\right]\Big\}+\frac{i}{\left(4\pi\right)^2}8q_\mu p_\nu\Big\{-\left[Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right] \\
&+\frac{1}{4}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)-Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]+\frac{1}{2}\left[Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]+\left[q^2\xi_{02}+\left(p\cdot q\right)\xi_{11}\right] \\
&-\frac{1}{2}\left[\left(p\cdot q\right)-\left(m_3-m_1\right)\left(m_2-m_1\right)\right]\left[\xi_{10}\right]+\frac{1}{4}\left[q^2-\left(m_3-m_1\right)^2\right]\left[\xi_{00}\right]-\frac{1}{2}\left[\eta_{00}\right]-q^2\left[\xi_{01}\right]\Big\} \\
&+\frac{i}{\left(4\pi\right)^2}8p_\mu q_\nu\Big\{-\left[Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right] \\
&+\frac{1}{4}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)+Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]-\frac{1}{2}\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]+\left[q^2\xi_{02}+\left(p\cdot q\right)\xi_{11}\right] \\
&+\left[q^2\xi_{11}+\left(p\cdot q\right)\xi_{20}\right]-\left[q^2\xi_{01}+\left(p\cdot q\right)\xi_{10}\right]-\frac{1}{2}\left[q^2-\left(p\cdot q\right)-\left(m_3-m_1\right)^2+\left(m_3-m_1\right)\left(m_2-m_1\right)\right]\left[\xi_{10}\right] \\
&-\frac{1}{2}\left[\eta_{00}\right]+\frac{1}{4}\left[q^2-\left(m_3-m_1\right)^2\right]\left[\xi_{00}\right]\Big\}+\left(m_3-m_1\right)\Big\{2\left(m_1+m_3\right)\Big\{\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]\Big\} \\
&+\frac{i}{\left(4\pi\right)^2}\Big\{-4g_{\mu\nu}\left(m_1+m_3\right)\left[\eta_{00}\right]+8\left(m_1+m_3\right)p_\mu p_\nu\left[\xi_{20}\right]+8\left(m_1+m_3\right)q_\mu q_\nu\left[\xi_{02}\right] \\
&+8\left(m_1+m_3\right)q_\mu p_\nu\left[\xi_{11}\right]+8\left(m_1+m_3\right)p_\mu q_\nu\left[\xi_{11}\right]\Big\}+\left[g_{\mu\nu}T^{SPP}\right]\Big\}.
\end{aligned}
$$

Given the obtained result, we now use the properties (19) and (20) to eliminate the ξ_{nm} functions having $n+m=2$ in favor of those having $n+m=1$. We get then

$$
\begin{aligned}
q_{\lambda}T^{VVV}_{\lambda\mu\nu} =& \left[A_{\mu\nu}\left(k_1,k_2\right)\right]-\left[A_{\mu\nu}\left(k_2,k_3\right)\right]+\frac{4}{3}\left(g_{\mu\nu}p^2-p_\mu p_\nu\right)\left[I_{\log}\left(\lambda^2\right)\right]-\frac{4}{3}\left[g_{\mu\nu}\left(p-q\right)^2-\left(p-q\right)_\mu\left(p-q\right)_\nu\right]\left[I_{\log}\left(\lambda^2\right)\right] \\
&-2g_{\mu\nu}\left[\left(m_1-m_2\right)^2-\left(m_2-m_3\right)^2\right]\left[I_{\log}\left(\lambda^2\right)\right]+\frac{i}{\left(4\pi\right)^2}4g_{\mu\nu}\Big\{2p^2\left[Z_2\left(m_1^2;p^2,m_2^2\right)-Z_1\left(m_1^2;p^2,m_2^2\right)\right] \\
&+\left(m_2^2-m_1^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2\right)\right]-\left(m_2-m_1\right)m_1\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] \\
&-2\left(p-q\right)^2\left[Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2\right)-Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2\right)\right] \\
&-\left(m_3^2-m_2^2\right)\left[Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2\right)\right]+\left(m_3-m_2\right)m_2\left[Z_0\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]\Big\} \\
&+\frac{i}{\left(4\pi\right)^2}8p_\mu p_\nu\Big\{\left[Z_2\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;\left(p-q\right)^2,m_3^2;\lambda^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\left[Z_2\left(m_1^2;p^2,m_2^2;\lambda^2\right)-Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\Big\}\\
&+\frac{i}{(4\pi)^2}8q_\mu q_\nu\left\{\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\right\}\\
&+\frac{i}{(4\pi)^2}8q_\mu p_\nu\left\{-\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\right.\\
&\left.+\frac{1}{4}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)-Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{1}{2}\left[q^2\xi_{01}+(p\cdot q)\xi_{10}\right]+\frac{1}{4}\left[q^2-(m_3-m_1)^2\right][\xi_{00}]\right\}\\
&+\frac{i}{(4\pi)^2}8p_\mu q_\nu\left\{-\left[Z_2\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\right.\\
&\left.+\frac{1}{4}\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)-Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]-\frac{1}{2}\left[q^2\xi_{01}+(p\cdot q)\xi_{10}\right]+\frac{1}{4}\left[q^2-(m_3-m_1)^2\right][\xi_{00}]\right\}\\
&+(m_3-m_1)\Big\{2(m_1+m_3)\Big\{\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]\Big\}+\frac{i}{(4\pi)^2}\Big\{-4g_{\mu\nu}(m_1+m_3)[\eta_{00}]\\
&+8p_\mu p_\nu(m_1+m_3)(\xi_{20}-\xi_{10})+8q_\mu q_\nu\left[(m_1+m_3)\xi_{02}-m_1\xi_{01}\right]+4q_\mu p_\nu\left[(m_1+m_3)(2\xi_{11}-\xi_{01})+(m_2-m_1)\xi_{10}\right]\\
&+4p_\mu q_\nu\left[(m_1+m_3)(2\xi_{11}-\xi_{01})-(m_2+m_1)\xi_{10}\right]\Big\}+\left[g_{\mu\nu}T^{SPP}\right]\Big\}.
\end{aligned}
$$

