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Nonlinear Spinor Field Equations in Gravitational Theory: Spherical Symmetric Soliton-Like Solutions

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ABSTRACT

This paper deals with an extension of a previous work [*Gravitation & Cosmology*, Vol. 4, 1998, pp. 107-113] to exact spherical symmetric solutions to the spinor field equations with nonlinear terms which are arbitrary functions of $S = \bar{\psi}\psi$, taking into account their own gravitational field. Equations with power and polynomial nonlinearities are studied in detail. It is shown that the initial set of the Einstein and spinor field equations with a power nonlinearity has regular solutions with spinor field localized energy and charge densities. The total energy and charge are finite. Besides, exact solutions, including soliton-like solutions, to the spinor field equations are also obtained in flat space-time.

Keywords: Lagrangian; Static Spherical Symmetric Metric; Field Equations; Einstein Equations; Dirac Equation; Energy-Momentum Tensor; Charge Density; Current Vector; Soliton-Like Solution

1. Introduction

The unification of quantum mechanics and general relativity into a theory of quantum gravity remains a hard (as yet) unsolved problem and physical phenomena requiring both general relativity and quantum theory for their description cannot be possibly completely understood. Such a challenge stimulates intense research activities in various field-theoretical models with full non-perturbative account of gravity. Among all these activities, the investigations of solitons in these theories, with a special emphasis on flat space theories, attracted a particular importance due to their properties. Indeed, the soliton sector in the flat space gauge theories is quite well understood, the most notable example being the t'Hooft-Polyakov magnetic monopole. For a review on some recent progress in the investigation of solitons and black holes in non-Abelian gauge theories coupled to gravity, see [1] and references therein. However, as is well known, the marriage of gravity and relativity leads to a curved space-time whose geometry is dynamical and is governed by the energy-matter distribution within it, a framework within which the gravitational interaction is the physical manifestation of any curvature in space and in space-time. The most fascinating offsprings of this union are undoubtedly, on the one hand, the cosmological theory of the history of our universe from its birth to its ultimate demise if ever, and on the other hand, the prediction for

regions of space-time to be so much curled up by their energy-matter content that even light can no longer escape from such black holes.

On the other hand, the marriage of relativity and quantum theory leads naturally to the quantum field theory description of the elementary particles and their interactions, at the most intimate presently accessible scales of space and energy, a fact made manifest by the value of the product $\hbar c \simeq 197$ Mev.fm. In fact, one offspring of this second union is the unification of matter and radiation, namely of particles with their corpuscular propagating properties and fields with their wavelike propagating properties. Particles, characterized through their energy, momentum and spin values in correspondence with the Poincaré symmetries of Minkowski space-time in the absence of gravity, are nothing but the relativistic energy-momentum quanta of a field, thereby implying a tremendous economy in the description of the physical universe, accounting for instance at once in terms of a single field filling all of space-time for the indistinguishability of identical particles and their statistics. Furthermore, quantum relativistic interactions are then understood simply as couplings between the various quantum fields locally in space-time, which translate in terms of particles as diverse exchanges of the associated quanta. Such a picture lends itself most ideally to a perturbative understanding of the fundamental interactions, which has proved to be so powerful beginning with quantum electrodynamics, up to the modern Standard Model of the

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strong and electroweak interactions. For more explanation on these profound concepts, quantum theory and relativity, which have culminated into relativistic space-time geometry and quantum gauge theory as the principles for gravity and the three other known fundamental interactions, see notes [2] on *The quantum geometer's universe*: *Particles*, *interactions and topology* delivered in 2001 by Govaerts at the Second International Workshop on Contemporary Problems in Mathematical Physics.

All these activities, diverse and complementary, made in this field [1-14], are also mainly motivated by the wide roles of Einstein and Dirac equations in modern physics, for example, for investigating the spin particle and for the necessity of analysis of synchrotronic radiation [11]. To this purpose, many systems have been subjects of considerable interest and studies. The pioneering investigation could be the work by Drill and Wheeler in 1957 [3], who considered the Dirac equation in a central gravitational field associated with a diagonal metric. Using a normal diagonal tetrad, these authors constructed the generalized angular momentum operator separating the variables in the Dirac equation. Later, in a remarkable paper, appeared in 1987 [12], entitled "*Criteria of separability of variables in the Dirac equation in gravitational fields*", Shishkin and Andrushkevich provided the necessary and sufficient conditions, based on rigorous theorems, for separability of the variables for a diagonal tetrad gauge, and deduced the operators that determine the dependence of the wave function on the separated variables. In the same year, Barut and Duru [10] gave exact solutions of the Dirac equation in spatially flat Robertson-Walker space-times for models of expanding universes and discussed the current decomposition. Henceforth the investigations go into diverse directions, considering various classes of models including different metrics, the general class of which is investigated by Hounkonnou and Mendy in 1999 [13]. Thus, for example, the usual Friedman-Lemaître-Roberston-Walker homogeneous and isotropic metric of standard cosmology belongs to this general class of metrics (whether in Cartesian or spherical coordinates), which also includes general classes of Kantowski-Sachs metrics for anisotropic cosmologies as well as some examples of metrics used in models for stellar gravitational collapse [14]. It may be worth pointing out that *a priori*, this class of metrics solves Einstein's equations for specific distributions of energy-momentum of matter in space-time, in the presence of which the study of the quantized Dirac field may be of interest. Such an avenue could be pursued. For details, see [13] and references therein.

