Integrability of Hamiltonian Systems with Two Degrees of Freedom and Homogenous Potential of Degree Zero

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Abstract

We provide necessary conditions in order that the Hamiltonian systems with Hamiltonian $H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V\left(q_1, q_2\right)$, and one of the following potentials

$$V_1 = \frac{a_0 q_1 + a_1 q_2}{a_2 q_1 + a_3 q_2},$$

$$V_2 = \frac{a_0 q_1^2 + a_1 q_1 q_2 + a_2 q_2^2}{a_3 q_1^2 + a_4 q_1 q_2 + a_5 q_2^2},$$

$$V_3 = \frac{a_0 q_2^n + a_1 q_2^{n-1} q_1 + \cdots + a_n q_1^n}{q_1^n},$$

are integrable in the Liouville sense.

Keywords

Hamiltonian System, Liouville Integrability, Darboux Points, Homogeneous Potentials of Degree Zero

1. Introduction

In this paper we consider classical Hamiltonian systems with two degrees of freedom for which the Hamiltonian is of the form

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V\left(q_1, q_2\right), \tag{1}$$

where V is a homogeneous function of degree 0. Although these systems arising from physics and applied sciences are generally understood to involve only real variables, we will assume that the Hamiltonian system associated to (1) is

defined on the complex symplectic manifold $M = \mathbb{C}^4 \setminus \Sigma$ equipped with the canonical symplectic form

$$\omega = \sum_{j=1}^{2} dq_j \wedge dp_j,$$

where Σ is a set of zero Lebesgue measure.

We will assume that the system of equations

$$\mathrm{grad} V(q_1, q_2) = V'(q_1, q_2) = (q_1, q_2),$$

has a nonzero solution $d = (d_1, d_2) \in \mathbb{C}^2$, called a *Darboux point* of the potential *V*.

A Hamiltonian system of two degrees of freedom is *integrable in the Liouville sense* if there exists a first integral *F*, defined in $\mathbb{C}^4$ except perhaps in a set of zero Lebesgue measure, which is independent with the Hamiltonian, *i.e.* the rank of the matrix

$$\begin{pmatrix} \dfrac{\partial H}{\partial q_1} & \dfrac{\partial H}{\partial q_2} & \dfrac{\partial H}{\partial p_1} & \dfrac{\partial H}{\partial p_2} \\ \dfrac{\partial F}{\partial q_1} & \dfrac{\partial F}{\partial q_2} & \dfrac{\partial F}{\partial p_1} & \dfrac{\partial F}{\partial p_2} \end{pmatrix}$$

is 2, except perhaps in a zero Lebesgue measure set of $\mathbb{C}^4$.

The aim of this paper is to find necessary conditions for the integrability of some homogeneous potentials of degree of homogeneity 0. More precisely, our main results are the following.

Proposition 1 *Assume that a Hamiltonian system of two degrees of freedom with Hamiltonian of the form* (1) *and potential*

$$V = \frac{a_0 q_1 + a_1 q_2}{a_2 q_1 + a_3 q_2}, \tag{2}$$

where $a_j = b_{j,1} + i b_{j,2} \in \mathbb{C}$ *for* $j = 0,1,2,3$, *is integrable in the Liouville sense, and that it has Darboux points. If* $a_3 \neq 0$, *then*

$$\begin{aligned} b_{01} &= \frac{b_{11}(b_{21}b_{31} + b_{22}b_{32})}{b_{31}^2 + b_{32}^2} - \frac{b_{12}(b_{22}b_{31} - b_{21}b_{32})}{b_{31}^2 + b_{32}^2}, \\ b_{02} &= -\frac{b_{11}(b_{21}b_{32} - b_{22}b_{31})}{b_{31}^2 + b_{32}^2} + \frac{b_{12}(b_{21}b_{31} + b_{22}b_{32})}{b_{31}^2 + b_{32}^2}. \end{aligned} \tag{3}$$

If $a_3 = 0$ *is a particular case of Proposition* 3.