Finally, using relation (17) we write

$$
\begin{aligned}
q^\lambda T^{VVV}_{\lambda\mu\nu}&=-2g_{\mu\nu}(m_1-m_2)^2\left[I_{\log}\left(\lambda^2\right)\right]+\frac{4}{3}\left(g_{\mu\nu}p^2-p_\mu p_\nu\right)\left[I_{\log}\left(\lambda^2\right)\right]\\
&+\frac{i}{(4\pi)^2}\Big\{8\left(g_{\mu\nu}p^2-p_\mu p_\nu\right)\left[Z_2\left(m_1^2;p^2,m_2^2\right)-Z_1\left(m_1^2;p^2,m_2^2\right)\right]+4g_{\mu\nu}\left(m_2^2-m_1^2\right)\left[Z_1\left(m_1^2;p^2,m_2^2\right)\right]\\
&-4g_{\mu\nu}(m_2-m_1)m_1\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\Big\}+\left[A_{\mu\nu}(k_1,k_2)\right]+2g_{\mu\nu}(m_2-m_3)^2\left[I_{\log}\left(\lambda^2\right)\right]\\
&-\frac{4}{3}\left[g_{\mu\nu}(p-q)^2-(p-q)_\mu(p-q)_\nu\right]\left[I_{\log}\left(\lambda^2\right)\right]\\
&+\frac{i}{(4\pi)^2}\Big\{-8\left[g_{\mu\nu}(p-q)^2-(p-q)_\mu(p-q)_\nu\right]\left[Z_2\left(m_2^2;(p-q)^2,m_3^2\right)-Z_1\left(m_2^2;(p-q)^2,m_3^2\right)\right]\\
&-4g_{\mu\nu}\left(m_3^2-m_2^2\right)\left[Z_1\left(m_2^2;(p-q)^2,m_3^2\right)\right]+4g_{\mu\nu}(m_3-m_2)m_2\left[Z_0\left(m_2^2;(p-q)^2,m_3^2;\lambda^2\right)\right]\Big\}\\
&-\left[A_{\mu\nu}(k_2,k_3)\right]+(m_3-m_1)\Big\{2(m_1+m_3)\Big\{\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]+g_{\mu\nu}\left[I_{\log}\left(\lambda^2\right)\right]\Big\}\\
&+\frac{i}{(4\pi)^2}\Big\{-4g_{\mu\nu}(m_1+m_3)[\eta_{00}]+8p_\mu p_\nu(m_1+m_3)(\xi_{20}-\xi_{10})+8q_\mu q_\nu\left[(m_1+m_3)\xi_{02}-m_1\xi_{01}\right]\\
&+4q_\mu p_\nu\left[(m_1+m_3)(2\xi_{11}-\xi_{01})+(m_2-m_1)\xi_{10}+m_1\xi_{00}\right]\\
&+4p_\mu q_\nu\left[(m_1+m_3)(2\xi_{11}-\xi_{01})-(m_2+m_1)\xi_{10}+m_1\xi_{00}\right]\Big\}+\left[g_{\mu\nu}T^{SPP}\right]\Big\},
\end{aligned}
$$

If we consider the results for the amplitudes VV and SVV, Equations (69) and (71), it is now easy to note that the expression above may be identified as being the relation (81). It is not difficult to verify the relations (82) and (83) by performing the same sequence of steps.

The procedure used above can also be adopted to state analogous constraints to the four-point Green function. As an example of such constraint we have

$$
\begin{aligned}
&(k_4-k_1)^\mu\left[T^{VVVV}_{\mu\nu\alpha\beta}\right]\\
&=\left[T^{VVV}_{\nu\alpha\beta}(k_1,k_2,k_3)\right]-\left[T^{VVV}_{\nu\alpha\beta}(k_2,k_3,k_4)\right]\\
&\quad+(m_4-m_1)\left[T^{SVVV}_{\nu\alpha\beta}\right].
\end{aligned}
$$

In order to show that the calculated four-point amplitude *VVVV* satisfies this relation, at first we contract Equation (73) with $(k_4-k_1)^\mu$ and eliminate the η_{ijk}

having $i + j + k = 1$ in favor of those having $i + j + k = 0$. The next step is to use the properties (34)-(43) in order to eliminate the ζ_{ijk} and ξ_{ijk} functions having $i + j + k = 4$ in favor of those having $i + j + k = 3$ and so on. The calculation is easy but involves a lot of algebra, therefore we will not show it explicitly. All the required ingredients have been given in the preceding sections.

8. Ambiguities and Symmetry relations

In the Section 6 we have evaluated, within the systematization proposed, Green's functions which are typical of the perturbative calculations. In particular, all the considered amplitudes appear in the context of Standard Model. In all the evaluated Green's functions, having degree of divergences higher than the logarithmic one, it is possible to note the presence of terms where the dependence on the internal momenta appear as arbitrary quantities (the summations of them). This is expected since a shift in the integrating momentum generates surfaces terms which implies that different choices for the label of the internal lines momenta lead to different amplitudes. This possible dependence on the choices for the labels of the internal lines momenta characterizes what we denominate as ambiguities. This situation is not acceptable just because, in this case, the power of predicttion of the theory is destroyed. In addition, fundamental space-time symmetries like the space-time homogeneity are not preserved in the perturbative calculations. It will not be surprising to find global and local gauge symmetries as well as internal symmetries violated in physical amplitudes having the space-time homogeneity broken. There is only one possibility to save such type of calculations: to eliminate the ambiguous terms in a consistent and universal way. Within the context of the adopted strategy the ambiguous terms are automatically separated and preserved so that it is easy to identify them.

In the case of one-point function it is simple to identify

$$\left[T^{S}\right]_{\text{ambiguous}} = 4m_1 k_1^{\alpha} k_1^{\beta}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right],$$

$$\begin{aligned}\left[T_{\mu}^{V}\right]_{\text{ambiguous}} &= -4\left(k_1\right)_{\xi}\left[\nabla_{\xi\mu}\left(\lambda^2\right)\right] \\ &+2\left(k_1^2+2\lambda^2-2m_1^2\right)\left(k_1\right)_{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right] \\ &-\frac{4}{3}\left(k_1\right)_{\xi}\left(k_1\right)_{\chi}\left(k_1\right)_{\tau}\left[\Box_{\xi\chi\tau\mu}\left(\lambda^2\right)\right] \\ &+2\left(k_1\right)_{\mu}\left(k_1\right)_{\xi}\left(k_1\right)_{\chi}\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right].\end{aligned}$$

In the two-point functions we get

$$\left[T^{SS}\right]_{\text{ambiguous}} = P_{\alpha}P_{\beta}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right],$$

$$\left[T_{\mu}^{SV}\right]_{\text{ambiguous}} = -2\left(m_2+m_1\right)P^{\alpha}\left[\Delta_{\alpha\mu}\left(\lambda^2\right)\right],$$

$$\left[T^{PP}\right]_{\text{ambiguous}} = -P^{\alpha}P^{\beta}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right],$$

$$\begin{aligned}\left[T_{\mu\nu}^{VV}\right]_{\text{ambiguous}} &= -\frac{1}{2}P^2\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] \\ &+\frac{1}{3}\left[3P^{\xi}P^{\chi}-p^{\xi}P^{\chi}+p^{\chi}P^{\xi}\right]\left[\Box_{\xi\chi\mu\nu}\left(\lambda^2\right)\right] \\ &-P_{\mu}P^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]-P_{\nu}P^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right] \\ &+\frac{1}{6}g_{\mu\nu}\left[-3P^{\xi}P^{\chi}-p^{\xi}P^{\chi}+p^{\chi}P^{\xi}\right]\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right].\end{aligned}$$