Moreover, it is also worthy of attention a previous study, which will be referred to *Part I* of the present work, where Adomou and Shikin [8] have obtained exact plane-symmetric solutions to the spinor field equations with nonlinear terms which are arbitrary functions of $S = \bar{\psi}\psi$, taking into account their own gravitational field. They have studied in detail equations with power and polynomial nonlinearities. They have shown that the initial set of the Einstein and spinor field equations with a power-law nonlinearity has regular solutions with a localized energy density of the spinor field only in the case of zero mass parameter in the spinor field, with a negative energy for the soliton-like configuration. They have also proved that the spinor field equation with a polynomial nonlinearity has a regular solution with positive energy. Their study has come out onto the non existence of soliton-like solutions in the flat space-time.

The present work, considered as *Part II* of all these investigated initiated in [8], aims at extending the results to exact spherical symmetric solutions. Here also equations with power and polynomial nonlinearities are thoroughly scrutinized.

The paper is organized as follows. Section 2 addresses the model with fundamental equations. We consider a self-consistent system to obtain spherical-symmetric solutions, taking into account the own gravitational field of particles. Section 3 deals with main results and their discussion; the solutions of the Einstein and nonlinear spinor field equations are derived. Besides, the regularity properties of the obtained solutions as well as the asymptotic behavior of the energy and charge densities are studied. Concluding remarks are outlined in Section 4.

2. Model and Fundamental Equations

We consider the Lagrangian of the self-consistent system of spinor and gravitational fields in the form [8]:

$$L = \frac{R}{2\kappa} + L_{sp} \qquad (2.1)$$

$$L_{sp} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi + L_N \quad (2.2)$$

where R is the scalar curvature; κ is Einstein's gravitational constant and $L_N = F(S)$ is an arbitrary function depending on $S = \bar{\psi}\psi$.

Instead of the static plane-symmetric metric chosen in [8], in the present analysis we opt for the static spherical symmetric metric in the form:

$$ds^2 = e^{2\gamma}dt^2 - e^{2\alpha}d\xi^2 - e^{2\beta}\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \quad (2.3)$$

α, β, γ being some functions depending only on $\xi = \frac{1}{r}$,

where r stands for the radial component of the spherical symmetric metric, and satisfying the coordinate condition

$$\alpha = 2\beta + \gamma. \qquad (2.4)$$

From the Lagrangian (2.1), through the variational

principle and usual algebraic manipulations, one can readily deduce the Einstein equations for the metric (2.3) under the condition (2.4), the spinor field equations for the functions ψ, $\bar{\psi}$, and the components of the metric spinor field energy-momentum tensor, respectively, in the form [3]:

$$G_0^0 = e^{-2\alpha}\left(2\beta'' - \beta'^2 - 2\beta'\gamma'\right) - e^{-2\beta} = -\kappa T_o^o \quad (2.5)$$

$$G_1^1 = e^{-2\alpha}\left(\beta'^2 + 2\beta'\gamma'\right) - e^{-2\beta} = -\kappa T_1^1 \quad (2.6)$$

$$G_2^2 = e^{-2\alpha}\left(\beta'' + \gamma'' - \beta'^2 - 2\beta'\gamma'\right) = -\kappa T_2^2 \quad (2.7)$$

$$G_3^3 = G_2^2 \quad (2.8)$$

$$T_2^2 = T_3^3 \quad (2.9)$$

$$i\gamma^\mu \nabla_\mu \psi - m\psi + L_N'\psi = 0 \quad (2.10)$$

$$i\nabla_\mu \bar{\psi}\gamma^\mu + m\bar{\psi} - L_N'\bar{\psi} = 0 \quad (2.11)$$