Proposition 2 *Assume that a Hamiltonian system of two degrees of freedom with Hamiltonian of the form* (1) *and potential*

$$V = \frac{a_0 q_1^2 + a_1 q_1 q_2 + a_2 q_2^2}{a_3 q_1^2 + a_4 q_1 q_2 + a_5 q_2^2}, \tag{4}$$

where $a_j = b_{i,1} + i b_{j,2} \in \mathbb{C}$ *for* $j = 0, \cdots, 5$, *is integrable in the Liouville sense, that it has Darboux points. If* $a_5 \neq 0$, *then*

$$b_{01} = \frac{b_{21}b_{31}b_{51} + b_{21}b_{32}b_{52} + b_{22}b_{31}b_{52} - b_{22}b_{32}b_{51}}{b_{51}^2 + b_{52}^2},$$

$$b_{02} = -\frac{b_{21}b_{31}b_{52} - b_{21}b_{32}b_{51} - b_{22}b_{31}b_{51} - b_{22}b_{32}b_{52}}{b_{51}^2 + b_{52}^2},$$

$$b_{11} = \frac{b_{21}b_{41}b_{51} + b_{21}b_{42}b_{52} + b_{22}b_{41}b_{52} - b_{22}b_{42}b_{51}}{b_{51}^2 + b_{52}^2},$$ (5)

$$b_{12} = \frac{-b_{21}b_{41}b_{52} + b_{21}b_{42}b_{51} + b_{22}b_{41}b_{51} + b_{22}b_{42}b_{52}}{b_{51}^2 + b_{52}^2}.$$

If $a_5 = 0$, then

$$b_{01} = b_{21}, \quad b_{02} = b_{22} \quad \text{and} \quad b_{11} = b_{12} = 0. \tag{6}$$

Proposition 3 *If a Hamiltonian system of two degrees of freedom with Hamiltonian of the form* (1) *and potential*

$$V = \frac{a_0 q_2^n + a_1 q_2^{n-1} q_1 + \cdots + a_n q_1^n}{q_1^n}, \tag{7}$$

where $a_j = b_{j,1} + ib_{j,2} \in \mathbb{C}$ for $j = 0, \cdots, n$, is integrable in the Liouville sense and has Darboux points, then it satisfies

$$b_{1,1} = -\sum_{k=1}^{[(n-1)/2]} b_{2k+1,1}(2k+1)^2(-1)^k, \quad b_{2,1} = \sum_{k=2}^{[n/2]} b_{2k,1}k^2(-1)^k,$$

$$b_{1,2} = -\sum_{k=1}^{[(n-1)/2]} b_{2k+1,2}(2k+1)^2(-1)^k, \quad b_{2,2} = \sum_{k=2}^{[n/2]} b_{2k,2}k^2(-1)^k. \tag{8}$$

These three propositions are proved in Section 3. The proof of Proposition 2 needs the help of an algebraic manipulator like mathematica for doing it, and a such manipulator also helps in the proof of Proposition 1 but there is not strictly necessary.

The integrability of other Hamiltonian systems with two degrees of freedom and different potentials have been studied in [1]-[30].

2. Preliminary Results

It was proved in Theorem 1.2 of [31] the following result for Hamiltonian systems with n degrees of freedom of the form (1) with homogeneous potentials of degree 0.

Theorem 4 *Assume that $V \in \mathbb{C}[q_1, \cdots, q_n]$ is homogeneous of degree 0 and that the following conditions are satisfied:*

1) *there exists a non-zero $d = (d_1, \cdots, d_n) \in \mathbb{C}^n$ such that $V'(d_1, \cdots, d_n) = (d_1, \cdots, d_n)$, and*

2) *the system is integrable in the Liouville sense with rational first integrals. Then*

1) *all eigenvalues of the Hessian matrix $V''(d_1, \cdots, d_n)$ are integers; and*

2) *the matrix $V''(d_1, \cdots, d_n)$ is diagonalizable.*

When $n = 2$ Theorem 4 has the following easier formulation given in section 5 of [31]

Theorem 5 *Assume that* $V \in \mathbb{C}[q_1, q_2]$ *is homogeneous of degree* 0 *and that the following conditions are satisfied*:

1) *there exists a non-zero* $d = (d_1, d_2) \in \mathbb{C}^2$ *such that* $V'(d_1, d_2) = (d_1, d_n)$, *and*

2) *the system is integrable in the Liouville sense with rational first integrals.*

Let $z = q_2/q_1$ *with* $q_1 \neq 0$, *be the affine coordinate on* $\mathbb{C}P^1$ *and set* $v(z) := V(1, z)$. *Then the Darboux points are* $\pm i$, *and the function* v *satisfies*

$$v'(z_*) + z_* v''(z_*) = 0. \tag{9}$$

Proof. Darboux points of V are non-zero solutions of equations

$$\frac{\partial V}{\partial q_1} = q_1, \quad \frac{\partial V}{\partial q_2} = q_2. \tag{10}$$

