

# Lie Symmetry Analysis, Optimal Systems and Explicit Solutions of the Dispersive Long Wave Equations

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## Abstract

In this paper, the dispersive long wave equation is studied by Lie symmetry group theory. Firstly, the Lie symmetries of this system are calculated. Secondly, one dimensional optimal systems of Lie algebra and all the symmetry reductions are obtained. Finally, based on the power series method and the extended Tanh function method, some new explicit solutions of this system are constructed.

## Keywords

Dispersive Long Wave Equations, Lie Symmetry Analysis, Optimal Systems, Power Series Method, Extended Tanh Function Method, Explicit Solutions

## 1. Introduction

In mathematical physics, many significant phenomena and dynamic processes can be represented by nonlinear partial differential equations (NLPDEs) [1] [2] [3] [4]. Therefore, it is very important to find the solution of NLPDEs. A wealth of effective methods have been developed to explore the solutions of the NLPDEs, such as Adomian decomposition method [5], Darboux and Backlund transformations [6], Hirota bilinear method [7] [8] [9], and Lie symmetry method [10] [11], etc. The Lie symmetry method for constructing explicit solutions of the NLPDEs has been regarded as one of the most widely applicable methods [12] [13] [14]. Its core idea is the invariance principle of the NLPDEs under the action of Lie point transformation group (point symmetry) [10]. In recent years, there has been a great deal of research and considerable development in the symmetry field of differential equations, in terms of the number of academic papers, books and new symbolic programs dedicated to this subject [15]-[20].

At present, there is no general method for solving NLPDEs. Although the symmetry method has a wide range of applications in solving methods, it still faces many difficulties and challenges to promote its development. However, the symmetry method and other methods (e.g. generalized simple equation method [21], generalized Tanh function method [22], homotopy perturbation method [23] and power series method [24], etc.) are effectively combined to reflect the complementarity of each other, which makes it possible to obtain exact solutions of some NLPDEs with physical significance, and attracts the attention and research of many scholars [25] [26] [27].

In the present paper, based on the Lie group method, we will investigate the dispersive long wave equations

$$\begin{cases} u_t + v_x + \frac{1}{2}(u^2)_x = 0, \\ v_t + (uv + u + u_{xx})_x = 0, \end{cases} \quad (1)$$

where  $u$  represents the amplitude of a surface wave, propagating along the  $x$ -axis with a horizontal velocity. It plays an important role in nonlinear physics [28] [29], considered as a good model for the study of bidirectional solitons in water waves. In [30] [31], Eckhaus and Boiti *et al.* presented the extensions of Equation (1) in higher-dimensional spaces. In [32], Zhang J F *et al.* discussed its some new multi-soliton solutions and travelling wave solutions using the extended homogeneous balance method, etc.

The outline of this paper is as follows: in Section 2, the Lie symmetry analysis is performed for the dispersive long wave equations; in Section 3, the optimal systems and the similarity reductions of Equation (1) are researched employing Lie group analysis in the last section; in Section 4, the exact solutions for the reduced equation are obtained by using the power series method and the extended Tanh method; and in Section 5, a brief summary is done to the full text.

## 2. Lie Symmetry Analysis

We first do some preparatory work on the concept of classical Lie symmetry of general NLPDEs. Consider the  $k$ th-order scalar NLPDEs of the form

$$f^\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \alpha = 1, 2, \dots, m, \quad (2)$$

where  $x = (x_1, x_2, \dots, x_n)$  denotes  $n$  independent variables,  $u = (u_1, u_2, \dots, u_m)$  denotes  $m$  independent variables, and  $u_{(j)} = u_{i_1 i_2 \dots i_j}^\alpha$  ( $j = 1, 2, \dots, k, i_s = 1, 2, \dots, j$ ) denote the partial derivatives of  $u^\alpha$  with respect to  $x_i$  ( $i = 1, 2, \dots, n$ ) up to  $j$ th-order, *i.e.*

$$u_{(j)}^\alpha = \frac{\partial^j u^\alpha}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_j}}.$$

Suppose that the one-parameter Lie group of point transformations

$$\begin{aligned} x_i^* &= X(x, u; \varepsilon) = x + \varepsilon \xi^i(x, u) + O(\varepsilon^2), \\ u_q^* &= U(x, u; \varepsilon) = u + \varepsilon \eta^q(x, u) + O(\varepsilon^2), \end{aligned} \quad (3)$$

where  $i = 1, 2, \dots, n$ ,  $q = 1, 2, \dots, m$ .  $\varepsilon$  is an infinitesimal parameter,  $\xi^i, \eta^\alpha$  are some smooth function with variables  $x, u$ .