$$T_{\mu\nu} = \frac{i}{4}\left(\bar{\psi}\gamma_\mu \nabla_\nu \psi + \bar{\psi}\gamma_\nu \nabla_\mu \psi - \nabla_\mu \bar{\psi}\gamma_\nu \psi - \nabla_\nu \bar{\psi}\gamma_\mu \psi\right) \\ - g_{\mu\nu}L_{sp} \quad (2.12)$$

where ∇_μ is the covariant spinor derivative [3]:

$$\nabla_\mu \psi = \frac{\partial \psi}{\partial \xi^\mu} - \Gamma_\mu \psi \; ; \; \Gamma_\mu(\xi)$$ are the spinor affine connection matrices. To define the matrices $\gamma^\mu(\xi)$, let us use the equalities

$$g_{\mu\nu}(\xi) = e_\mu^{(a)}(\xi)e_\nu^{(b)}(\xi)\eta_{ab} \; ; \gamma_\mu(\xi) = e_\mu^{(a)}(\xi)\bar{\gamma}_a \quad (2.13)$$

where $\eta_{ab} = \text{diag}(1, -1, -1, -1)$; $\bar{\gamma}_a$ are the Dirac's matrices in flat space-time; $e_\mu^{(a)}(\xi)$ are tetradic 4-vectors. Then we get:

$$\gamma^0 = e^{-\gamma}\bar{\gamma}^0 \; ; \gamma^1 = e^{-\alpha}\bar{\gamma}^1 \; ; \gamma^2 = e^{-\beta}\bar{\gamma}^2 \; ; \gamma^3 = \frac{e^{-\beta}\bar{\gamma}^3}{\sin\theta}. \quad (2.14)$$

The matrices $\Gamma_\mu(\xi)$ are then determined as follows:

$$\Gamma_\mu = \frac{1}{4}g_{\rho\delta}\left(\partial_\mu e_\sigma^{(b)} \cdot e_{(b)}^\rho - \Gamma_{\mu\sigma}^\rho\right)\gamma^\delta\gamma^\sigma; \quad (2.15)$$

$$\Gamma_0 = -\frac{1}{2}\bar{\gamma}^0\bar{\gamma}^1 e^{-2\beta}\gamma' \; ; \; \Gamma_1 = 0 \; ; \; \Gamma_2 = \frac{1}{2}\bar{\gamma}^2\bar{\gamma}^1 e^{-\gamma-\beta}\beta' \quad (2.16)$$

$$\Gamma_3 = \frac{1}{2}\left(\bar{\gamma}^3\bar{\gamma}^1 e^{-\beta-\gamma}\beta'\sin\theta + \bar{\gamma}^3\bar{\gamma}^2\cos\theta\right). \quad (2.17)$$

The matrices $\bar{\gamma}^a$ are chosen as in [3]. Using the spinor field equations, we can rewrite L_{sp} in the form

$$L_{sp} = -\frac{1}{2}\left(\bar{\psi}\frac{\partial L_N}{\partial \bar{\psi}} + \frac{\partial L_N}{\partial \psi}\psi\right) + L_N = -SL_N' + L_N, \quad (2.18)$$

with the spinor

$$\psi = \begin{pmatrix} V_1 & V_2 & V_3 & V_4 \end{pmatrix}.$$

Taking into account (2.18), let us write explicitly the

nonzero components of the tensor T_ν^μ:

$$T_0^0 = T_2^2 = T_3^3 = -L_{sp} = SL_N' - L_N \quad (2.19)$$

setting the condition $\bar{V}_1 V_4 + \bar{V}_2 V_3 = \bar{V}_3 V_2 + \bar{V}_4 V_1$,

$$T_1^1 = \frac{i}{2}\left(\bar{\psi}\gamma^1 \nabla_1 \psi - \nabla_1 \bar{\psi}\gamma^1 \psi\right) + SL_N' - L_N \quad (2.20)$$

