

A New Formulation of Electrodynamics

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ABSTRACT

A new formulation of electromagnetism based on linear differential commutator brackets is developed. Maxwell equations are derived, using these commutator brackets, from the vector potential \vec{A} , the scalar potential ϕ and the Lorentz gauge connecting them. With the same formalism, the continuity equation is written in terms of these new differential commutator brackets.

Keywords: Mathematical Formulation, Maxwell's Equations

1. Introduction

Maxwell equations are first order differential equations in space and time. They are compatible with Lorentz transformation which guarantees its applicability to any inertial frame. A symmetric space-time formulation of any theory will generally guarantee the universality of the theory. With this motivation, we adopt a differential commutator bracket involving first order space and time derivative operators to formulate the Maxwell equations and quantum mechanics. This is in addition to our recent quaternionic formulation of physical laws, where we have shown that many physical equations are found to emerge from a unified view of physical variables [1]. In such a formulation, we have found that Maxwell equations emerge from a single equation. Maxwell equations were originally written in terms of quaternions. They were initially written in twenty equations [a]. However, later on Maxwell equations are then written in terms of vector in the way that we are familiar today. In our present formulation, Maxwell equations are described by a set of two wave equations representing the evolution of the electric and magnetic fields. This is instead of having four equations. We aim in this paper to write down (derive) the physical equations by vanishing differential commutator brackets. We know that second order partial derivatives commute for space-space variables. We don't assume here this property is a priori for space and time. To guarantee this, we eliminate the time derivative of a quantity that is acted by a space (∇) derivative followed by a time derivative, and vice-versa. In expanding the differential commutator bracket, we don't commute time and space derivative, but rather eliminate the time de-

riivative by the space derivative, and vice versa. This differential commutator bracket may enlighten us to quantize these physical quantities. By employing the differential commutator brackets of the vector \vec{A} and scalar potential ϕ , we have derived Maxwell equations without invoking any a priori physical law. Maxwell arrives at his theory of electromagnetism by combing the Gauss, Faraday and Ampere laws. For mathematical consistency, he modified Ampere's law. He then came with the known Maxwell equations.

2. Relativistic Prelude

From Lorentz transformations one obtain,

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right). \quad (1)$$

We see that the commutator bracket

$$[\Delta t, \Delta x] = [\Delta t', \Delta x'] \quad (2)$$

where we have taken into account in the order of multiplication of the space and time differences, $(\Delta x, \Delta t)$. This shows that the commutator is Lorentz invariant. This is a new invariant quantity in relativity. We, however, already knew that the square interval is Lorentz invariant, i.e., $(\Delta S)^2 = (\Delta S')^2$ [2]. It follows from Equation (1) that the differential commutator bracket $\left[\frac{\partial}{\partial t}, \vec{\nabla}\right] = 0$ is Lorentz invariant too, i.e., $\left[\frac{\partial}{\partial t'}, \vec{\nabla}'\right] = \left[\frac{\partial}{\partial t}, \vec{\nabla}\right]$. We know that the spatial second order derivatives of a function, $f = f(x, y)$, is commutative, i.e.,

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. We wonder if the commutations of space and time derivatives are equally valid for all physical quantities. Motivated by this hypothesis, we propose the following differential commutator brackets to formulate the physical laws. In particular, we apply these differential commutator brackets, in this work to derive the continuity equation, Maxwell equations.

3. Differential Commutators Algebra

Define the three linear differential commutator brackets as follows:

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \right] = 0, \quad \left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] = 0, \quad \left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] = 0. \quad (3)$$

Equation (3) is correct, since partial derivatives commute, *i.e.*, $\frac{\partial^2}{\partial t \partial x} \varphi = \frac{\partial^2}{\partial x \partial t} \varphi$. For a scalar ψ and a vector \vec{G} , one defines the three brackets as follows:¹