In the case of three-point functions we found

$$\left[T_{\lambda}^{VSS}\right]_{\text{ambiguous}} = -2\left(k_3+k_1\right)^{\xi}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right],$$

$$\left[T_{\lambda}^{VPP}\right]_{\text{ambiguous}} = 2\left(k_3+k_1\right)^{\xi}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right],$$

$$\left[T_{\mu}^{PVP}\right]_{\text{ambiguous}} = 2\left(k_2+k_1\right)^{\xi}\left[\Delta_{\mu\xi}\left(\lambda^2\right)\right],$$

$$\left[T_{\nu}^{PPV}\right]_{\text{ambiguous}} = 2\left(k_3+k_2\right)^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right],$$

$$\begin{aligned}\left[T_{\lambda\mu\nu}^{VVV}\right]_{\text{amb}} &= -\frac{4}{3}\left(k_1+k_2+k_3\right)^{\xi}\left[\Box_{\xi\mu\nu\lambda}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}g_{\mu\lambda}\left(2k_2+2k_3-k_1\right)^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}g_{\lambda\nu}\left(2k_1+2k_2-k_3\right)^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}g_{\mu\nu}\left(2k_1+2k_3-k_2\right)^{\xi}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}\left(2k_1+2k_2-k_3\right)_{\mu}\left[\Delta_{\lambda\nu}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}\left(2k_1-k_2+2k_3\right)_{\lambda}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right] \\ &+\frac{2}{3}\left(-k_1+2k_2+2k_3\right)_{\nu}\left[\Delta_{\mu\lambda}\left(\lambda^2\right)\right].\end{aligned}$$

In all the above listed ambiguous terms it can be noted that they invariably appear as multiplying the objects ∇, Δ and, $\Box$. All these terms present simultaneously scale ambiguities because such objects are dependent on the arbitrary mass scale λ. This is due to the fact that in all amplitudes the obtained expression is independent of the parameter λ^2 if the terms containing the objects ∇, Δ and, $\Box$ are absent. This statement can be verified directly by differentiating the expression or changing the scale to another one, like for example one of the involved fermionic masses, through the scale properties of the finite function and of the basic divergent objects $I_{\log}\left(\lambda^2\right)$ and $I_{\text{quad}}\left(\lambda^2\right)$. The referred properties are

$$I_{\text{quad}}\left(m_1^2\right) = I_{\text{quad}}\left(\lambda^2\right)+\left(m_1^2-\lambda^2\right)\left[I_{\log}\left(\lambda^2\right)\right] + \frac{i}{(4\pi)^2}\left[m_1^2-\lambda^2+m_1^2\ln\left(\frac{\lambda^2}{m_1^2}\right)\right],$$

$$I_{\log}\left(m_1^2\right)=I_{\log}\left(\lambda^2\right)-\frac{i}{(4\pi)^2}\ln\left(\frac{m_1^2}{\lambda^2}\right),$$

$$Z_k\left(m_1^2;p^2,m_2^2;\lambda_2^2\right)=\left[Z_k\left(m_1^2;p^2,m_2^2;\lambda_1^2\right)\right]+\frac{1}{k+1}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right),$$

$$\eta_{nm}\left(m_1^2;p,m_2^2;q,m_3^2;\lambda_2^2\right)=\eta_{nm}\left(m_1^2;p,m_2^2;q,m_3^2;\lambda_1^2\right)+\frac{1}{(n+1)(m+1)}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right).$$

This means that there are terms in the expressions for the perturbative amplitudes which are nonambiguous relative to the choice for the internal lines momenta, but are ambiguous relative to the choice for the common scale for the finite and divergent parts. This aspect can be easily noted in the considered amplitudes. In the VV two-point function

$$\begin{aligned}&\left[T_{\mu\nu}^{VV}\right]_{\text{scale\_amb}}\\&=4\left[\nabla_{\mu\nu}\left(\lambda^2\right)\right]-2\left(2\lambda^2-m_1^2-m_2^2\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]\\&-\frac{5}{6}p^2\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]-\frac{5}{6}g_{\mu\nu}p^{\xi}p^{\chi}\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right]\\&+\frac{1}{3}p_{\mu}p^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]+\frac{1}{3}p_{\nu}p^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\\&+\frac{1}{3}p^{\xi}p^{\chi}\left[\Box_{\xi\chi\mu\nu}\left(\lambda^2\right)\right].\end{aligned}$$

In the SVV amplitude

$$\left[T_{\mu\nu}^{SVV}\right]_{\text{scale\_amb}}=2\left(m_1+m_3\right)\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right],$$

and in the $VVVV$ amplitude

$$\begin{aligned}&\left[T_{\alpha\beta\mu\nu}^{VVVV}\right]_{\text{scale\_amb}}\\&=\frac{4}{3}\left[\Box_{\mu\nu\alpha\beta}\left(\lambda^2\right)\right]+\frac{2}{3}g_{\mu\alpha}\left[\Delta_{\beta\nu}\left(\lambda^2\right)\right]\\&-\frac{4}{3}g_{\mu\beta}\left[\Delta_{\nu\alpha}\left(\lambda^2\right)\right]-\frac{4}{3}g_{\mu\nu}\left[\Delta_{\alpha\beta}\left(\lambda^2\right)\right]\\&+\frac{2}{3}g_{\nu\beta}\left[\Delta_{\mu\alpha}\left(\lambda^2\right)\right]-\frac{4}{3}g_{\alpha\beta}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]\\&-\frac{4}{3}g_{\nu\alpha}\left[\Delta_{\mu\beta}\left(\lambda^2\right)\right].\end{aligned}$$

In such examples the listed terms are independent of the choices for the internal momenta. They can be converted in ambiguities through their evaluation in intermediary steps within the context of traditional regularization techniques. Again we can note that all the potentially scale ambiguous terms are combinations of the objects ∇, Δ and, $\Box$.

Let us now consider the symmetry relations. It is easy to see that the situation is completely similar to the question of ambiguities considered above. There are two types of impositions coming from the symmetries for the amplitudes. The general ones, coming from Lorentz and CPT, present in the Furry's theorem, whose implications states that all amplitude which has an odd number of external vectors and only one species of fermion at the internal lines must vanish identically, and that coming from the divergence of the fermionic vector current which states a precise relation with the corresponding scalar current. The first of the impositions mentioned above implies that the amplitude $T_{\nu}^{V}\left(k_1\right)$ must be identically zero, which means that it is required

$$\begin{aligned}0=&-4\left(k_1\right)_{\xi}\left[\nabla_{\xi\mu}\left(\lambda^2\right)\right]\\&+2\left(k_1^2+2\lambda^2-2m_1^2\right)\left(k_1\right)_{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\\&-\frac{4}{3}\left(k_1\right)_{\xi}\left(k_1\right)_{\chi}\left(k_1\right)_{\tau}\left[\Box_{\xi\chi\tau\mu}\left(\lambda^2\right)\right]\\&+2\left(k_1\right)_{\mu}\left(k_1\right)_{\xi}\left(k_1\right)_{\chi}\left[\Delta_{\xi\chi}\left(\lambda^2\right)\right].\end{aligned}$$