**Theorem 1.** [10] (The Infinitesimal Invariant Formula for NLPDEs) If

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \quad (4)$$

is the infinitesimal generator of the one-parameter Lie group of transformations for (3), and the  $k$ -th prolongation of the infinitesimal generator is

$$X^{(k)} = X + \eta_i^{(1)\alpha} \frac{\partial}{\partial u_i^\alpha} + \dots + \eta_{i_1, i_2, \dots, i_k}^{(k)\alpha} \frac{\partial}{\partial u_{i_1, i_2, \dots, i_k}^\alpha} \quad (5)$$

where the prolongation of the infinitesimals satisfy the following recurrence relation

$$\begin{aligned} \eta_i^{(1)\alpha} &= D_i \eta^\alpha - (D_i \xi^j) u_j^\alpha, i = 1, 2, \dots, n, \\ \eta_{i_1, i_2, \dots, i_k}^{(k)\alpha} &= D_{i_k} \eta_{i_1, i_2, \dots, i_{k-1}}^{(k-1)\alpha} - (D_{i_k} \xi^j) u_{i_1, i_2, \dots, i_{k-1}, j}^\alpha, i_k = 1, 2, \dots, k (k \geq 2) \end{aligned}$$

where  $D_i$  denotes the total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots + u_{i_1 i_2 \dots i_j}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \dots i_j}^\alpha}, i = 1, 2, \dots, n.$$

That one-parameter Lie group of transformations (3) is the Lie symmetry of Equation (2), if and only if

$$X^{(k)} f^\alpha(x, u, u_{(1)}, \dots, u_{(k)}) \Big|_{f^\alpha(x, u, u_{(1)}, \dots, u_{(k)})=0} = 0.$$

Next, we calculate the Lie symmetry of Equation (1). With regard to the infinitesimal generator of Equation (1), it can be expressed from (4) as the following form

$$X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} + \phi \frac{\partial}{\partial v}. \quad (6)$$

Applying the Theorem 1 to Equation (1), we have

$$\begin{cases} X^{(3)} \left[ u_t + v_x + \frac{1}{2}(u^2)_x \right] \Big|_{u_t + v_x + \frac{1}{2}(u^2)_x = 0, v_t + (uv + u + u_{xx})_x = 0} = 0, \\ X^{(3)} \left[ v_t + (uv + u + u_{xx})_x \right] \Big|_{u_t + v_x + \frac{1}{2}(u^2)_x = 0, v_t + (uv + u + u_{xx})_x = 0} = 0. \end{cases} \quad (7)$$

By simplifying (7), we can get the following overdetermined equations about  $\xi, \tau, \eta, \phi$

$$\begin{cases} \xi_u = \xi_v = 0, \tau_x = \tau_u = \tau_v = 0, \eta_x = \eta_t = \eta_v = 0, \\ \phi_x = \phi_t = \phi_u = 0, 2\eta_u + 2v\eta_u - \phi = 0, \\ 2\xi_x + 2v\xi_x + \phi = 0, \tau_t + v\tau_t + \phi = 0, \\ 2\eta + 2v\eta - 2\xi_t - 2v\xi_t - u\phi = 0, \phi - \phi_v - v\phi_v = 0. \end{cases} \quad (8)$$

From (8) it is easy to calculate that the only solution of this system is

$$\xi = k_4 x + k_3 t + k_1, \tau = 2k_4 t + k_2, \eta = -k_4 u + k_3, \phi = -2k_4 v - 2k_4, \quad (9)$$

where  $k_1, k_2$  and  $k_3$  are arbitrary constants. Accordingly, the symmetry groups of Equation (1) can be written as

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, X_4 = x \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - 2(v+1) \frac{\partial}{\partial v}. \quad (10)$$

The infinitesimal generators (10) correspond to a four-parameter Lie group of nontrivial point transformations acting on  $(x, t, u, v)$ -space.

### 3. Optimal System and Symmetry Reductions

#### 3.1. Optimal System

In this section, we study how to construct the one-dimensional optimal system of Equation (1) in order to obtain more abundant group invariant solutions. The basic method of constructing it is to simplify the expression of Lie algebra by using a variety of adjoint transformations on the most general expression of Lie algebra. The adjoint transformation is expressed as the following series form

$$\begin{aligned} Ad(\exp(\varepsilon X_i)) X_j \\ = X_j - \varepsilon [X_i, X_j] + \frac{\varepsilon^2}{2} [X_i, [X_i, X_j]] - \frac{\varepsilon^3}{3!} [X_i, [X_i, [X_i, X_j]]] + \dots, \end{aligned}$$

where  $\varepsilon$  is a parameter, and  $[X_i, X_j]$  is the usual commutator, given by

$$[X_i, X_j] = X_i X_j - X_j X_i.$$

Hence we can get the following commutator **Table 1** and the adjoint representation **Table 2**.

According to the method of constructing one dimensional optimal system in [11], we set up the following non-zero vector field with arbitrary coefficients  $a_1, a_2, a_3$  and  $a_4$ , which is a Lie algebras made up of (10)

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4,$$

and simplify the coefficients of the vector as much as possible. Without loss of generality, suppose first that  $a_4 \neq 0$  and set up  $a_4 = 1$ , then the vector  $X$  becomes  $X = a_1 X_1 + a_2 X_2 + a_3 X_3 + X_4$ . To eliminate the coefficient of  $X_1$ , we use  $X_1$  to act on  $X$  by means of the adjoint operation, *i.e.*

$$X' = Ad(\exp(\varepsilon_1 X_1)) X = a_2 X_2 + a_3 X_3 + X_4,$$

where the group parameter  $\varepsilon_1 = a_1$ . Then continue to eliminate  $X_2, X_3$  by using one after another  $X_2, X_3$  to act on  $X'$ , the vector becomes

$$X'' = Ad(\exp(\varepsilon_3 X_3)) Ad(\exp(\varepsilon_2 X_2)) X' = X_4,$$

where the group parameters  $\varepsilon_2 = a_2/2, \varepsilon_3 = -a_3$ . It can be seen easily that the vector form can not be simplified much more. Secondly, suppose that  $a_4 = 0, a_3 \neq 0$  and set up  $a_3 = 1$ , the vector  $X$  becomes  $X = a_1 X_1 + a_2 X_2 + X_3$ . To eliminate the coefficient of the vector  $X_1$  we use  $X_2$  to act on  $X$  by means of the adjoint operation, *i.e.*

$$X''' = Ad(\exp(\varepsilon_4 X_2)) X = a_2 X_2 + X_3,$$

**Table 1.** Commutator table.

$[X_i, X_j]$	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	0	0	0	$X_1$
$X_2$	0	0	$X_1$	$2X_2$
$X_3$	0	$-X_1$	0	$-X_3$
$X_4$	$-X_1$	$-2X_2$	$X_3$	0