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \right] \psi = \frac{\partial}{\partial t} (\bar{\nabla} \psi) - \bar{\nabla} \left(\frac{\partial \psi}{\partial t} \right), \quad (4)$$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] \vec{G} = \frac{\partial}{\partial t} (\bar{\nabla} \cdot \vec{G}) - \bar{\nabla} \cdot \left(\frac{\partial \vec{G}}{\partial t} \right), \quad (5)$$

and

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \vec{G} = \frac{\partial}{\partial t} (\bar{\nabla} \times \vec{G}) - \bar{\nabla} \times \left(\frac{\partial \vec{G}}{\partial t} \right). \quad (6)$$

It follows that

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\psi \vec{G}) = \psi \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] \vec{G} \right\} + \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \right] \psi \right\} \cdot \vec{G}, \quad (7)$$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] (\psi \vec{G}) = \psi \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \vec{G} \right\} + \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \right] \psi \right\} \times \vec{G}, \quad (8)$$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\vec{G} \times \vec{F}) = \vec{F} \cdot \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \vec{G} \right\} - \vec{G} \cdot \left\{ \left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \vec{F} \right\}, \quad (9)$$

for any vector \vec{F} . The differential commutator brackets above satisfy the distribution rule

$$\left[\hat{A} \hat{B}, \hat{C} \right] = \hat{A} \left[\hat{B}, \hat{C} \right] + \left[\hat{A}, \hat{C} \right] \hat{B}, \quad (10)$$

where $\hat{A}, \hat{B}, \hat{C}$ are $\bar{\nabla}, \frac{\partial}{\partial t}$. It is evident that the differential commutator brackets identities follow the same ordinary vector identities. We call the three differential commutator brackets in Equation (3) the grad-commutator bracket,

the dot-commutator bracket and the cross-commutator bracket respectively. The prime idea here is to replace the time derivative of a quantity by the space derivative $\bar{\nabla}$ of another quantity, and vice-versa, so that the time derivative of a quantity is followed by a time derivative with which it commutes. We assume here that space and time derivatives don't commute. With this minimal assumption, we have shown here that all physical laws are determined by vanishing differential commutator bracket.

4. The Continuity Equation

Using quaternionic algebra [3], we have recently found that generalized continuity equations can be written as

$$\bar{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \quad (11)$$

$$\bar{\nabla} (\rho c^2) + \frac{\partial \vec{J}}{\partial t} = 0, \quad (12)$$

and

$$\bar{\nabla} \times \vec{J} = 0. \quad (13)$$

Now consider the dot-commutator of $\rho \vec{J}$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\rho \vec{J}) = \frac{\partial}{\partial t} (\bar{\nabla} \cdot (\rho \vec{J})) - \bar{\nabla} \cdot \left(\frac{\partial (\rho \vec{J})}{\partial t} \right) = 0. \quad (14)$$

Using Equations (11)-(13), one obtains

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\rho \vec{J}) = c^2 \rho \left(\frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} - \nabla^2 \rho \right) + \left(\frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} - \nabla^2 \vec{J} \right) \cdot \vec{J} = 0. \quad (15)$$

For arbitrary ρ and \vec{J} , Equation (15) yields the two wave equations

$$\frac{1}{c^2} \frac{\partial^2 \rho}{\partial t^2} - \nabla^2 \rho = 0, \quad (16)$$

and

$$\frac{1}{c^2} \frac{\partial \vec{J}}{\partial t} - \nabla^2 \vec{J} = 0. \quad (17)$$

Equations (16) and (17) show that the charge and current density satisfy a wave equation traveling at speed of light in vacuum. It is remarkable to know that these two equations are already obtained in [3]. Hence, the current-charge density wave equations are equivalent to

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\rho \vec{J}) = 0. \quad (18)$$

5. Maxwell's Equations

Maxwell's equations are first order differential equations in space and time of the electromagnetic field, viz.,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \tag{19}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}, \tag{20}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{21}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{22}$$

These equations show that charge (ρ) and current (\vec{J}) densities are the sources of the electromagnetic field. Differentiating Equation (20) and using Equation (21), one obtains

$$\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = -\mu_0 \left(\vec{\nabla}(\rho c^2) + \frac{\partial \vec{J}}{\partial t} \right). \tag{23}$$