Due to the same reasons, the theorem states that the amplitude for the process $V\to VV$, which is the VVV amplitude symmetrized in the final state,

$$T_{\lambda\mu\nu}^{V\to VV}=T_{\lambda\mu\nu}^{VVV}\left(k_1,k_2,k_3\right)+T_{\lambda\nu\mu}^{VVV}\left(l_1,l_2,l_3\right),$$

must vanish for the case of equal masses. The arbitrary internal momenta for the second channel obey, $q=l_2-l_1$ and $p=l_3-l_1$. This means that it is required

$$\begin{aligned}0=&-\frac{4}{3}\left(k_1+k_2+k_3\right)^{\xi}\left[\Box_{\xi\mu\nu\lambda}\left(\lambda^2\right)\right]\\&+\frac{2}{3}g_{\mu\lambda}\left(2k_2+2k_3-k_1\right)^{\xi}\left[\Delta_{\xi\nu}\left(\lambda^2\right)\right]\\&+\frac{2}{3}g_{\lambda\nu}\left(2k_1+2k_2-k_3\right)^{\xi}\left[\Delta_{\xi\mu}\left(\lambda^2\right)\right]\\&+\frac{2}{3}g_{\mu\nu}\left(2k_1+2k_3-k_2\right)^{\xi}\left[\Delta_{\xi\lambda}\left(\lambda^2\right)\right]\\&+\frac{2}{3}\left(2k_1+2k_2-k_3\right)_{\mu}\left[\Delta_{\lambda\nu}\left(\lambda^2\right)\right]\\&+\frac{2}{3}\left(2k_1-k_2+2k_3\right)_{\lambda}\left[\Delta_{\mu\nu}\left(\lambda^2\right)\right]\\&+\frac{2}{3}\left(-k_1+2k_2+2k_3\right)_{\nu}\left[\Delta_{\mu\lambda}\left(\lambda^2\right)\right]\\&+\left\{k_i\to l_i,\mu\to\nu\right\}.\end{aligned}$$

Concerning the symmetry relations coming from the proportionality of the divergence of the fermionic vector current with the scalar current, we note that in the VV two-point function we get

$$\begin{aligned} p^{\nu}T_{\mu\nu}^{VV}(k_1,k_2) &= (m_2-m_1)\Big\{-2(m_2-m_1)\,p_\mu\left[I_{\log}(\lambda^2)\right] \\ &\quad +4(m_2+m_1)\,p_\mu\left[Z_1\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right] \\ &\quad -4m_1p_\mu\left[Z_0\left(m_1^2;p^2,m_2^2;\lambda^2\right)\right]\Big\} \\ &\quad +p^{\nu}\left[A_{\mu\nu}(k_1,k_2)\right]. \end{aligned}$$

By comparing to the result (67) for the VS amplitude we can identify

$$\begin{aligned} p^{\nu}T_{\mu\nu}^{VV}(k_1,k_2) &= p^{\nu}\left[A_{\mu\nu}(k_1,k_2)\right] \\ &\quad +2\left(m_2^2-m_1^2\right)(k_2+k_1)^{\xi}\left[\Delta_{\xi\mu}(\lambda^2)\right] \\ &\quad +(m_2-m_1)\left[T_{\nu}^{VS}(k_1,k_2)\right], \end{aligned}$$

which means that the symmetry relation is broken by the terms which are all combination of the objects ∇, Δ and, $\Box$. In fact this result requires the same as the Fur-ry's theorem, a vanishing value for the vector one point function. Following this line of reasoning we note that the SVV amplitude possesses a symmetry violating term which is independent of the choice for the internal lines momenta

$$\left[p^{\mu}T_{\mu\nu}^{S\to VV}\right]_{\text{sym\_break}} = 2(m_1+m_3)(p+q)^{\mu}\left[\Delta_{\mu\nu}(\lambda^2)\right].$$

The same occurs for the $VV \to VV$ process where the violating term is proporcional to $\Box_{\tau\mu\nu\lambda}(\lambda^2)$ with a nonanbiguous coefficient.

In view of the above comments and others omitted, it is very simple to conclude that all these unwanted problems can be removed from the amplitudes in a consistent way. There are simple but powerful arguments. If we consider that a perturbative solution for the amplitudes of a QFT must be compatible with the space-time homogeneity or it does not make any sense, if we cannot admit that the scale independence can be broken by any method or strategy adopted to give some meaning for the perturbative amplitudes and if we also cannot admit that an acceptable interpretation for the perturbative solution breaks symmetry relations of the underlying theory, then it becomes necessary to impose a set of properties for the divergent Feynman integrals in order to recover these symmetries, due to the fact that the perturbative series is not automatically translational and scale invariant and symmetry preserving. Fortunately all these problems can be solved simultaneously. It is enough to impose that

$$\nabla = \Delta = \Box = 0.$$

We can look at these conditions as a set of properties required to a regularization method in order to produce consistent results or we can think that this is the set of properties required to the perturbative series in order to get the space-time homogeneity maintained in the calculated expressions (among others). Due to these reasons we denominated them as Consistency Relations. Such conditions can be easily understood. In fact the definition of the objects ∇, Δ and, $\Box$ has been conveniently made in order to get clean and sound clarifications. First note that

$$\frac{\partial}{\partial k_\nu}\left\{\frac{k_\mu}{\left(k^2-\lambda^2\right)}\right\} = \frac{g_{\mu\nu}}{\left(k^2-\lambda^2\right)} - \frac{2k_\nu k_\mu}{\left(k^2-\lambda^2\right)^2}, \tag{87}$$

$$\frac{\partial}{\partial k_\nu}\left\{\frac{k_\mu}{\left(k^2-\lambda^2\right)^2}\right\} = \frac{g_{\mu\nu}}{\left(k^2-\lambda^2\right)^2} - \frac{4k_\nu k_\mu}{\left(k^2-\lambda^2\right)^3}, \tag{88}$$

$$\begin{aligned} \frac{\partial}{\partial k_\alpha}\left\{\frac{k_\beta k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}\right\} &= g_{\alpha\beta}\frac{k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3} + g_{\alpha\nu}\frac{k_\beta k_\mu}{\left(k^2-\lambda^2\right)^3} \\ &\quad + g_{\alpha\mu}\frac{k_\beta k_\nu}{\left(k^2-\lambda^2\right)^3} - \frac{6k_\mu k_\nu k_\alpha k_\beta}{\left(k^2-\lambda^2\right)^4}, \end{aligned} \tag{89}$$

so that we can identify

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\partial}{\partial k_\nu}\left\{\frac{k_\mu}{\left(k^2-\lambda^2\right)}\right\} = -\nabla_{\mu\nu}, \tag{90}$$