It is convenient to consider Darboux points in the projective line $\mathbb{C}P^1$. Let $z = q_2/q_1$ with $q_1 \neq 0$, be the affine coordinate on $\mathbb{C}P^1$. Then we can rewrite system (10) in the form

$$v'(z)z = -q_1^2, \quad v'(z) = zq_1^2, \tag{11}$$

where $v(z) := V(1, z)$. From the above formulae it follows that z_* is a Darboux point of V if and only if $z_* \in \{-i, i\}$, and $v'(z_*) \neq 0$. Thus the location of the Darboux points does not depend on the potential.

If z_* is the affine coordinate of the Darboux point $d = (d_1, d_2)$ of V, then the Hessian matrix $V''(d)$ expressed in this coordinate has the form

$$V''(d) = \begin{pmatrix} -v''(z_*)x_*^{-2} - 2 & -[v'(z_*) + z_* v''(z_*)]x_*^{-2} \\ -[v'(z_*) + z_* v''(z_*)]x_*^{-2} & v''(z_*)x_*^{-2} \end{pmatrix},$$

where

$$x_* = -v'(z_*)z_* = v'(z_*)/z_*.$$

Vector d is an eigenvector of $V''(d)$ with corresponding eigenvalue $\lambda = -1$. As the trace of $V''(d)$ is -2, $\lambda = -1$ is the only eigenvalue of $V''(d)$. Thus the first hypothesis of Theorem 4 is satisfied, and the second also by our assumptions, then the matrix $V''(d)$ is diagonalizable, and since its eigenvalues are -1 and -1, we get that $V''(d)$ is diagonal. Hence the second condition of Theorem 4 is satisfied if and only if (9) holds.

3. Proof of the Propositions

Proof of Proposition 1. We will apply Theorem 5 to potential (2). Then it has the Darboux points $\pm i$. Since

$$v(z) = \frac{a_0 + a_1 z}{a_2 + a_3 z},$$

we have

$$v'(z) + z v''(z) = \frac{(a_1 a_2 - a_0 a_3)(a_2 - a_3 z)}{(a_2 + a_3 z)^3}.$$

Taking $a_j = b_{j,1} + i b_{j,2}$ for $j = 0, \cdots, 3$, the condition (9) implies

$$\frac{\left(b_{21}+ib_{22}-i\left(b_{31}+ib_{32}\right)\right)\left(\left(b_{11}+ib_{12}\right)\left(b_{21}+ib_{22}\right)-\left(b_{01}+ib_{02}\right)\left(b_{31}+ib_{32}\right)\right)}{\left(b_{21}+ib_{22}+i\left(b_{31}+ib_{32}\right)\right)^3}=0 \quad (12)$$

and

$$\frac{\left(b_{21}+ib_{22}+i\left(b_{31}+ib_{32}\right)\right)\left(\left(b_{11}+ib_{12}\right)\left(b_{21}+ib_{22}\right)-\left(b_{01}+ib_{02}\right)\left(b_{31}+ib_{32}\right)\right)}{\left(b_{21}+ib_{22}-i\left(b_{31}+ib_{32}\right)\right)^3}=0. \quad (13)$$

Taking the real part and the imaginary part of (12) we get

$$-b_{01}b_{21}^2b_{31}-2b_{01}b_{21}b_{31}b_{32}-b_{01}b_{22}^2b_{31}-2b_{01}b_{22}b_{32}^2+b_{01}b_{31}^3+b_{01}b_{31}b_{32}^2$$
$$+b_{02}b_{21}^2b_{32}-2b_{02}b_{21}b_{31}^2+b_{02}b_{22}^2b_{32}-2b_{02}b_{22}b_{31}b_{32}-b_{02}b_{31}^2b_{32}-b_{02}b_{32}^3$$
$$+b_{11}b_{21}^3+b_{11}b_{21}b_{22}^2+2b_{11}b_{21}b_{22}b_{31}-b_{11}b_{21}b_{31}^2-b_{11}b_{21}b_{32}^2+2b_{11}b_{22}^2b_{32}$$
$$-b_{12}b_{21}^2b_{22}+2b_{12}b_{21}^2b_{31}+2b_{12}b_{21}b_{22}b_{32}-b_{12}b_{22}^3+b_{12}b_{22}b_{31}^2+b_{12}b_{22}b_{32}^2=0$$