Similarly, differentiating Equation (21) and using Equation (20), one obtains

$$\frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} - \nabla^2 \vec{B} = \mu_0 (\vec{\nabla} \times \vec{J}). \tag{24}$$

These two equations state that the electromagnetic field propagates with speed of light in two cases:

- 1) charge and current free medium (vacuum), *i.e.*, $\rho = 0, \vec{J} = 0$, or
- 2) if the two equations

$$\vec{\nabla}(\rho c^2) + \frac{\partial \vec{J}}{\partial t} = 0, \tag{25}$$

and

$$(\vec{\nabla} \times \vec{J}) = 0, \tag{26}$$

besides the familiar continuity equation in Equation (11)

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \tag{27}$$

are satisfied. Equation (23) and (24) resemble Einstein's general relativity equation where space-times geometry is induced by the distribution of matter present. We see here that the electromagnetic field is produced by any charge and current densities distribution (in space and time). Now define the electromagnetic vector \vec{F} as

$$\vec{F} = \vec{B} - \frac{i}{c} \vec{E} \tag{28}$$

Adding Equation (25) and Equation (26) according to Equation (28), one obtains

$$\begin{aligned} & \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (\vec{B} - \frac{i}{c} \vec{E}) - \nabla^2 (\vec{B} - \frac{i}{c} \vec{E}) \\ &= \mu_0 \left[\frac{i}{c} \left(\vec{\nabla}(\rho c^2) + \frac{\partial \vec{J}}{\partial t} \right) + \vec{\nabla} \times \vec{J} \right]. \end{aligned} \tag{29}$$

Applying Equations(25), (26) (see [3]) in Equation (29) yields

$$\frac{1}{c^2} \frac{\partial^2 \vec{F}}{\partial t^2} - \nabla^2 \vec{F} = 0, \tag{30}$$

This is a wave equation propagating with speed of light in vacuum (c). Hence, Maxwell wave equations can be written as a pure single wave equation of an electromagnetic sourceless complex vector field \vec{F} . We call Equations (25)-(27) the generalized continuity equations. We have recently obtained these generalized continuity equations by adopting quaternionic formalism for fluid mechanics [3]. It is challenging to check whether any real fluid satisfies these equations or not. We have recently shown that Schrodinger, Dirac and Klein-Gordon and diffusion equations are compatible with these generalized continuity equations [3]. Using Equations (19) and (20), the electric field dot-commutator bracket yields

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \cdot \right] \vec{E} = \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0. \tag{31}$$

This is the familiar continuity equation. Hence, the continuity equation in the commutator bracket form can be written as

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \cdot \right] \vec{E} = 0. \tag{32}$$

Similar, using Equations (21) and (22), the magnetic field dot-commutator bracket yields

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \cdot \right] \vec{B} = \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{B}) - \vec{\nabla} \cdot \left(\frac{\partial \vec{B}}{\partial t} \right) = 0. \tag{33}$$

The electric field cross-commutator bracket gives

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \times \right] \vec{E} = \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t} \right) = 0. \tag{34}$$

Using Equations (20) and (21), this yields

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \times \right] \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} - \nabla^2 \vec{B} - \mu_0 (\vec{\nabla} \times \vec{J}) = 0. \tag{35}$$

This equation is nothing but Equation (24) above. Similarly, the magnetic field cross-commutator bracket gives

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \times \right] \vec{B} = \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) - \vec{\nabla} \times \left(\frac{\partial \vec{B}}{\partial t} \right) = 0. \tag{36}$$

Using Equations (20) and (21) this yields,

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{B} = \frac{1}{c^2} \frac{\partial^2 \bar{E}}{\partial t^2} - \nabla^2 \bar{E} + \mu_0 \left(\bar{\nabla}(\rho c^2) + \frac{\partial \bar{J}}{\partial t} \right) = 0. \quad (37)$$

This equation is nothing but Equation (23) above. Hence, Equations (35) and (37), *i.e.*,

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{E} = 0, \quad \left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{B} = 0. \quad (38)$$

represent the combined Maxwell equations. In terms of the vector \bar{F} defined in Equation (33), the wave equation in Equation (35) can be written as

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{F} = 0, \quad (39)$$

which is also evident from Equation (28).