Using the obtained expressions for $\Gamma_\mu(\xi)$ in (2.15)-(2.17), we can expand (2.10) as

$$ie^{-\alpha}\bar{\gamma}^1\left[\partial_\xi + \frac{1}{2}\alpha'\right]\psi + \frac{i}{2}e^{-\beta}\bar{\gamma}^2\psi\cot\theta - m\psi + L_N'\psi = 0 \quad (2.21)$$

yielding the following set of equations:

$$V_4' + \frac{1}{2}\alpha'V_4 - \frac{i}{2}e^{\alpha-\beta}V_4\cot\theta - ie^\alpha\left(L_N' - m\right)V_1 = 0 \quad (2.22)$$

$$V_3' + \frac{1}{2}\alpha'V_3 + \frac{i}{2}e^{\alpha-\beta}V_3\cot\theta - ie^\alpha\left(L_N' - m\right)V_2 = 0 \quad (2.23)$$

$$-V_2' - \frac{1}{2}\alpha'V_2 + \frac{i}{2}e^{\alpha-\beta}V_2\cot\theta - ie^\alpha\left(L_N' - m\right)V_3 = 0 \quad (2.24)$$

$$-V_1' - \frac{1}{2}\alpha'V_1 - \frac{i}{2}e^{\alpha-\beta}V_1\cot\theta - ie^\alpha\left(L_N' - m\right)V_4 = 0. \quad (2.25)$$

3. Results and Discussion

From the set of Equations (2.22)-(2.25), we infer that the invariant function

$$S = \bar{\psi}\psi = V_1^*V_1 + V_2^*V_2 - V_3^*V_3 - V_4^*V_4$$

satisfies a first order differential equation:

$$\frac{dS}{d\xi} + \alpha'S = 0 \quad (3.1)$$

giving the evident solution

$$S = Ce^{-\alpha(\xi)}, \quad (3.2)$$

C being a constant. Combining the spinor field Equation (2.21) with its conjugate expression results the following expression for (2.20):

$$T_1^1 = mS - L_N. \quad (3.3)$$

The difference $\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ of the Einstein equations with (2.19) leads to

$$\beta'' - \gamma'' = e^{2\beta+2\gamma} \quad (3.4)$$

which can be transformed into a Liouville equation (see [7], page 30) to produce the solutions:

$$\beta(\xi) = \frac{A}{4}\left(1 + \frac{2}{G}\right)\ln\frac{A}{GT^2(h, \xi + \xi_1)} = \left(1 + \frac{2}{G}\right)\gamma(\xi) \quad (3.5)$$

$$\gamma(\xi) = \frac{A}{4}\ln\frac{A}{GT^2(h,\xi+\xi_1)} \qquad (3.6)$$

where the quantity A is expressed in terms of the Newton's gravitational constant G as:

$$A = \frac{G}{G+1}.$$

$$T(h,\xi+\xi_1) = \begin{cases} \dfrac{1}{h}\sinh\left[h(\xi+\xi_1)\right], & h>0 \\ \xi+\xi_1, & h=0 \\ \dfrac{1}{h}\sin\left[h(\xi+\xi_1)\right], & h<0, \end{cases} \qquad (3.7)$$

h being an integration constant and ξ_1 another non zero integration constant. Taking into account (3.5) and (3.6), we get from (2.4) the following relations:

$$\alpha(\xi) = \frac{A}{2}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)} \qquad (3.8)$$

and

$$\beta(\xi) = \frac{2+G}{4+3G}\alpha(\xi); \quad \gamma(\xi) = \frac{G}{4+3G}\alpha(\xi). \qquad (3.9)$$

Substituting (3.8) into (2.6), we obtain the Einstein equation $\begin{pmatrix}1\\1\end{pmatrix}$ in the form

$$\alpha'^2 = \frac{(4+3G)^2}{3G^2+8G+4}e^{2\alpha}\left[e^{\frac{-2G-4}{4+3G}\alpha} - \kappa(mS-L_N)\right]. \qquad (3.10)$$

Since $\alpha' = -\dfrac{1}{S}\dfrac{\mathrm{d}S}{\mathrm{d}\xi}$ with the invariant $S = Ce^{-\alpha}$, from (3.10), we get:

$$\frac{\mathrm{d}S}{\mathrm{d}\xi} = \pm\frac{(4+3G)S}{\sqrt{3G^2+8G+4}}\left(\frac{C}{S}\right)\sqrt{\left(\frac{S}{C}\right)^{\frac{4+2G}{4+3G}} - \kappa(mS-L_N)}. \qquad (3.11)$$

With the knowledge of $\beta(\xi)$, $\gamma(\xi)$ and $\alpha(\xi)$ from the relations (3.5), (3.6) and (3.8), respectively, the invariant $S(\xi)$ as well as the solutions of the Einstein equations can be completely determined. Furthermore, considering the concrete expression of the invariant $S(\xi)$, namely $S(\xi) = Ce^{-\alpha(\xi)}$, we can establish the regularity properties of the obtained solutions. Studying the distribution of the energy per unit invariant volume $T_0^0\sqrt{-3g}$, we can also deduce their localization proper-

ties.