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\partial}{\partial k_\nu}\left\{\frac{k_\mu}{\left(k^2-\lambda^2\right)^2}\right\} = -\Delta_{\mu\nu}, \tag{91}$$

$$\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\partial}{\partial k_\alpha}\left\{\frac{k_\beta k_\mu k_\nu}{\left(k^2-\lambda^2\right)^3}\right\} = -\frac{1}{4}\Box_{\alpha\beta\mu\nu}. \tag{92}$$

The factor 4 in the last condition is justified by the symmetrization in the Lorentz indexes. In order to give symmetrical role to all indexes four terms need to be introduced in the left hand side given the factor 24 to the fourlinear in loop momentum integral. Frequently it is convenient to write such integral in symmetrized form. We adopted the definition of the object $\Box$ in a non-simmetrized way only to reduce the mathematical expressions. Note that through the Gauss theorem these quantities are identified as surfaces terms. It becomes clear now that if these conditions are not imposed the perturbative calculations simply does not make any sense. It is on the other hand simple to verify that these conditions are satisfied in the presence of any distribution. Without these conditions being fulfilled space-time, local and gauge symmetries are violated as well as the amplitudes may be ambiguous quantities. The prescription is universal since in other dimensions as well as in theories or models where higher degree of divergences are present analogous conditions can be identified. This interpretation of the perturbative calculations provides us the required consistency. The calculated amplitudes are ambiguities free and symmetry preserving.

If one agrees with the arguments put above then the adoption of a regularization become completely unnecessary for any purposes in the perturbative calculations. All the required manipulations and calculations, including the renormalization, can be performed, following our strategy, without any mention to the word regularization. And, which is better, the results are so consistent as desirable and no restrictions of applicability exist.

9. Generalizations of the Finite Functions and Their Relationship

Through the proposed method to manipulate and calculate divergent integrals, in the above sections we have been learning how to systematize the finite parts of the one, two, three, and four-point integrals which are present in the relevant amplitudes belonging to fundamental theories. It is not hard to see that this systematization could be generalized to amplitudes with an arbitrary number of points. In this section we discuss some aspects of this generalization. We begin by defining the set of functions

$$\begin{aligned}&\xi^{(n)}_{i_1,\cdots,i_k}\left(m_1;p_1,m_2;\cdots;p_k,m_{k+1};\lambda^2\right)\\&=\int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_k\; x_1^{i_1}\cdots x_k^{i_k}\frac{[Q]^n}{n!}\left\{\ln\frac{Q}{\left(-\lambda^2\right)}-\psi\left(n+1\right)-\gamma\right\},\\&\text{if}\quad n=0,1,2,\cdots\end{aligned}\tag{93}$$

$$\begin{aligned}&\xi^{(n)}_{i_1,\cdots,i_k}\left(m_1;p_1,m_2;\cdots;p_k,m_{k+1}\right)\\&=\int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_k\; x_1^{i_1}\cdots x_k^{i_k}\left[Q\right]^n\quad\text{if}\quad n=-1,-2,-3,\cdots\end{aligned}\tag{94}$$

where $k=1,2,3,\cdots$, and

$$\psi\left(n+1\right)+\gamma=\sum_{l=1}^{n}\frac{1}{l},$$

with γ being the Euler-Mascheroni constant. The following shorthand notation has also been used

$$\int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_k\equiv\int_0^1\mathrm{d}x_1\int_0^{1-x_1}\mathrm{d}x_2\cdots\int_0^{1-\Sigma}\mathrm{d}x_k,\ \text{with}\ \Sigma=\sum_{i=1}^{k-1}x_i,$$

$$\begin{aligned}Q&\equiv Q\left(m_1;p_1,m_2,x_1;\cdots;p_k,m_{k+1},x_k\right)\\&=\sum_{i,j=1}^{k}\left(p_i\cdot p_j\right)x_i\left(\delta_{ij}-x_j\right)\\&\quad+\sum_{i=1}^{k}\left(m_1^2-m_{i+1}^2\right)x_i-m_1^2.\end{aligned}$$

with δ representing a Kronecker delta symbol. All finite parts of the one-loop Feynman integrals with an arbitrary number of points, handled by the proposed approach, can be systematized through this set of functions. We recognize that Equation (93) is the generalization of definitions (12), (16) and (33) and Equation (94) is the generalization of Equations (15), (32) and (31).

In the preceding sections we have systematically evaluated the one, two, three, and four-point vector amplitudes and verified their Ward identities. Within our approach, the verification of the Ward identities is greatly simplified by using a set of identities characteristic of $\xi^{(n)}_{i_1,\cdots,i_k}$, like those given by Equations (17)-(29). In order to obtain such identities for an arbitrary number of points first we note that

$$\begin{aligned}&p_1^2\left[\xi^{(n)}_{i_1+1,i_2,\cdots,i_k}\right]+\left(p_1\cdot p_2\right)\left[\xi^{(n)}_{i_1,i_2+1,i_3,\cdots,i_k}\right]+\cdots+\left(p_1\cdot p_k\right)\left[\xi^{(n)}_{i_1,i_2,\cdots,i_{k-1},i_{k+1}}\right]\\&=-\frac{1}{2}\int_0^1\mathrm{d}x_1\cdots\mathrm{d}x_k\; x_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}\left(\frac{\partial Q}{\partial x_1}\right)\frac{[Q]^n}{n!}\left\{\ln\frac{Q}{-\lambda^2}-\psi\left(n+1\right)-\gamma\right\}+\frac{\left(p_1^2+m_1^2-m_2^2\right)}{2}\left[\xi^{(n)}_{i_1,i_2,\cdots,i_k}\right],\end{aligned}$$

where $n\geq 0$ and $i_1,i_2,\cdots,i_k=0,1,2,\cdots$. After an integration by parts, the first term on the right hand side of the above equation may be rewritten as

$$\int_0^1\mathrm{d}x_1\cdots\mathrm{d}x_k\frac{\partial}{\partial x_1}\left\{x_1^{i_1}\cdots x_k^{i_k}\frac{Q^{n+1}}{(n+1)!}\left[\ln\frac{Q}{-\lambda^2}-\psi\left(n+2\right)-\gamma\right]\right\}-i_1\int_0^1\mathrm{d}x_1\cdots\mathrm{d}x_k\; x_1^{i_1-1}\cdots x_k^{i_k}\frac{Q^{n+1}}{(n+1)!}\left[\ln\frac{Q}{-\lambda^2}-\psi\left(n+2\right)-\gamma\right].$$