**Table 2.** Adjoint representation table.

$Ad$	$X_1$	$X_2$	$X_3$	$X_4$
$X_1$	$X_1$	$X_2$	$X_3$	$X_4 - \varepsilon X_1$
$X_2$	$X_1$	$X_2$	$X_3 - \varepsilon X_1$	$X_4 - 2\varepsilon X_2$
$X_3$	$X_1$	$X_2 + \varepsilon X_1$	$X_3$	$X_4 + \varepsilon X_3$
$X_4$	$X_1 e^\varepsilon$	$X_2 e^{2\varepsilon}$	$X_3 e^{-\varepsilon}$	$X_4$

where the group parameter  $\varepsilon_4 = a_1$ . Obviously, it can not continue to simplify by using adjoint operators. Thirdly, suppose that  $a_4 = 0, a_3 = 0, a_2 \neq 0$  and set up  $a_2 = 1$ , the vector is already the simplest form as  $X = a_1 X_1 + X_2$ . Last suppose that  $a_4 = 0, a_3 = 0, a_2 \neq 0, a_1 \neq 0$  and set up  $a_1 = 1$ , that can only be  $X = X_1$ .

To summarize, we state the result that the one-dimensional optimal system of symmetry groups (10) is

$$\{X_1, X_2, X_3, X_4, aX_1 + X_2, aX_2 + X_3\}$$

where  $a$  is arbitrary constant.

### 3.2. Symmetry Reductions

In the present section, we present all possible similarity reduction forms of Equation (1), which is an indispensable step to solve the NLPDEs by the symmetry method.

For the symmetry  $X_4$ , the corresponding characteristic equation is

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{-u} = \frac{dv}{-2(v+1)}, \quad (11)$$

hence we can get a similarity independent variable from (11) defined as  $\zeta = xt^{-1/2}$  and group invariant solutions defined as

$$u(x, t) = t^{-1/2} F(\zeta), v(x, t) = -1 + t^{-1} H(\zeta),$$

which satisfy the following reduced equation

$$\begin{cases} F + \zeta F' - 2FF' - 2H' = 0, \\ 2H - 2HF' + \zeta H' - 2FH' - 2F^{(3)} = 0, \end{cases}$$

where  $F' = \frac{dF}{d\zeta}, H' = \frac{dH}{d\zeta}$ .

For other symmetries in the optimal system, the reduction method is the same as  $X_4$ . The results are shown in **Table 3**.

#### 4. Explicit Solution of the Dispersive Long Wave Equation

In the third section, we obtain the one-dimensional optimal system of Equation (1), and give the reduction equation corresponding to each symmetry in the optimal system in Table 3. The reduction equations corresponding to  $X_1$  and  $X_3$  can be easily solved by Mathematica, where the process is omitted. For other symmetries in the optimal system, it is very difficult to get directly through the calculation software. In this connection, we will use two methods to solve the rest of reduction equations, namely, the power series method and the extended Tanh function method.

##### 4.1. Explicit Power Series Solutions of the Reduction Equation (A)

The power series method is a useful approach to solve higher order ordinary differential equations. A large number of solutions for ordinary differential equations can be constructed by utilizing the method.

Suppose that the power series solution is the following form

$$F(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^n, H(\zeta) = \sum_{n=0}^{\infty} s_n \zeta^n, \quad (12)$$

where  $c_n, s_n$  is undetermined coefficient. Substituting (12) into (A), we get

$$\begin{cases} \sum_{n=1}^{\infty} c_n \zeta^n + c_0 + \sum_{n=1}^{\infty} n c_n \zeta^n - 2 \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k c_{n+1-k} \zeta^n \\ - 2c_0 c_1 - 2 \sum_{n=1}^{\infty} (n+1) s_{n+1} \zeta^n - 2s_1 = 0, \\ 2 \sum_{n=1}^{\infty} s_n \zeta^n + 2s_0 - 2 \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \zeta^n - 2s_0 c_1 \\ + \sum_{n=1}^{\infty} n s_n \zeta^n - 2 \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \zeta^n - 2c_0 s_1 \\ - 2 \sum_{n=1}^{\infty} (n+1)(n+2)(n+3) c_{n+3} \zeta^n - 12c_3 = 0. \end{cases}$$

Through comparing the coefficients of  $\zeta$ , we can easily get the following results when  $n = 0$ ,

$$s_1 = \frac{c_0 - 2c_0 c_1}{2}, c_3 = \frac{s_0 - s_0 c_1 - c_0 s_1}{6}, \quad (13)$$

when  $n \geq 1$ ,

$$\begin{aligned} s_{n+1} &= \frac{1}{2(n+1)} \left[ (n+1)c_n - 2 \sum_{k=0}^n (n+1-k) c_k c_{n+1-k} \right], \\ c_{n+3} &= \frac{1}{2(n+1)(n+2)(n+3)} \left[ (2+n)s_n - 2 \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \right. \\ &\quad \left. - 2 \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \right]. \end{aligned} \quad (14)$$