We can get a concrete form of the functions $V_\rho(\xi)$ by solving Equations (2.22)-(2.25) in a more compact form if we pass to the functions $W_\rho(\xi) = e^{\frac{1}{2}\alpha(\xi)}V_\rho(\xi)$, $\rho = 1, 2, 3, 4$:

$$W_4' - \frac{i}{2}e^{\alpha-\beta}W_4\cot\theta - ie^{\alpha}\left(-m+L_N'\right)W_1 = 0 \qquad (3.12)$$

$$W_3' + \frac{i}{2}e^{\alpha-\beta}W_3\cot\theta - ie^{\alpha}\left(-m+L_N'\right)W_2 = 0 \qquad (3.13)$$

$$W_2' - \frac{i}{2}e^{\alpha-\beta}W_2\cot\theta + ie^{\alpha}\left(-m+L_N'\right)W_3 = 0 \qquad (3.14)$$

$$W_1' + \frac{i}{2}e^{\alpha-\beta}W_1\cot\theta + ie^{\alpha}\left(-m+L_N'\right)W_4 = 0 \qquad (3.15)$$

where

$$W_\rho' = \left(V_\rho' + \frac{1}{2}\alpha'V_\rho\right)e^{\frac{1}{2}\alpha}. \qquad (3.16)$$

Re-express Equations (3.12)-(3.15) under forms depending on functions of the argument $S(\xi)$, *i.e.* $U_\rho(S) = W_\rho(\xi)$, $S(\xi) = Ce^{-\alpha(\xi)}$. Then we get for the functions $U_\rho(S)$ the following set of equations:

$$\frac{\mathrm{d}U_4}{\mathrm{d}S} - iB(S)U_4 - iQ(S)U_1 = 0 \qquad (3.17)$$

$$\frac{\mathrm{d}U_3}{\mathrm{d}S} + iB(S)U_3 - iQ(S)U_2 = 0 \qquad (3.18)$$

$$\frac{\mathrm{d}U_2}{\mathrm{d}S} - iB(S)U_2 + iQ(S)U_3 = 0 \qquad (3.19)$$

$$\frac{\mathrm{d}U_1}{\mathrm{d}S} + iB(S)U_1 + iQ(S)U_4 = 0 \qquad (3.20)$$

where

$$B(S) = \frac{1}{2}\frac{\left(\frac{C}{S}\right)^{\frac{2+2G}{4+3G}}\cot\theta}{\frac{\mathrm{d}S}{\mathrm{d}\xi}}; \quad Q(S) = \frac{\left(\frac{C}{S}\right)\left(-m+L_N'\right)}{\frac{\mathrm{d}S}{\mathrm{d}\xi}} \qquad (3.21)$$

with $\dfrac{\mathrm{d}S}{\mathrm{d}\xi}$ determined by (3.11).

Differentiating now Equations (3.17)-(3.20) and substituting Equations (3.20) and (3.17) into the result, we obtain second-order differential equations obeyed by the functions $U_4(S)$ and $U_1(S)$:

$$U_4'' - \frac{Q'(S)}{Q(S)}U_4' + \left[B^2(S) - Q^2(S) + i\frac{B(S)Q'(S) - Q(S)B'(S)}{Q(S)}\right]U_4 = 0 \qquad (3.22)$$

$$U_1'' - \frac{Q'(S)}{Q(S)}U_1' + \left[B^2(S) - Q^2(S) + i\frac{Q(S)B'(S) - B(S)Q'(S)}{Q(S)}\right]U_1 = 0 \qquad (3.23)$$

Summing (3.22) and (3.23) and setting $U = U_1 + U_4$ afford the differential equation:

$$U'' - \frac{Q'(S)}{Q(S)}U' + \left[B^2(S) - Q^2(S)\right]U = 0, \qquad (3.24)$$

which, under the condition $B^2(S) = (1-\varepsilon)Q^2(S)$, with $0 < \varepsilon \leq 1$, yields the solution

$$U_1 + U_4 = \alpha_0 \cosh N_1(S), \quad \alpha_0 = const, \qquad (3.25)$$

where $N_1(S) = \sqrt{\varepsilon}\int Q(S)\mathrm{d}S + R_1; R_1 = const$. Substracting Equations (3.17) and (3.20) and taking into account (3.25), we obtain

$$U_1 - U_4 = -i\alpha_0 \frac{\sqrt{1-\varepsilon}+1}{\sqrt{\varepsilon}}\sinh N_1(S). \qquad (3.26)$$

It then follows, from the Equations (3.25) and (3.26), that

$$U_1(S) = \alpha_1\left[\cosh N_1(S) - i\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\sinh N_1(S)\right] \quad (3.27)$$

and

$$U_4(S) = \alpha_1\left[\cosh N_1(S) + i\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\sinh N_1(S)\right] \quad (3.28)$$

with $\alpha_1 = \dfrac{\alpha_0}{2}$.