The first term is a total derivative in x_1. So, performing the integral over x_1, we write the above expression as

$$\begin{aligned}&\delta_{k,1}\delta_{i_1,0}\left\{\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right)-\psi\left(n+2\right)-\gamma\right]\right\}-\frac{\left(-m_1^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_1^2}{\lambda^2}\right)-\psi\left(n+2\right)-\gamma\right]\\&+\delta_{k,1}\left(1-\delta_{i_1,0}\right)\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right)-\psi\left(n+2\right)-\gamma\right]+\delta_{i_1,0}\left(1-\delta_{k,1}\right)\left[\xi''^{(n+1)}_{i_2,i_3,\cdots,i_k}-\xi'^{(n+1)}_{i_2,i_3,\cdots,i_k}\right]-i_1\\&+\left(1-\delta_{k,1}\right)\left(1-\delta_{i_1,0}\right)\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{k-1}=0}^{l_{k-2}}\mathcal{C}^{i_1}_{l_1\cdots l_{k-1}}\left[\xi''^{(n+1)}_{i_2+l_1-l_2,i_3+l_2-l_3,\cdots,i_k+l_{k-1}}\right],\end{aligned}$$

where

$$\xi \equiv \xi\left(m_1; p_1, m_2; \cdots; p_k, m_{k+1}; \lambda^2\right),$$

$$\xi' \equiv \xi\left(m_2; p_2, m_3; \cdots; p_k, m_{k+1}; \lambda^2\right),$$

$$\xi'' \equiv \xi\left(m_2; p_1 - p_2, m_3; \cdots; p_1 - p_k, m_{k+1}; \lambda^2\right),$$

$$\mathcal{C}^{i_1}_{l_1 \cdots l_{k-1}} = \frac{(-1)^{l_1} i_1!}{(i_1 - l_1)!(l_1 - l_2)! \cdots (l_{k-2} - l_{k-1})! l_{k-1}!}.$$

Finally, we get a recurrence relation

$$\begin{aligned}
&p_1^2 \left[\xi^{(n)}_{i_1+1,\cdots,i_k}\right] + (p_1 \cdot p_2)\left[\xi^{(n)}_{i_1,i_2+1,i_3,\cdots,i_k}\right] + \cdots + (p_1 \cdot p_k)\left[\xi^{(n)}_{i_1,i_2,\cdots,i_k+1}\right] \\
&= \left(1-\delta_{k,1}\right)\left(1-\delta_{i_1,0}\right)\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{k-1}=0}^{l_{k-2}} \mathcal{C}^{i_1}_{l_1\cdots l_{k-1}}\left[\xi''^{(n+1)}_{i_2+l_1-l_2,i_3+l_2-l_3,\cdots,i_k+l_{k-1}}\right] + \delta_{i_1,0}\left(1-\delta_{k,1}\right)\left[\xi''^{(n+1)}_{i_2,i_3,\cdots,i_k} - \xi'^{(n+1)}_{i_2,i_3,\cdots,i_k}\right] \\
&+ \delta_{k,1}\delta_{i_1,0}\left\{\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\right\} - \frac{\left(-m_1^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_1^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\Bigg\} \\
&+ \delta_{k,1}\left(1-\delta_{i_1,0}\right)\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right] + \frac{1}{2}\left(p_1^2 + m_1^2 - m_2^2\right)\left[\xi^{(n)}_{i_1,\cdots,i_k}\right].
\end{aligned}$$

Extending these relations for functions with arbitrary n is straightforward. The result is very similar

$$\begin{aligned}
&p_1^2 \left[\xi^{(n)}_{i_1+1,\cdots,i_k}\right] + (p_1 \cdot p_2)\left[\xi^{(n)}_{i_1,i_2+1,i_3,\cdots,i_k}\right] + \cdots + (p_1 \cdot p_k)\left[\xi^{(n)}_{i_1,i_2,\cdots,i_k+1}\right] \\
&= \left(1-\delta_{k,1}\right)\frac{\left(1-\delta_{i_1,0}\right)}{\mathcal{N}}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{k-1}=0}^{l_{k-2}} \mathcal{C}^{i_1}_{l_1\cdots l_{k-1}}\left[\xi''^{(n+1)}_{i_2+l_1-l_2,i_3+l_2-l_3,\ldots,i_k+l_{k-1}}\right] + \delta_{i_1,0}\frac{\left(1-\delta_{k,1}\right)}{\mathcal{N}}\left[\xi''^{(n+1)}_{i_2,i_3,\cdots,i_k} - \xi'^{(n+1)}_{i_2,i_3,\cdots,i_k}\right] \\
&+ \delta_{k,1}\frac{\delta_{i_1,0}}{\mathcal{N}}\left\{\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\right\} - \frac{\left(-m_1^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_1^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\Bigg\} \\
&+ \delta_{k,1}\frac{\left(1-\delta_{i_1,0}\right)}{\mathcal{N}}\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right] + \frac{1}{2}\left(p_1^2 + m_1^2 - m_2^2\right)\left[\xi^{(n)}_{i_1,\cdots,i_k}\right],
\end{aligned} \tag{95}$$

with

$$\mathcal{N} = \begin{cases} n+1 & \text{if } n < -1 \\ 1 & \text{if } n \geq -1 \end{cases}$$

The symmetry of $\xi^{(n)}_{i_1,\cdots,i_k}$ functions by interchanging momenta and masses

$$\begin{aligned}
&\xi^{(n)}_{i_1,\cdots,i_k}\left(p_1 \leftrightarrow p_j, m_j \leftrightarrow m_{j+1}\right) \\
&= \xi^{(n)}_{(i_1 \leftrightarrow i_j)}\left(m_1; p_1, m_2; \cdots; p_k, m_{k+1}; \lambda^2\right)
\end{aligned}$$

where $j = 2,3,4,\cdots$

may be used to get more $k-1$ similar identities. If we perform this operation in Equation (95) we get a system of linear equations given by

$$\begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \cdots & p_1 \cdot p_k \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \cdots & p_2 \cdot p_k \\ \vdots & \vdots & \ddots & \vdots \\ p_k \cdot p_1 & p_k \cdot p_2 & \cdots & p_k \cdot p_k \end{pmatrix}\begin{pmatrix} \xi^{(n)}_{i_1+1,i_2,\cdots,i_k} \\ \xi^{(n)}_{i_1,i_2+1,i_3,\cdots,i_k} \\ \vdots \\ \xi^{(n)}_{i_1,i_2,\ldots,i_{k-1},i_k+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{pmatrix},$$

with

$$\begin{aligned}
b_1 &= \left(1-\delta_{k,1}\right)\frac{\left(1-\delta_{i_1,0}\right)}{\mathcal{N}}\sum_{l_1=0}^{i_1}\sum_{l_2=0}^{l_1}\cdots\sum_{l_{k-1}=0}^{l_{k-2}} \mathcal{C}^{i_1}_{l_1\cdots l_{k-1}}\left[\eta''^{(n+1)}_{i_2+l_1-l_2,i_3+l_2-l_3,\cdots i_k+l_{k-1}}\right] + \delta_{i_1,0}\frac{\left(1-\delta_{k,1}\right)}{\mathcal{N}}\left\{\left[\eta''^{(n+1)}_{i_2,i_3,\cdots,i_k}\right] - \left[\eta'^{(n+1)}_{i_2,i_3,\cdots,i_k}\right]\right\} \\
&+ \delta_{k,1}\frac{\delta_{i_1,0}}{\mathcal{N}}\left\{\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\right\} + \delta_{k,1}\frac{\delta_{i_1,0}}{\mathcal{N}}\left\{\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\right\} \\
&- \frac{\left(-m_1^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_1^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right]\Bigg\} + \delta_{k,1}\frac{\left(1-\delta_{i_1,0}\right)}{\mathcal{N}}\frac{\left(-m_2^2\right)^{n+1}}{(n+1)!}\left[\ln\left(\frac{m_2^2}{\lambda^2}\right) - \psi(n+2) - \gamma\right] + \frac{1}{2}\left(p_1^2 + m_1^2 - m_2^2\right)\left[\eta^{(n)}_{i_1,\cdots,i_k}\right],
\end{aligned}$$