6. Derivation of Maxwell Equations from the Vector and Scalar Potentials, \bar{A} , φ

The electric and magnetic fields are defined by the vector \bar{A} and the scalar potential φ as follows

$$\bar{E} = -\bar{\nabla} \varphi - \frac{\partial \bar{A}}{\partial t}, \quad \bar{B} = \bar{\nabla} \times \bar{A}. \quad (40)$$

These are related by the Lorentz gauge as

$$\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0. \quad (41)$$

Comparing this equation with Equation (11) reveals that the continuity equation is nothing but a gauge condition. This means that a new current density \bar{J}' can be found so that the equation is still intact. We have recently explored such a possibility which showed that it is true [3]. With this motivation the physicality of the gauge \bar{A} exhibited by Aharonov–Bohm effect is tantamount to that of the current density \bar{J} [5]. The grad-commutator bracket of the scalar potential φ

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \right] \varphi = \frac{\partial}{\partial t} (\bar{\nabla} \varphi) - \bar{\nabla} \left(\frac{\partial \varphi}{\partial t} \right) = 0. \quad (42)$$

Using Equations (40) and (41), one obtains

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \right] \varphi = \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} - \mu_0 \bar{J} = 0. \quad (43)$$

This yields the wave equation of the vector field \bar{A} as

$$\frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} - \nabla^2 \bar{A} = \mu_0 \bar{J}. \quad (44)$$

Similarly, the dot-commutator bracket of the vector \bar{A}

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] \bar{A} = \frac{\partial}{\partial t} (\bar{\nabla} \cdot \bar{A}) - \bar{\nabla} \cdot \left(\frac{\partial \bar{A}}{\partial t} \right) = 0. \quad (45)$$

Using Equations (40) and (41), one obtains

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] \bar{A} = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi - \frac{\rho}{\epsilon_0} = 0. \quad (46)$$

This yields the wave equation of φ

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\epsilon_0}. \quad (47)$$

The cross-commutator bracket of the scalar potential φ

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{A} = \frac{\partial}{\partial t} (\bar{\nabla} \times \bar{A}) - \bar{\nabla} \times \left(\frac{\partial \bar{A}}{\partial t} \right) = 0. \quad (48)$$

Using Equation (40), one finds

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] \bar{A} = \frac{\partial \bar{B}}{\partial t} + \bar{\nabla} \times \bar{E} = 0. \quad (49)$$

This yields the Faraday's equation,

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}. \quad (50)$$

It is interesting to arrive at this result by using the definition in Equation (40) only. Now consider the dot-commutator bracket of $\varphi \bar{A}$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \cdot \right] (\varphi \bar{A}) = \frac{\partial}{\partial t} (\bar{\nabla} \cdot (\varphi \bar{A})) - \bar{\nabla} \cdot \left(\frac{\partial}{\partial t} (\varphi \bar{A}) \right) = 0. \quad (51)$$

Using Equations (40), (41) and the vector identities

$$\begin{aligned} \bar{\nabla} \cdot (\varphi \bar{G}) &= (\bar{\nabla} \varphi) \cdot \bar{G} + \varphi (\bar{\nabla} \cdot \bar{G}), \\ \bar{\nabla} \times (\bar{\nabla} \times \bar{G}) &= \bar{\nabla} (\bar{\nabla} \cdot \bar{G}) - \nabla^2 \bar{G}, \end{aligned} \quad (52)$$

Equation (51) yields

$$\varphi \left(\bar{\nabla} \cdot \bar{E} - \frac{\rho}{\epsilon_0} \right) - c^2 \left(\bar{\nabla} \times \bar{B} - \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} - \mu_0 \bar{J} \right) \cdot \bar{A} = 0. \quad (53)$$

For arbitrary φ and \bar{A} , Equation (53) yields the two equations

$$\bar{\nabla} \cdot \bar{E} = \frac{\rho}{\epsilon_0}, \quad (54)$$

and

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \frac{1}{c^2} \frac{\partial \bar{E}}{\partial t}. \quad (55)$$

Equations (54) and (55) are the Gauss and Ampere equations.