Analogously operating on Equations (3.18) and (3.19), we arrive at

$$U_2(S) = \alpha_2\left[\sinh N_2(S) - i\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\cosh N_2(S)\right] \tag{3.29}$$

and

$$U_3(S) = \alpha_2\left[\sinh N_2(S) + i\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\cosh N_2(S)\right], \tag{3.30}$$

with $\alpha_2 = const$,

$$N_2(S) = -\sqrt{\varepsilon}\int Q(S)\mathrm{d}S + R_2, \quad R_2 = const. \qquad (3.31)$$

As mentioned in [8], it is worth considering a self-consistent solution to the linear spinor field equation (Dirac's equation), in view of its comparison with solutions to nonlinear spinor equations and of a better insight of the role of nonlinear terms in the nonlinear field equations in the formation of regular localized soliton-like solutions. For this purpose, $L_N = 0$ and we have from (2.21):

$$ie^{-\alpha}\overline{\gamma}^1\left[\partial_\xi + \frac{1}{2}\alpha'\right]\psi + \frac{i}{2}e^{-\beta}\overline{\gamma}^2\psi\cot\theta - m\psi = 0. \quad (3.32)$$

In this case, the relation (3.32) giving $S(\xi)$ becomes:

$$S(\xi) = C\exp\left\{-\frac{A}{2}\left(\frac{3}{2} + \frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\}. \quad (3.33)$$

From (3.5), (3.6) and (3.8), we get:

$$e^{2\gamma(\xi)} = \exp\left\{\frac{A}{2}\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\} \qquad (3.34)$$

$$e^{2\beta(\xi)} = \exp\left\{\frac{A}{2}\left(1+\frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\} \qquad (3.35)$$

$$e^{2\alpha(\xi)} = \exp\left\{A\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\} \qquad (3.36)$$

showing that the invariant S and the functions $g_{00} = e^{2\gamma}$, $g_{11} = -e^{2\alpha}$, $g_{22} = -e^{2\beta}$, $g_{33} = -e^{2\beta}\sin^2\theta$ are regular. In the case under consideration we have $T_0^0(\xi) = 0$, *i.e.* the energy density is localized.

Using (3.11), (3.21) and (3.31), we get:

$$N_{1,2}(S)$$
$$= \frac{\sqrt{C^a\varepsilon(3G^2+8G+4)}}{4+3G}\int\frac{-m+L_N'}{S\sqrt{S^a - C^a\kappa(mS - L_N)}}\mathrm{d}S, \tag{3.37}$$

with $a = \dfrac{4+2G}{4+3G} \cong 1$.

Let us find the explicit form of $V_\rho(\xi)$, $\rho = 1,2,3,4$. To this end, we retrieve the expressions of $N_1(S)$ and $N_2(S)$ from (3.37), knowing that $L_N = 0$. Without loss of generality, let us set $a = 1$. Then,

$$N_{1,2}(S) = \frac{2m\sqrt{\varepsilon C(3G^2+8G+4)}}{(4+3G)\sqrt{(1-C\kappa m)S}} + R_{1,2}. \qquad (3.38)$$

Substituting $S(\xi)$ from (3.33) into (3.38), we get

$$N_{1,2}(\xi) = \frac{2m\sqrt{\varepsilon(3G^2+8G+4)}}{(4+3G)\sqrt{1-C\kappa m}}$$
$$\times \exp\left\{\frac{A}{4}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\} + R_{1,2}, \tag{3.39}$$

with $R_{1,2} = const$.

We then replace the expressions of $N_1(\xi)$ and $N_2(\xi)$ from (3.39) into (3.27)-(3.30) and get an explicit form of $U_\rho(\xi)$, $\rho = 1,2,3,4$, and subsequently the expressions of $V_\rho(\xi) = U_\rho(\xi)e^{-\frac{1}{2}\alpha(\xi)}$:

$$V_1(\xi) = \alpha_1\exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2(h,\xi+\xi_1)}\right\}$$
$$\times\left[\cosh N_1(\xi) - i\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\sinh N_1(\xi)\right] \tag{3.40}$$

$$V_2(\xi) = \alpha_2 \exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\}$$

$$\times\left[\sinh N_2(\xi) - i\frac{\sqrt{1-\varepsilon}-1}{\sqrt{\varepsilon}}\cosh N_2(\xi)\right] \tag{3.41}$$