with the j-esimo term given by $b_j = b_1\left(p_1 \leftrightarrow p_j; m_j \leftrightarrow m_{j+1}\right)$ and $i_1 \leftrightarrow i_j$. If, in a particular kinematical situation, the matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & \cdots & p_1 \cdot p_k \\ p_2 \cdot p_1 & p_2 \cdot p_2 & \cdots & p_2 \cdot p_k \\ \vdots & \vdots & \ddots & \vdots \\ p_k \cdot p_1 & p_k \cdot p_2 & \cdots & p_k \cdot p_k \end{pmatrix},$$

has $\det \boldsymbol{A} \neq 0$, the solution of the above system of linear equations can be written in a formal way by

$$\xi^{(n)}_{i_1+1, i_2, \cdots, i_k} = \frac{1}{\det \boldsymbol{A}} \sum_{j=1}^{k} \Delta_{1j} b_j,$$

$$\xi^{(n)}_{i_1, i_2+1, i_3, \cdots, i_k} = \frac{1}{\det \boldsymbol{A}} \sum_{j=1}^{k} \Delta_{2j} b_j,$$

$$\vdots$$

$$\xi^{(n)}_{\cdots, i_{l-1}, i_l+1, i_{l+1}, \cdots} = \frac{1}{\det \boldsymbol{A}} \sum_{j=1}^{k} \Delta_{lj} b_j,$$

where Δ_{ij} is the cofactor of $a_{ij} = p_i \cdot p_j$. By recursive use of the above relation it is possible to reduce all functions $\xi^{(n)}_{i_1, \cdots, i_k}$ to functions with $i_1 = i_2 = \cdots = i_k = 0$. This type of reduction is useful, for example, in applications where we are interested in numerical results because within this procedure we have to manipulate only a low number of mathematical structures saving, in this way, considerable computational time.

10. Conclusions

In the present work we considered general aspects involved in the calculations of perturbative amplitudes of QFT's. A very general procedure is presented for this purpose. The work can be considered as an extension of a previous one where only one species of fermion has been considered [2]. In addition, the calculations in the present contribution have been done by adopting an arbitrary scale parameter putting the calculations in the most general way. All the arbitrariness involved in the calculations were preserved in intermediary steps. The adoption of a regularization was avoided, the internal momenta are assumed as arbitrary and the common scale for the finite and divergent parts was taken as arbitrary too. Divergent integrals were not really evaluated. Only very general properties of such quantities were used. This became possible through a convenient interpretation of the Feynman rules. The perturbative amplitudes after written for one value of the loop (unrestricted) momentum are not integrated before a convenient representation for the propagators is assumed. When the integration is taken all the dependence on the internal arbitrary momenta is present in finite integrals. In the divergent ones no physical quantity is present. Only the arbitrary scale appears there. The divergent parts are written as a combination of standard mathematical objects which are never really integrated and the finite parts are written, after the integration is performed, in terms of finite structure functions. So, two very general types of systematization are proposed;

1) Divergent parts. The divergent content of one loop amplitudes perturbative amplitudes belonging to fundamental theories can be written as a combination of five objects; $I_{\log}\left(\lambda^2\right)$, $I_{\text{quad}}\left(\lambda^2\right)$, $\square_{\alpha\beta\mu\nu}\left(\lambda^2\right)$, $\Delta_{\mu\nu}\left(\lambda^2\right)$ and $\nabla_{\mu\nu}\left(\lambda^2\right)$.

2) Finite parts. The finite content can be written as a combination of only three functions $Z_0\left(p^2; m^2\right)$, $\xi_{00}\left(p, q\right)$ and $\zeta_{000}\left(p, q, r\right)$ for amplitudes having two, three and four internal propagators.

All self energies, decays and elastic scattering of two fields can be calculated by using the results presented here as well as their symmetry relations can be verified. The results written in terms of the systematization above can be used in the context of regularizations since all the operations performed are valid in the presence of any reasonable regularization distribution. All we need to evaluate is the standard divergent objects.

As a last comment we argue that if we want to give some meaning to the perturbative calculations we have to impose that the space-time homogeneity and the scale independence need to be recovered. Otherwise, the amplitudes become completely arbitrary quantities as well as local and gauge symmetries may be violated (invariably by the ambiguous terms). If we agree with this argument, our procedure makes this job easy. All we need is to impose that the conveniently defined objects $\square$, $\Delta_{\mu\nu}$ and $\nabla_{\mu\nu}$ become identically vanishing. This assumption can be viewed as completely reasonable since these objects can be identified as surfaces terms which are really vanishing quantities in the presence of any distribution. The same will occur by assuming the analytic continuation of the integrals to a continuum and complex dimension which is the ingredient of the dimensional regularization. So, in any consistent interpretation of the perturbative amplitudes only the basic divergences $I_{\log}\left(\lambda^2\right)$ and $I_{\text{quad}}\left(\lambda^2\right)$ will remain in a calculated divergent amplitude. They need not to be calculated since they will be absorbed in the renormalization of physical parameters. The calculation of beta functions can be done by using the scale properties of such objects.

All these comments allow us to conclude that within the context of our strategy the amplitudes are automatically ambiguities free and symmetry preserving and no regularization method needs to be used for any purpose. The strategy, in addition, is universal since it can be applied to any theory or model, renormalizable or not, and formulated in odd and even space-time dimensions in an absolutely identical way. And, which is still better, the results are as consistent as desirable. Investigations involving higher space-time dimensions (odd and even) as well as nonrenormalizable theories in four dimensions are presently under way and the obtained results are in accordance with our best expectations.

In addition, other authors have been made investigations by using the procedure adopted in the present work. In particular in [31] the authors explored some very interesting aspects of the systematization proposed in [2] concluding that there are important advantages relative to the traditional ones.

REFERENCES

[1] O. A. Battistel and G. Dallabona, "Scale Ambiguities in Perturbative Calculations and the Value for the Radiatively Induced Chern-Simons Term in Extended QED," *Physical Review D*, Vol. 72, No. 4, 2005, Article ID: 045009.