The sequence  $\{s_n\}_1^{\infty}, \{c_n\}_3^{\infty}$  can be uniquely determined by (13) and (14) and depend on the other undetermined coefficients  $s_0, c_i (i = 0, 1, 2)$ . It is easy to

**Table 3.** Reduction of the nonlinear long wave equation.

Infinitesimal generator	Similarity variables	Reduction equation
$X_1$	$\zeta = t,$ $u(x, t) = F(\zeta), v(x, t) = H(\zeta).$	$F'' = 0,$ $H'' = 0.$
$X_2$	$\zeta = x,$ $u(x, t) = F(\zeta), v(x, t) = H(\zeta).$	$FF' + H' = 0,$ $F' + HF' + FH' + F^{(3)} = 0. \quad (D)$
$X_3$	$\zeta = t,$ $u(x, t) = x/t + F(\zeta), v(x, t) = H(\zeta).$	$F + \zeta F' = 0,$ $H + \zeta H' + 1 = 0.$
$X_4$	$\zeta = xt^{-1/2},$ $u(x, t) = t^{-1/2}F(\zeta), v(x, t) = -1 + t^{-1}H(\zeta).$	$F + \zeta F' - 2FF' - 2H' = 0,$ $2H - 2HF' + \zeta H' - 2FH' - 2F^{(3)} = 0. \quad (A)$
$X_1 + X_2$	$\zeta = x - t,$ $u(x, t) = F(\zeta), v(x, t) = H(\zeta).$	$-F' + FF' + H' = 0,$ $F' + HF' - H' + FH' + F^{(3)} = 0. \quad (B)$
$X_2 + X_3$	$\zeta = -t^2 + 2x/2,$ $u(x, t) = t + F(\zeta), v(x, t) = H(\zeta).$	$1 + FF' + H' = 0,$ $F' + HF' + FH' + F^{(3)} = 0. \quad (C)$

prove that the power series solution is convergent by references [13], so the reduction Equation (A) has the following power series solution

$$\begin{aligned}
 F(\zeta) &= c_0 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \sum_{n=1}^{\infty} c_{n+3}\zeta^{n+3} \\
 &= c_0 + c_1\zeta + c_2\zeta^2 + \frac{s_0 - s_0c_1 - c_0s_1}{6}\zeta^3 \\
 &\quad + \frac{1}{2(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \left( 2s_n + ns_n - 2 \sum_{k=0}^n (n+1-k)s_k c_{n+1-k} \right. \\
 &\quad \left. - 2 \sum_{k=0}^n (n+1-k)c_k s_{n+1-k} \right) \zeta^{n+3}, \\
 H(\zeta) &= s_0 + s_1\zeta + \sum_{n=1}^{\infty} s_{n+1}\zeta^{n+1} \\
 &= s_0 + \frac{c_0 - 2c_0c_1}{2}\zeta + \frac{1}{2(n+1)} \sum_{n=1}^{\infty} \left( c_n + nc_n - 2 \sum_{k=0}^n (n+1-k)c_k c_{n+1-k} \right) \zeta^{n+1}.
 \end{aligned}$$

And then we get the following power series solution of Equation (1)

$$\begin{aligned}
 u(x, t) &= t^{-1/2} \left[ c_0 + c_1 \left( xt^{-1/2} \right) + c_2 \left( xt^{-1/2} \right)^2 + \frac{s_0 - s_0c_1 - c_0s_1}{6} \left( xt^{-1/2} \right)^3 \right. \\
 &\quad \left. + \frac{1}{2(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \left( 2s_n + ns_n - 2 \sum_{k=0}^n (n+1-k)s_k c_{n+1-k} \right. \right. \\
 &\quad \left. \left. - 2 \sum_{k=0}^n (n+1-k)c_k s_{n+1-k} \right) \left( xt^{-1/2} \right)^{n+3} \right], \\
 v(x, t) &= -1 + t^{-1} \left[ s_0 + \frac{c_0 - 2c_0c_1}{2} \left( xt^{-1/2} \right) \right. \\
 &\quad \left. + \frac{1}{2(n+1)} \sum_{n=1}^{\infty} \left( (n+1)c_n - 2 \sum_{k=0}^n (n+1-k)c_k c_{n+1-k} \right) \left( xt^{-1/2} \right)^{n+1} \right],
 \end{aligned}$$

where  $s_0, c_n (n = 0, 1, 2)$  are arbitrary constant.

## 4.2. Explicit Solutions of the Reduction Equation (B) Using Extended Tanh Function Method

The extended Tanh function method is a very effective method for solving some nonlinear evolution equations proposed in recent years [33]. The method is based on the Tanh function expansion method and using the general Riccati equation as an auxiliary equation. It can transform the solution of complex equations into the solution of nonlinear algebraic equations by traveling wave transformation. Next we use this method to find the traveling wave solutions of the reduction Equation (B) in Table 3.