$$V_3(\xi) = \alpha_2 \exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\}$$

$$\times\left[\sinh N_2(\xi) + i\frac{\sqrt{1-\varepsilon}-1}{\sqrt{\varepsilon}}\cosh N_2(\xi)\right]; \tag{3.42}$$

$$V_4(\xi) = \alpha_1 \exp\left\{-\frac{A}{4}\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\}$$

$$\times\left[\cosh N_1(\xi) + i\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\sinh N_1(\xi)\right]. \tag{3.43}$$

$$N_{1,2}(S) = \frac{\sqrt{C\varepsilon\left(3G^2+8G+4\right)}}{4+3G}$$

$$\times\left\{\frac{2m}{\sqrt{C\kappa\lambda}+\sqrt{C\kappa\lambda S^2+\left(1-C\kappa m\right)S}}+2\sqrt{\frac{\lambda}{C\kappa}}\ln\left[\frac{2C\kappa\lambda}{1-C\kappa m}S+1+\frac{2\sqrt{C\kappa\lambda}}{1-C\kappa m}\sqrt{C\kappa\lambda S^2+\left(1-C\kappa\right)S}\right]\right\}+R_{1,2} \tag{3.45}$$

that we substitute into (3.27)-(3.30) to get an explicit expression of $U_\rho(\xi)$ and subsequently the initial functions $V_\rho(\xi) = U_\rho(\xi)e^{-\frac{1}{2}\alpha(\xi)}$, $\rho = 1,2,3,4$.

Let us compute the distribution of the spinor field energy density per unit invariant volume $f(\xi) = T_0^0(\xi)\sqrt{-3g}$. From (2.19) and (3.33) we have the following expression for $T_0^0(\xi)$:

$$T_0^0(\xi) = \lambda S^2(\xi)$$

$$= \lambda C^2 \exp\left\{-A\left(\frac{3}{2}+\frac{2}{G}\right)\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\}, \tag{3.46}$$

permitting to write

which represent nothing but the regular localized soliton-like solutions.

In the sequel, we deal with a concrete type of nonlinear spinor field equations which have the virtue that $L_N = \lambda S^n$, where λ is a nonlinearity parameter, $n \geq 2$. It is convenient to separately analyze the two cases $n = 2$ and $n > 2$:

- $n = 2$: $L_N = \lambda S^2$ and we have the nonlinear spinor field equation

$$ie^{-\alpha}\overline{\gamma}^1\left(\partial_\xi + \frac{1}{2}\alpha'\right)\psi + \frac{i}{2}e^{-\beta}\overline{\gamma}^2\psi\cot\theta$$

$$-m\psi + 2\lambda\left(\overline{\psi}\psi\right)\psi = 0. \tag{3.44}$$

The equalities (3.33)-(3.36) remain valid. Let us find an explicit form of $V_\rho(\xi)$, $\rho = 1,2,3,4$. For that, we deduce from (3.37) the function $N_1(S)$ and $N_2(S)$:

$$f(\xi) = T_0^0(\xi)e^{\alpha(\xi)+2\beta(\xi)}\sin\theta$$

$$= \lambda C^2 \sin\theta \exp\left\{-\frac{A}{4}\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\}, \tag{3.47}$$

inferring that the quantities g_{00}, g_{11}, g_{22} and V_ρ are regular and, from (3.47), the total energy $E = \int_0^{\xi_c} T_0^0(\xi)\sqrt{-3g}\,d\xi$ is finite. Therefore, the equation (3.44) possesses a soliton-like solution.

- $n > 2$: $L_N = \lambda S^n$ and the energy density is
$$T_0^0 = \lambda\left(n-1\right)S^n. \tag{3.48}$$

From (3.33), the distribution of the spinor field energy density per unit invariant volume takes the form

$$f(\xi) = T_0^0(\xi)\sqrt{-3g}$$

i.e.

$$f(\xi) = \lambda\left(n-1\right)C^n \sin\theta \exp\left\{\frac{A}{4G}\left[-n\left(4+3G\right)+5G+8\right]\cdot\ln\frac{A}{GT^2\left(h,\xi+\xi_1\right)}\right\} \tag{3.49}$$

showing that the spinor field energy density per unit invariant volume f is localized and the total energy