[2] O. A. Battistel and G. Dallabona, "A Systematization for One-Loop 4D Feynman Integrals," *The European Physical Journal C*, Vol. 45, No. 3, 2006, pp. 721-743. doi:10.1140/epjc/s2005-02437-0

[3] G. Passarino and M. Veltman, "One Loop Corrections for e^+e^- Annihilation into $\mu^+\mu^-$ in the Weinberg Model," *Nuclear Physics B*, Vol. 160, No. 1, 1979, pp. 151-207. doi:10.1016/0550-3213(79)90234-7

[4] W. L. van Neerven and J. A. Vermaseren, "Large Loop Integrals," *Physics Letters B*, Vol. 137, No. 3-4, 1984, pp. 241-244.

[5] G. J. Oldenborgh and J. A. Vermaseren, "New Algorithms for One Loop Integrals," *Zeitschrift für Physik C Particles and Fields*, Vol. 46, No. 3, 1990, pp. 425-437.

[6] A. I. Davydychev, "A Simple Formula for Reducing Feynman Diagrams to Scalar Integrals," *Physics Letters B*, Vol. 263, No. 1, 1991, pp. 107-111.

[7] Z. Bern, L. J. Dixon and D. A. Kosower, "Dimensionally Regulated One Loop Integrals," *Physics Letters B*, Vol. 302, No. 2-3, 1993, pp. 299-308.

[8] O. V. Tasarov, "Connection between Feynman Integrals Having Different Values of the Space-Time Dimension," *Physical Review D*, Vol. 54, No. 10, 1996, pp. 6479-6490.

[9] R. G. Stuart, "Algebraic Reduction of One Loop Feynman Diagrams to Scalar Integrals," *Computer Physics Communications*, Vol. 48, No. 3, 1988, pp. 367-389. doi:10.1016/0010-4655(88)90202-0

[10] J. Campbell, E. Glover and D. Miller, "One Loop Tensor Integrals in Dimensional Regularization," *Nuclear Physics B*, Vol. 498, No. 1-2, 1997, pp. 397-442. doi:10.1016/S0550-3213(97)00268-X

[11] G. Devaraj and R. G. Stuart, "Reduction of One Loop Tensor Form-Factors to Scalar Integrals: A General Scheme," *Nuclear Physics B*, Vol. 519, No. 1-2, 1998, pp. 483-513. doi:10.1016/S0550-3213(98)00035-2

[12] J. Fleischer, F. Jegerlehner and O. V. Tasarov, "Algebraic Reduction of One Loop Feynman Graph Amplitudes," *Nuclear Physics B*, Vol. 566, No. 1-2, 2000, pp. 423-440. doi:10.1016/S0550-3213(99)00678-1

[13] G.'t Hooft and M. Veltman, "Scalar One Loop Integrals," *Nuclear Physics B*, Vol. 153, 1979, pp. 365-401. doi:10.1016/0550-3213(79)90605-9

[14] T. Binoth, J. P. Guillet and G. Heinrich, "Reduction Formalism for Dimensionally Regulated One-Loop N-Point Integrals," *Nuclear Physics B*, Vol. 572, No. 1-2, 2000, pp. 361-386. doi:10.1016/S0550-3213(00)00040-7

[15] T. Binoth, J. P. Guillet, G. Heinrich and C. Schubert, "Calculation of One Loop Hexagon Amplitudes in the Yukawa Model," *Nuclear Physics B*, Vol. 615, No. 1-3, 2001, pp. 385-401. doi:10.1016/S0550-3213(01)00436-9

[16] A. Denner and S. Dittmaier, "Reduction of One Loop Tensor Five Point Integrals," *Nuclear Physics B*, Vol. 658, No. 1-2, 2003, pp. 175-202. doi:10.1016/S0550-3213(03)00184-6

[17] G. Duplancic and B. Nizic, "Reduction Method for Dimensionally Regulated One Loop N Point Feynman Integrals," *European Physical Journal*, Vol. 35, 2004, pp. 105-118.

[18] G. Duplancic and B. Nizic, "Dimensionally Regulated One Loop Box Scalar Integrals with Massless Internal Lines," *European Physical Journal*, Vol. 20, 2001, pp. 357-370.

[19] G. Duplancic and B. Nizic, "IR Finite One Loop Box Scalar Integral with Massless Internal Lines," *European Physical Journal*, Vol. 24, 2002, pp. 385-391.

[20] F. del Aguila and R. Pittau, "Recursive Numerical Calculus of One-Loop Tensor Integrals," *Journal of High Energy Physics*, Vol. 7, 2004, p. 17.

[21] W. T. Giele and E. W. N. Glover, "A Calculational Formalism for One Loop Integrals," *Journal of High Energy Physics*, Vol. 8, 2004, p. 29.

[22] R. Britto and B. Feng, "Integral Coefficients for One-Loop Amplitudes," *Journal of High Energy Physics*, Vol. 2, 2008, p. 95.

[23] A. Denner and S. Dittmaier, "Reduction Schemes for One-Loop Tensor Integrals," *Nuclear Physics B*, Vol. 734, No. 1-2, 2006, pp. 62-115. doi:10.1016/j.nuclphysb.2005.11.007

[24] T. Binoth, J. Ph. Guillet, G. Heinrich, E. Pilon and C. Schubert, "An Algebraic/Numerical Formalism for One-Loop Multi-Leg Amplitudes," *Journal of High Energy Physics*, Vol. 10, 2005, p. 15.

[25] Y. Kurihara, "Dimensionally Regularized One-Loop Tensor-Integrals with Massless Internal Particles," *European Physical Journal*, Vol. 45, No. 2, 2006, pp. 427-444. doi:10.1140/epjc/s2005-02428-1

[26] C. Anastasiou, E. W. Nigel Glover and C. Oleari, "Scalar One Loop Integrals Using the Negative Dimension Approach," *Nuclear Physics B*, Vol. 572, No. 1-2, 2000, pp. 307-360. doi:10.1016/S0550-3213(99)00637-9

[27] P. Mastrolia, G. Ossola, C. G. Papadopoulos and R. Pittau, "Optimizing the Reduction of One-Loop Amplitudes," *Journal of High Energy Physics*, Vol. 6, 2008, p. 30.

[28] R. Keith Ellis and G. Zanderighi, "Scalar One-Loop Inte-

grals for QCD," *Journal of High Energy Physics*, Vol. 2, 2008, p. 2.

[29] G. Ossola, C. G. Papadopoulos and R. Pittau, "On the Rational Terms of the One-Loop Amplitudes," *Journal of High Energy Physics*, Vol. 5, 2008, p. 4.

[30] O. A. Battistel, "Uma Nova Estratégia Para Manipulações e Cálculos Envolvendo Divergências em T.Q.C.," Ph.D. Thesis, Universidade Federal de Minas Gerais, Belo Horizonte, 1999.

[31] Y. Sun and H.-R. Chang, "One Loop Integrals Reduction," 2012.