Suppose that the solution of the reduction Equation (B) can be expressed as the form

$$\begin{cases} F(\varsigma) = a_0 + \sum_{i=1}^m (a_i \phi^i(\varsigma) + A_i \phi^{-i}(\varsigma)), \\ H(\varsigma) = b_0 + \sum_{j=1}^n (b_j \phi^j(\varsigma) + B_j \phi^{-j}(\varsigma)), \end{cases} \quad (15)$$

where  $a_0, b_0, a_i, A_i (i=1, \dots, m), b_j, B_j (j=0, 1, \dots, n)$  are undetermined constants, and function  $\phi = \phi(\varsigma)$  satisfies

$$\phi' = \lambda + \rho\phi + \omega\phi^2, \quad (16)$$

where  $\lambda, \rho, \omega$  are arbitrary constant. By solving Equation (16), we can know that the solution of function  $\phi$  can be divided into 4 categories, and amount to 27 solutions [22].

1) when  $\rho^2 - 4\lambda\omega > 0$  and  $\rho\omega \neq 0$  (or  $\lambda\omega \neq 0$ ),

$$\begin{aligned} \phi_1 &= -\frac{1}{2\omega} \left( \rho + \sqrt{\theta} \tanh \left[ \frac{\sqrt{\theta}}{2} \varsigma \right] \right), \\ \phi_2 &= -\frac{1}{2\omega} \left( \rho + \sqrt{\theta} \left( \tanh [\sqrt{\theta} \varsigma] \pm i \operatorname{sech} [\sqrt{\theta} \varsigma] \right) \right), \\ \phi_3 &= -\frac{1}{2\omega} \left( \rho + \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta}}{2} \varsigma \right] \right), \\ \phi_4 &= -\frac{1}{2\omega} \left( \rho + \sqrt{\theta} \left( \coth [\sqrt{\theta} \varsigma] \pm i \operatorname{csch} [\sqrt{\theta} \varsigma] \right) \right), \\ \phi_5 &= -\frac{1}{4\omega} \left[ 2\rho + \sqrt{\theta} \left( \tanh \left[ \frac{\sqrt{\theta}}{4} \varsigma \right] + \coth \left[ \frac{\sqrt{\theta}}{4} \varsigma \right] \right) \right], \\ \phi_6 &= \frac{1}{2\omega} \left( -\rho + \frac{\sqrt{(B^2 - A^2)(\theta)} - A\sqrt{\theta} \cosh [\sqrt{\theta} \varsigma]}{A \sinh [\sqrt{\theta} \varsigma] + B} \right), \\ \phi_7 &= \frac{1}{2\omega} \left( -\rho + \frac{\sqrt{(B^2 - A^2)(\theta)} + A\sqrt{\theta} \sinh [\sqrt{\theta} \varsigma]}{A \cosh [\sqrt{\theta} \varsigma] + B} \right), \end{aligned}$$

where  $A, B$  are two nonzero constants, and satisfy  $B^2 - A^2 > 0$ .



$$\begin{aligned}\phi_8 &= \frac{2\lambda \cosh[\sqrt{\theta}\zeta/2]}{\sqrt{\theta} \sinh[\sqrt{\theta}\zeta/2] - \rho \cosh[\sqrt{\theta}\zeta/2]}, \\ \phi_9 &= \frac{-2\lambda \sinh[\sqrt{\theta}\zeta/2]}{\rho \sinh[\sqrt{\theta}\zeta/2] - \sqrt{\theta} \cosh[\sqrt{\theta}\zeta/2]}, \\ \phi_{10} &= \frac{2\lambda \cosh[\sqrt{\theta}\zeta]}{\sqrt{\theta} \sinh[\sqrt{\theta}\zeta] - \rho \cosh[\sqrt{\theta}\zeta] \pm i\sqrt{\theta}}, \\ \phi_{11} &= \frac{2\lambda \sinh[\sqrt{\theta}\zeta]}{-\rho \sinh[\sqrt{\theta}\zeta] + \sqrt{\theta} \cosh[\sqrt{\theta}\zeta] \pm \sqrt{\theta}}, \\ \phi_{12} &= \frac{4\lambda \sinh[\sqrt{\theta}\zeta/4] \cosh[\sqrt{\theta}\zeta/4]}{-2\rho \sinh[\sqrt{\theta}\zeta/4] \cosh[\sqrt{\theta}\zeta/4] + 2\sqrt{\theta} \cosh^2[\sqrt{\theta}\zeta/4] - \sqrt{\theta}}.\end{aligned}$$

2) when  $\rho^2 - 4\lambda\omega < 0$  and  $\rho\omega \neq 0$  (or  $\lambda\omega \neq 0$ ),

$$\begin{aligned}\phi_{13} &= \frac{1}{2\omega} \left( -\rho + \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}}{2} \zeta \right] \right), \\ \phi_{14} &= \frac{1}{2\omega} \left[ -\rho + \sqrt{-\theta} \left( \tan[\sqrt{-\theta}\zeta] \pm \sec[\sqrt{-\theta}\zeta] \right) \right], \\ \phi_{15} &= -\frac{1}{2\omega} \left( \rho + \sqrt{-\theta} \cot \left[ \frac{\sqrt{-\theta}}{2} \zeta \right] \right), \\ \phi_{16} &= -\frac{1}{2\omega} \left( \rho + \sqrt{-\theta} \left( \cot[\sqrt{-\theta}\zeta] \pm \csc[\sqrt{-\theta}\zeta] \right) \right), \\ \phi_{17} &= \frac{1}{4\omega} \left[ -2\rho + \sqrt{-\theta} \left( \tan \left[ \frac{\sqrt{-\theta}}{4} \zeta \right] - \cot \left[ \frac{\sqrt{-\theta}}{4} \zeta \right] \right) \right], \\ \phi_{18} &= \frac{1}{2\omega} \left( -\rho + \frac{\pm \sqrt{(A^2 - B^2)(-\theta)} - A\sqrt{-\theta} \cos[\sqrt{-\theta}\zeta]}{A \sinh[\sqrt{-\theta}\zeta] + B} \right), \\ \phi_{19} &= \frac{1}{2\omega} \left( -\rho + \frac{\sqrt{(A^2 - B^2)(-\theta)} + A\sqrt{-\theta} \sinh[\sqrt{-\theta}\zeta]}{A \cosh[\sqrt{-\theta}\zeta] + B} \right),\end{aligned}$$

where  $A, B$  are two nonzero constants, and satisfy  $A^2 - B^2 > 0$ .