$E = \int_0^{\xi_c} T_0^0(\xi)\sqrt{-3g}\,d\xi$ is finite. To compute V_ρ, $\rho = 1,2,3,4$, we need the functions $N_1(S)$ and $N_2(S)$:

$$N_{1,2}(S) = \frac{\sqrt{C\varepsilon\left(3G^2+8G+4\right)}}{4+3G}\times\left\{\frac{2n}{C\kappa\left(n-2\right)}\sqrt{C\kappa\lambda S^{n-2}+\left(1-C\kappa m\right)\frac{1}{S}}+\frac{1}{\sqrt{C\kappa\lambda}}\left(\frac{C\kappa\lambda}{1-C\kappa m}\right)^{\frac{n}{2(n-1)}}\left[\frac{n}{n-2}\left(\frac{1}{C\kappa}-m\right)-m\right]\right.$$

$$\left.\times\left[B_S\left(\frac{n}{2(n-1)};1-\frac{1}{2(n-1)}\right)-2\left(\frac{1-C\kappa m}{C\kappa\lambda S^{n-1}+1-C\kappa m}\right)^{\frac{n}{2(n-1)}}\times\left(1+\frac{1-C\kappa m}{C\kappa\lambda S^{n-1}}\right)^{\frac{1}{2(n-1)}}\right]\right\}+R_{1,2}, \tag{3.50}$$

where

$$B_s\left(\frac{n}{2(n-1)};1-\frac{1}{2(n-1)}\right)=\int_0^S y^{\frac{n}{2(n-1)}-1}(1-y)^{\frac{-1}{2(n-1)}}\,\mathrm{d}y \tag{3.51}$$

and

$$y=\frac{1-C\kappa m}{C\kappa\lambda t^{n-1}+1-C\kappa m}, \tag{3.52}$$

that we substitute into (3.27)-(3.30) to get an explicit expression for $U_\rho(\xi)$, and then we readily compute the initial functions

$$V_\rho(\xi)=U_\rho(\xi)e^{-\frac{1}{2}\alpha(\xi)} \tag{3.53}$$

for $\rho=1,2,3,4$. Using the solutions (3.27)-(3.30), we deduce the components of the spinor current vector $j^\mu=\bar{\psi}\gamma^\mu\psi$ as follows:

$$j^o=2e^{-\gamma-\alpha}\left\{\alpha_1^2\left[\cosh^2 N_1(S)+\left(\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right)^2\sinh^2 N_1(S)\right]+\alpha_2^2\left[\sinh^2 N_2(S)+\left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right)^2\cosh^2 N_2(S)\right]\right\}$$

$$j^1=2e^{-2\alpha}\left\{\alpha_1^2\left[\cosh^2 N_1(S)-\left(\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right)^2\sinh^2 N_1(S)\right]+\alpha_2^2\left[\sinh^2 N_2(S)-\left(\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\right)^2\cosh^2 N_2(S)\right]\right\} \tag{3.54}$$

$$j^2=4e^{-\beta-\alpha}\left(\alpha_1^2\frac{1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\cosh N_1(S)\cdot\sinh N_1(S)-\alpha_2^2\frac{-1+\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}}\sinh N_2(S)\cosh N_2(S)\right)$$

$$j^3=0.$$

Since the configuration is static, only the component j^0 is nonzero. The constants in the solution of the spinor field equation are obtained from the equations $j^1=0$ and $j^2=0$, thus giving $\alpha_1=\alpha_2$, $N_2(S)=-N_1(S)$ and $\varepsilon=1$. The component j^0 defines the charge density of the spinor field whose the chronometric invariant form is characterized by:

$$q=\left(j_o j^o\right)^{1/2}=4a^2 e^{-\alpha}\cosh 2N(S) \tag{3.55}$$

where $a=\alpha_1=\alpha_2$, $N(S)=N_1(S)=-N_2(S)$, $\varepsilon=1$. The total charge of the spinor field is:

$$Q=\int_o^{\xi_c}q\sqrt{-3g}\,\mathrm{d}\xi, \tag{3.56}$$

ξ_c being the center of the field configuration.

The relations (3.33), (3.39), (3.45), (3.55) and (3.56) infer that the charge density of the spinor field is localized, and the total charge is a finite quantity, when $L_N=0$, or λS^2, or λS^n, $n>2$.

4. Concluding Remarks

In this paper, we have obtained exact spherical symmetric solutions to the spinor and gravitational field equations and studied their regularity properties as well as the localization properties of both the energy and charge densities in different configurations, when $L_N=0$, λS^2 and λS^n.

In all these cases, the solutions are regular; the energy and charge densities are localized. The total energy and charge of the spinor field are finite quantities. The study of the set of all regular spherical solutions with a possible

criterion of their classification could deserve some interest. Such investigation will be in the core of the forthcoming paper.

5. Acknowlegements

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