and

$$-b_{01}b_{21}^2b_{32}+2b_{01}b_{21}b_{31}^2-b_{01}b_{22}^2b_{32}+2b_{01}b_{22}b_{31}b_{32}+b_{01}b_{31}^2b_{32}+b_{01}b_{32}^3$$
$$-b_{02}b_{21}^2b_{31}-2b_{02}b_{21}b_{31}b_{32}-b_{02}b_{22}^2b_{31}-2b_{02}b_{22}b_{32}^2+b_{02}b_{31}^3+b_{02}b_{31}b_{32}^2$$
$$+b_{11}b_{21}^2b_{22}-2b_{11}b_{21}^2b_{31}-2b_{11}b_{21}b_{22}b_{32}+b_{11}b_{22}^3-b_{11}b_{22}b_{31}^2-b_{11}b_{22}b_{32}^2$$
$$+b_{12}b_{21}^3+b_{12}b_{21}b_{22}^2+2b_{12}b_{21}b_{22}b_{31}-b_{12}b_{21}b_{31}^2-b_{12}b_{21}b_{32}^2+2b_{12}b_{22}^2b_{32}=0.$$

Taking the real part and the imaginary part of (13) we get

$$-b_{01}b_{21}^2b_{31}+2b_{01}b_{21}b_{31}b_{32}-b_{01}b_{22}^2b_{31}+2b_{01}b_{22}b_{32}^2+b_{01}b_{31}^3+b_{01}b_{31}b_{32}^2$$
$$+b_{02}b_{21}^2b_{32}+2b_{02}b_{21}b_{31}^2+b_{02}b_{22}^2b_{32}+2b_{02}b_{22}b_{31}b_{32}-b_{02}b_{31}^2b_{32}-b_{02}b_{32}^3$$
$$+b_{11}b_{21}^3+b_{11}b_{21}b_{22}^2-2b_{11}b_{21}b_{22}b_{31}-b_{11}b_{21}b_{31}^2-b_{11}b_{21}b_{32}^2-2b_{11}b_{22}^2b_{32}$$
$$-b_{12}b_{21}^2b_{22}-2b_{12}b_{21}^2b_{31}-2b_{12}b_{21}b_{22}b_{32}-b_{12}b_{22}^3+b_{12}b_{22}b_{31}^2+b_{12}b_{22}b_{32}^2=0$$

and

$$-b_{01}b_{21}^2b_{32}-2b_{01}b_{21}b_{31}^2-b_{01}b_{22}^2b_{32}-2b_{01}b_{22}b_{31}b_{32}+b_{01}b_{31}^2b_{32}+b_{01}b_{32}^3$$
$$-b_{02}b_{21}^2b_{31}+2b_{02}b_{21}b_{31}b_{32}-b_{02}b_{22}^2b_{31}+2b_{02}b_{22}b_{32}^2+b_{02}b_{31}^3+b_{02}b_{31}b_{32}^2$$
$$+b_{11}b_{21}^2b_{22}+2b_{11}b_{21}^2b_{31}+2b_{11}b_{21}b_{22}b_{32}+b_{11}b_{22}^3-b_{11}b_{22}b_{31}^2-b_{11}b_{22}b_{32}^2$$
$$+b_{12}b_{21}^3+b_{12}b_{21}b_{22}^2-2b_{12}b_{21}b_{22}b_{31}-b_{12}b_{21}b_{31}^2-b_{12}b_{21}b_{32}^2-2b_{12}b_{22}^2b_{32}=0.$$

Solving these four equations we obtain that the condition (3) given in the statement of Proposition 1.