$$\begin{aligned}\phi_{20} &= \frac{2\lambda \cos[\sqrt{-\theta}\zeta/2]}{\sqrt{-\theta} \sin[\sqrt{-\theta}\zeta/2] + \rho \cos[\sqrt{-\theta}\zeta/2]}, \\ \phi_{21} &= \frac{-2\lambda \sin[\sqrt{-\theta}\zeta/2]}{-\rho \sin[\sqrt{-\theta}\zeta/2] - \sqrt{-\theta} \cos[\sqrt{-\theta}\zeta/2]}, \\ \phi_{22} &= \frac{2\lambda \cos[\sqrt{-\theta}\zeta]}{\sqrt{-\theta} \sin[\sqrt{-\theta}\zeta] + \rho \cos[\sqrt{-\theta}\zeta] \pm \sqrt{-\theta}},\end{aligned}$$

$$\phi_{23} = \frac{2\lambda \sin[\sqrt{-\theta}\zeta]}{-\rho \sin[\sqrt{-\theta}\zeta] + \sqrt{-\theta} \cosh[\sqrt{-\theta}\zeta] \pm \sqrt{-\theta}},$$

$$\phi_{24} = \frac{4\lambda \sin[\sqrt{-\theta}\zeta/4] \cos[\sqrt{-\theta}\zeta/4]}{-2\rho \sin[\sqrt{-\theta}\zeta/4] \cos[\sqrt{-\theta}\zeta/4] + 2\sqrt{-\theta} \cos^2[\sqrt{-\theta}\zeta/4] - \sqrt{-\theta}}.$$

Above formula  $\phi_1 \sim \phi_{24}$ , the symbol  $\theta$  is expressed as  $\theta = \rho^2 - 4\lambda\omega$ .

3) when  $\lambda = 0$  and  $\rho\omega \neq 0$ ,

$$\phi_{25} = \frac{-\rho b}{\omega(b + \cosh[\rho\zeta] - \sinh[\rho\zeta])},$$

$$\phi_{26} = -\frac{\rho(\cosh[\rho\zeta] + \sinh[\rho\zeta])}{\omega(b + \cosh[\rho\zeta] - \sinh[\rho\zeta])},$$

where  $b$  is a arbitrary constant.

4) when  $\omega \neq 0$  and  $\lambda = \rho = 0$ ,

$$\phi_{27} = \frac{1}{\omega\zeta + c},$$

where  $c$  is a arbitrary constant, and  $\zeta = x - t$ .

Considering the homogeneous equilibrium between the highest order linear term and the nonlinear term in the reduction Equation (B), we can obtain  $m=1, n=2$ . As a result, the trial Equations (16) reduces to

$$\begin{cases} F(\zeta) = a_0 + a_1\phi(\zeta) + \frac{A_1}{\phi(\zeta)}, \\ H(\zeta) = b_0 + b_1\phi(\zeta) + b_2\phi^2(\zeta) + \frac{B_1}{\phi(\zeta)} + \frac{B_2}{\phi^2(\zeta)}. \end{cases} \quad (17)$$

In order to determine the values of undetermined coefficients  $a_0, a_1, b_0, b_1, b_2$ , substituting (16) and (17) into the reduction Equation (B) and merging the polynomial of the same power of  $\phi$ , and setting up each polynomial coefficient to zero, we can get the following nonlinear algebraic equations

$$\begin{cases} \phi^0: -\lambda A_1^2 - 2\lambda B_2 = 0, -6\lambda^3 A_1 - 3\lambda A_1 B_2 = 0 \\ \phi^1: -\lambda A_1 - \lambda a_0 A_1 - \rho A_1^2 - \lambda B_1 - 2\rho B_2 = 0, \\ \quad -12\lambda^2 \rho A_1 - 2\lambda A_1 B_1 + 2\lambda B_2 - 2\lambda a_0 B_2 - 3\rho A_1 B_2 = 0 \\ \phi^2: \rho A_1 - \rho a_0 A_1 - \omega A_1^2 - \rho B_1 - 2\omega B_2 = 0, \\ \quad -\lambda A_1 - 7\lambda \rho^2 A_1 - 8\lambda^2 \omega A_1 - \lambda A_1 b_0 + \lambda B_1 - \lambda a_0 B_1 \\ \quad - 2\rho A_1 B_1 + 2\rho B_2 - 2\rho a_0 B_2 - \lambda a_1 B_2 - 3\omega A_1 B_2 = 0 \\ \quad \vdots \\ \phi^6: \omega a_1^2 + 2\omega b_2 = 0, \\ \quad \omega a_1 + 7\rho^2 \omega a_1 + 8\lambda \omega^2 a_1 + \omega a_1 b_0 - \omega b_1 + \omega a_0 b_1 + 2\rho a_1 b_1 \\ \quad - 2\rho b_2 + 2\rho a_0 b_2 + 3\lambda a_1 b_2 + \omega A_1 b_2 = 0 \\ \phi^7: 0 = 0, 12\rho \omega^2 a_1 + 2\omega a_1 b_1 - 2\omega b_2 + 2\omega a_0 b_2 + 3\rho a_1 b_2 = 0 \\ \phi^8: 0 = 0, 6\omega^3 a_1 + 3\omega a_1 b_2 = 0 \end{cases}$$