Similarly, the cross-commutator bracket of $\varphi \bar{A}$

$$\left[\frac{\partial}{\partial t}, \bar{\nabla} \times \right] (\varphi \bar{A}) = \frac{\partial}{\partial t} (\bar{\nabla} \times (\varphi \bar{A})) - \bar{\nabla} \times \left(\frac{\partial}{\partial t} (\varphi \bar{A}) \right) = 0. \quad (56)$$

Using Equations (40), (41) and the vector identity

$$\vec{\nabla} \times (\varphi \vec{G}) = (\vec{\nabla} \varphi) \times \vec{G} + \varphi (\vec{\nabla} \times \vec{G}), \quad (57)$$

Equation (56) yields

$$\varphi \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) - c^2 \left(\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} - \mu_0 \vec{J} \right) \times \vec{A} = 0. \quad (58)$$

For arbitrary φ and \vec{A} , Equation (58) yields the two equations

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad (59)$$

and

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}. \quad (60)$$

Once again, Equations (59) and (60) are the Faraday and Ampere equations, respectively. Hence, the four Maxwell equations are completed. To sum up, Maxwell equations are the commutator brackets

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \cdot \right] (\varphi \vec{A}) = 0, \quad \left[\frac{\partial}{\partial t}, \vec{\nabla} \times \right] (\varphi \vec{A}) = 0. \quad (61)$$

7. Energy Conservation Equation

In electromagnetism, the energy conservation equation for electromagnetic field is written as

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}, \quad (62)$$

where

$$u = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2, \quad \vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}. \quad (63)$$

The energy conservation equation of the electromagnetic field can be easily obtain using the following vector identity

$$\vec{\nabla} \cdot (\vec{F} \times \vec{G}) = \vec{G} \cdot (\vec{\nabla} \times \vec{F}) - \vec{F} \cdot (\vec{\nabla} \times \vec{G}). \quad (64)$$

Let now $\vec{E} = \vec{F}$, $\vec{G} = \vec{B}$, so that Equation (64) becomes

Appendix

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \times \right] (\vec{F} \times \vec{G}) = \vec{F} \times \left(\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{G}) - \vec{\nabla} \times \frac{\partial \vec{G}}{\partial t} \right) + \left(\vec{\nabla} \times \frac{\partial \vec{F}}{\partial t} - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{F}) \right) \times \vec{G} \quad (A1)$$

$$\left[\frac{\partial}{\partial t}, \vec{\nabla} \cdot \right] (\vec{F} \cdot \vec{G}) = \vec{F} \cdot \left(\frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{G}) - \vec{\nabla} \cdot \frac{\partial \vec{G}}{\partial t} \right) + \left(\vec{\nabla} \cdot \frac{\partial \vec{F}}{\partial t} - \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{F}) \right) \vec{G} \quad (A2)$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}). \quad (65)$$

Employing Equations (20), (21) and (63), Eq.(65) yields

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{S} = -\vec{J} \cdot \vec{E}, \quad (66)$$

which is the familiar energy conservation equation of the electromagnetic field [5].

8. Concluding Remarks

By introducing three vanishing linear differential commutator brackets for scalar and vector fields, φ and \vec{A} and the Lorentz gauge connecting them, we have derived the Maxwell's equations and the continuity equation without resort to any other physical equation. Using different vector identities, we have found that no any independent equation can be generated from the three differential commutators brackets.

9. Acknowledgements

Equations (14) and (15) are in the form of coupled wave equations known as inhomogeneous Helmholtz equations. We see that the current density J enters into these equations in a relatively complicated way, and for this reason these equations are not readily soluble in general. This work is supported by the university of Khartoum research fund. We are grateful for this support.

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