Proof of Proposition 2. We will apply Theorem 5 to potential (4). First assume that $a_5\neq 0$. Therefore this potential has the Darboux points $\pm i$. Since

$$v(z)=\frac{a_0+a_1z+a_2z^2}{a_3+a_4z+a_5z^2},$$

we obtain

$$v'(z)+zv''(z)$$
$$=\frac{1}{\left(a_3+z\left(a_4+a_5z\right)\right)^3}\left(a_0\left(z\left(a_4^2+3a_4a_5z+4a_5^2z^2\right)-a_3\left(a_4+4a_5x\right)\right)\right.$$
$$+a_1\left(a_3^2-a_3z\left(a_4+6a_5z\right)+a_5z^3\left(a_5z-a_4\right)\right)$$
$$\left.+a_2z\left(4a_3^2+a_3z\left(3a_4-4a_5x\right)+a_4z^2\left(a_4-a_5z\right)\right)\right)$$
$$=0.$$

By condition (9) on v and since $a_j = b_{j,1} + ib_{j,2}$ for $j = 0, \cdots, 5$, we get

$$\frac{i}{\left(b_{31} + ib_{32} + i(b_{41} + ib_{42}) - b_{51} - b_{52}\right)^3} \Big((b_{01} + ib_{02}) \big(i(b_{31} + ib_{32})(b_{41} + ib_{42})$$
$$- 4(b_{31} + ib_{32})(b_{51} + ib_{52}) + 3i(b_{41} + ib_{42})(b_{51} + ib_{52}) + (b_{41} + ib_{42})^2$$
$$- 4(b_{51} + ib_{52})^2 \big) + (b_{21} + ib_{22})\big(3i(b_{31} + ib_{32})(b_{41} + ib_{42}) + 4(b_{31} + ib_{32})^2 \quad (14)$$
$$+ 4(b_{31} + ib_{32})(b_{51} + ib_{52}) + i(b_{41} + ib_{42})(b_{51} + ib_{52}) - (b_{41} + ib_{42})^2 \big)$$
$$+ (b_{11} + ib_{12})\big(-i(b_{31} + ib_{32})(b_{41} + ib_{42}) + 6(b_{31} + ib_{32})(b_{51} + ib_{52})$$
$$+ (b_{31} + ib_{32})^2 + i(b_{41} + ib_{42})(b_{51} + ib_{52}) + (b_{51} + ib_{52})^2 \big) \Big) = 0$$

and

$$\frac{1}{\left(b_{31} + ib_{32} - i(b_{41} + ib_{42}) - b_{51} - ib_{52}\right)^3} \Big((b_{11} + ib_{12}) \big(i(b_{31} + ib_{32})(b_{41} + ib_{42})$$
$$+ 6(b_{31} + ib_{32})(b_{51} + ib_{52}) + (b_{31} + ib_{32})^2 - i(b_{41} + ib_{42})(b_{51} + ib_{52})$$
$$+ (b_{51} + ib_{52})^2 \big) - i\big((b_{01} + ib_{02})\big(-i(b_{31} + ib_{32})(b_{41} + ib_{42}) + 4(b_{31} + ib_{32})^2$$
$$- 4(b_{31} + ib_{32})(b_{51} + ib_{52}) - 3i(b_{41} + ib_{42})(b_{51} + ib_{52}) + (b_{41} + ib_{42})^2 \quad (15)$$
$$- 4(b_{51} + ib_{52})^2 \big) + (b_{21} + ib_{22})\big((b_{31} + ib_{32})\big(4(b_{51} + ib_{52}) - 3i(b_{41} + ib_{42}) \big)$$
$$- (b_{41} + ib_{42})\big(b_{41} + ib_{42} + i(b_{51} + ib_{52}) \big) \big) \big) \Big) = 0.$$

Taking the real part and the imaginary parts of (14) and (15) and solving the corresponding equations with the help of an algebraic manipulator such as Mathematica we get the unique solution given in (5).

Now we assume that $a_5 = 0$. In this case

$$v(z) = \frac{a_0 + a_1 z + a_2 z^2}{a_3 + a_4 z}.$$

Then

$$v'(z) + zv''(z)$$
$$= \frac{1}{\left(b_{31} + ib_{32} + (b_{41} + ib_{42})z\right)^3} \Big((b_{11} + ib_{12})(b_{31} + ib_{32})\big(b_{31} + ib_{32}(b_{41} + ib_{42})z \big)$$
$$+ (b_{01} + ib_{02})(b_{41} + ib_{42})\big(-b_{31} - ib_{32} + (b_{41} + ib_{42})z \big)$$
$$+ (b_{21} + ib_{22})z\big(4(b_{31} + ib_{32})^2 + 3(b_{41} + ib_{42})z(b_{31} + ib_{32})(b_{41} + ib_{42})^2 z^2 \big) \Big).$$