By solving the above system with the help of Mathematic, we can get the following results

$$\begin{aligned} a_0 &= 1 \pm \rho, a_1 = \pm 2\omega, A_1 = \pm 2\lambda, \\ b_0 &= -1, b_1 = -2\rho\omega, b_2 = -2\omega^2, B_1 = -2\lambda\rho, B_2 = -2\lambda^2. \end{aligned} \quad (18)$$

Now, substituting (18) into (19), we obtain explicit solutions of Equation (1) as follow.

$$\begin{cases} u_k(x, t) = 1 \pm \rho \pm 2\omega\phi_k(\varsigma) \pm \frac{2\lambda}{\phi_k(\varsigma)}, \\ v_k(x, t) = -1 - 2\rho\omega\phi_k(\varsigma) - 2\omega^2\phi_k^2(\varsigma) - \frac{2\lambda\rho}{\phi_k(\varsigma)} - \frac{2\lambda^2}{\phi_k^2(\varsigma)}, \end{cases} \quad (19)$$

where  $k = 1, 2, \dots, 27, \varsigma = x - t$ , and selecting any hyperbolic function in  $\phi_1 \sim \phi_{27}$ , for example,

$$\phi(\varsigma) = \frac{1}{2\omega} \left( -\rho + \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}}{2} \varsigma \right] \right).$$

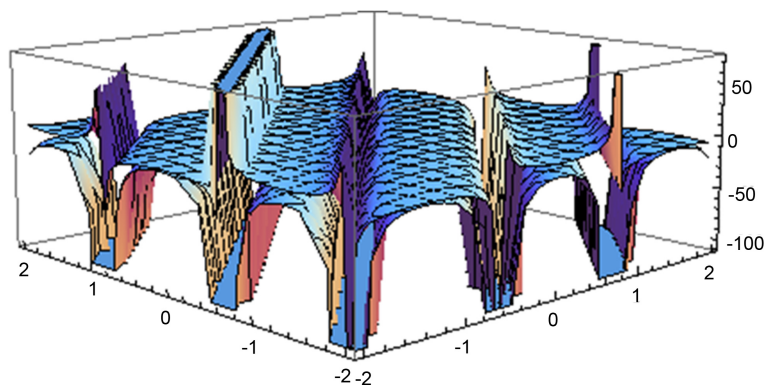
The explicit solutions (19) become as

$$\begin{cases} u(x, t) = 1 \pm \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}\varsigma}{2} \right] \mp \frac{4\lambda\omega}{\rho - \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}\varsigma}{2} \right]}, \\ v(x, t) = -1 + \frac{\rho^2}{2} + \frac{1}{2}\theta \tan^2 \left[ \frac{\sqrt{-\theta}\varsigma}{2} \right] - \frac{8\lambda^2\omega^2}{\left( -\rho + \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}\varsigma}{2} \right] \right)^2} \\ \quad - \frac{4\lambda\rho\omega}{-\rho + \sqrt{-\theta} \tan \left[ \frac{\sqrt{-\theta}\varsigma}{2} \right]}, \end{cases}$$

where  $\varsigma = x - t, \theta = \rho^2 - 4\lambda\omega < 0$  (see **Figure 1**).

### 4.3. Explicit Power Series Solutions of the Reduction Equation (C)

In this section, we study the power series solution of the reduction Equation (C)



**Figure 1.** Plot of explicit solution above with  $\lambda = 4, \rho = -0.2, \omega = 0.3$ .

in the form of (12). Substituting (12) into the reduction Equation (C), we get

$$\left\{ \begin{aligned} & 1 + \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k c_{n+1-k} \varsigma^n + c_0 c_1 + \sum_{n=1}^{\infty} (n+1) s_{n+1} \varsigma^n + s_1 = 0, \\ & \sum_{n=1}^{\infty} (n+1) c_{n+1} \varsigma^n + c_1 + \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \varsigma^n \\ & + s_0 c_1 + \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \varsigma^n + c_0 s_1 \\ & + \sum_{n=1}^{\infty} (n+1)(n+2)(n+3) c_{n+3} \varsigma^n + 6c_3 = 0. \end{aligned} \right.$$

Through comparing the coefficients of  $\varsigma$ , we can easily get the following results.

where  $n = 0$ ,

$$s_1 = -1 - c_0 c_1, c_3 = -\frac{c_1 + s_0 c_1 + c_0 s_1}{6},$$

where  $n \geq 1$ ,

$$\begin{aligned} s_{n+1} &= -\frac{1}{(n+1)} \sum_{k=0}^n (n+1-k) c_k c_{n+1-k}, \\ c_{n+3} &= -\frac{1}{(n+1)(n+2)(n+3)} \left[ (n+1) c_{n+1} + \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \right. \\ & \quad \left. + \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \right]. \end{aligned}$$

Accordingly, the power series solution of the reduction Equation (C) is as follows

$$\begin{aligned} F(\varsigma) &= c_0 + c_1 \varsigma + c_2 \varsigma^2 - \frac{c_1 + s_0 c_1 + c_0 s_1}{6} \varsigma^3 \\ & \quad - \frac{1}{(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \left[ (n+1) c_{n+1} + \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \right. \\ & \quad \left. + \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \right] \varsigma^{n+3}, \\ H(\varsigma) &= s_0 - (1 + c_0 c_1) \varsigma - \frac{1}{(n+1)} \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k c_{n+1-k} \varsigma^{n+1}. \end{aligned}$$

And then we get the following power series solution of Equation (1)

$$\begin{aligned} u(x, t) &= t + c_0 + c_1 \left( \frac{-t^2 + 2x}{2} \right) + c_2 \left( \frac{-t^2 + 2x}{2} \right)^2 \\ & \quad - \frac{c_1 + s_0 c_1 + c_0 s_1}{6} \left( \frac{-t^2 + 2x}{2} \right)^3 \\ & \quad - \frac{1}{(n+1)(n+2)(n+3)} \sum_{n=1}^{\infty} \left[ (n+1) c_{n+1} + \sum_{k=0}^n (n+1-k) s_k c_{n+1-k} \right. \\ & \quad \left. + \sum_{k=0}^n (n+1-k) c_k s_{n+1-k} \right] \left( \frac{-t^2 + 2x}{2} \right)^{n+3}, \end{aligned}$$

$$v(x, t) = s_0 - (1 + c_0 c_1) \left( \frac{-t^2 + 2x}{2} \right) - \frac{1}{(n+1)} \sum_{n=1}^{\infty} \sum_{k=0}^n (n+1-k) c_k c_{n+1-k} \left( \frac{-t^2 + 2x}{2} \right)^{n+1}.$$

#### 4.4. Explicit Solutions of the Reduction Equation (D) Using Extended Tanh Function Method

Using extended tanh function method, similar to the solving of the reduction Equation (B), we obtain the following results:

$$a_0 = \pm \rho, a_1 = \pm 2\omega, A_1 = \pm 2\lambda, b_0 = -1, b_1 = -2\rho\omega, \\ b_2 = -2\omega^2, B_1 = -2\lambda\rho, B_2 = -2\lambda^2.$$

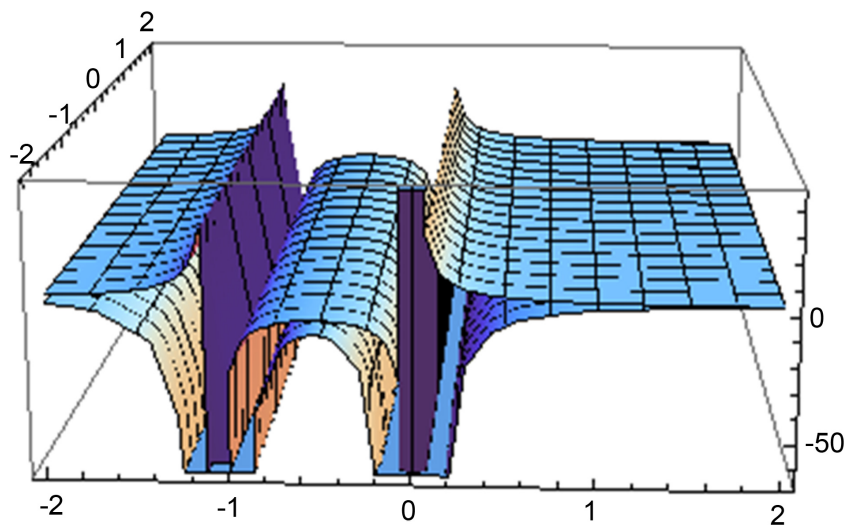
and selecting the following hyperbolic function

$$\phi = -\frac{1}{2\omega} \left( \rho + \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta}}{2} \varsigma \right] \right)$$

We obtain explicit solutions of the Equation (1.1) as follow

$$\begin{cases} u(x, t) = \mp \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta} \varsigma}{2} \right] \mp \frac{4\lambda\omega}{\rho - \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta} \varsigma}{2} \right]}, \\ v(x, t) = -1 + \frac{\rho^2}{2} - \frac{1}{2} \theta \coth^2 \left[ \frac{\sqrt{\theta} \varsigma}{2} \right] - \frac{8\lambda^2 \omega^2}{\left( -\rho + \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta} \varsigma}{2} \right] \right)^2} \\ + \frac{4\lambda\rho\omega}{\rho + \sqrt{\theta} \coth \left[ \frac{\sqrt{\theta} \varsigma}{2} \right]}, \end{cases}$$

where  $\varsigma = x - t, \theta = \rho^2 - 4\lambda\omega < 0$  (see **Figure 2**).



**Figure 2.** Plot of explicit solution above with  $\lambda = 4, \rho = 2, \omega = 0.2$ .

## 5. Conclusion

In the field of physics and engineering mechanics, it is particularly important to solve nonlinear differential equations. In the work, the Lie group analysis method has been employed to investigate the dispersive long wave equations. Based on this method, the vector fields and symmetry reductions have been obtained for the system. Since it is difficult to solve the reduction equations directly, the power series method and the extended Tanh function method have been used to construct more explicit solutions, which can enrich the exact solutions of the dispersive long wave equations. The basic idea is efficient and powerful in solving wide classes of nonlinear differential equations.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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