By condition (9) on v and since $a_j = b_{j,1} + ib_{j,2}$ for $j = 0, \cdots, 4$, we get

$$\frac{b_{01} + ib_{02} + i(b_{11} + ib_{12}) - b_{21} - ib_{22}}{b_{31} + ib_{32} + i(b_{41} + ib_{42})} = 0 \quad (16)$$

and

$$\frac{b_{01} + ib_{02} - i(b_{11} + ib_{12}) - b_{21} - ib_{22}}{b_{31} + ib_{32} - i(b_{41} + ib_{42})} = 0. \quad (17)$$

Taking the real part and the imaginary parts of (16) and (17) and solving the

corresponding equations with the help of an algebraic manipulator such as Mathematica we get the unique solution given in (6).

Proof of Proposition 3. We will apply Theorem 5 to potential (7). Then its Darboux points are $\pm i$. Since

$$V = \frac{1}{q_1^n} \sum_{j=0}^{n} a_j q_2^j q_1^{n-j}, \quad \text{then } v(z) = \sum_{j=0}^{n} a_j z^j,$$

and consequently

$$v'(z) + z v''(z) = \sum_{j=1}^{n} j^2 a_j z^{j-1}.$$

Now applying condition (9) on v and taking into account that $a_j = b_{j,1} + i b_{j,2}$ for $j = 1, \cdots, n$, we get

$$\sum_{j=1}^{n} j^2 (b_{j,1} + i b_{j,2}) i^{j-1} = 0 \quad \text{and} \quad \sum_{j=1}^{n} j^2 (b_{j,1} + i b_{j,2})(-1)^{j-1} i^{j-1} = 0,$$

which yields

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,1} (2k+1)^2 (-1)^k - i \sum_{k=1}^{[n/2]} b_{2k,1} (2k)^2 (-1)^k$$
$$+ i \sum_{k=0}^{[(n-1)/2]} b_{2k+1,2} (2k+1)^2 (-1)^k + \sum_{k=1}^{[n/2]} b_{2k,2} (2k)^2 (-1)^k = 0$$

and

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,1} (2k+1)^2 (-1)^k + i \sum_{k=1}^{[n/2]} b_{2k,1} (2k)^2 (-1)^k$$
$$+ i \sum_{k=0}^{[(n-1)/2]} b_{2k+1,2} (2k+1)^2 (-1)^k - \sum_{k=1}^{[n/2]} b_{2k,2} (2k)^2 (-1)^k = 0.$$

Therefore

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,1} (2k+1)^2 (-1)^k + \sum_{k=1}^{[n/2]} b_{2k,2} (2k)^2 (-1)^k = 0,$$

$$-\sum_{k=1}^{[n/2]} b_{2k,1} (2k)^2 (-1)^k + \sum_{k=0}^{[(n-1)/2]} b_{2k+1,2} (2k+1)^2 (-1)^k = 0,$$

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,1} (2k+1)^2 (-1)^k - \sum_{k=1}^{[n/2]} b_{2k,2} (2k)^2 (-1)^k = 0,$$

$$\sum_{k=1}^{[n/2]} b_{2k,1} (2k)^2 (-1)^k + \sum_{k=0}^{[(n-1)/2]} b_{2k+1,2} (2k+1)^2 (-1)^k = 0.$$

Hence

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,1} (2k+1)^2 (-1)^k = 0,$$

$$\sum_{k=1}^{[n/2]} b_{2k,2} (2k)^2 (-1)^k = 0,$$

$$\sum_{k=1}^{[n/2]} b_{2k,1} (2k)^2 (-1)^k = 0,$$

$$\sum_{k=0}^{[(n-1)/2]} b_{2k+1,2} (2k+1)^2 (-1)^k = 0.$$

This yields (8).

4. Conclusions

We have characterized the Liouville integrability of the Hamiltonian systems with Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V\left(q_1, q_2\right),$$

and one of the following potentials

$$V_1 = \frac{a_0 q_1 + a_1 q_2}{a_2 q_1 + a_3 q_2},$$

$$V_2 = \frac{a_0 q_1^2 + a_1 q_1 q_2 + a_2 q_2^2}{a_3 q_1^2 + a_4 q_1 q_2 + a_5 q_2^2},$$

$$V_3 = \frac{a_0 q_2^n + a_1 q_2^{n-1} q_1 + \cdots + a_n q_1^n}{q_1^n}.$$

For doing this we have used the Darboux theory of integrability.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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