Efficient Iterative Method for Solving the General Restricted Linear Equation

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Abstract

An iterative method is developed for solving the solution of the general restricted linear equation. The convergence, stability, and error estimate are given. Numerical experiments are presented to demonstrate the efficiency and accuracy.

Keywords

Linear Equation, Iterative Method, Error Estimate

1. Introduction

Let $\mathbb{C}_r^{m \times n}$ be the set of all $m \times n$ complex matrices with rank r. For any $A \in \mathbb{C}_r^{m \times n}$, let $\|A\|_2, \mathcal{R}(A)$, and $\mathcal{N}(A)$ be matrix spectral norm, range space and null space, respectively. Let $\rho(A)$ be the spectral radius of the matrix A. For any $A \in C_r^{m \times n}$, if there exists a matrix X such that $XAX = X$, then X is called a {2}-inverse (or an outer inverse) of A [1].

The restricted linear equation is widely applied in many practical problems [2] [3] [4]. In this paper, we consider the general restricted linear equations as

$$Ax = b, \quad x \in T, \tag{1}$$

where $A \in \mathbb{C}_r^{m \times n}$ and T is a subspace of $\mathbb{C}^n$. As the conclusion given in [2], (1) has a unique solution if and only if

$$b \in AT, \quad T \bigcap \mathcal{N}(A) = \{0\}. \tag{2}$$

In recent years, some numerical methods have been developed to solve such as problems (1). The Cramer rule method is given in [2] and then this method is developed for computing the unique solution of restricted matrix equations

over the quaternion skew field in [5]. An iterative method is investigated for finding some solution of (1) in [6]. In [7], a subproper and regular splittings iterative method is constructed. The PCR algorithm is applied for parallel computing the solution of (1) in [8]. In [4], a new iterative method is developed and its convergence analysis is also considered. The result on condensed Cramer's rule is given for solving the general solution to the restricted quaternion matrix equation in [9]. In [10] [11], authors develop the determinantal representation of the generalized inverse $A_{T,S}^{(1)}$ for the unique solution of (1). The non-stationary Richardson iterative method is given for solving the general restricted linear equation (1) in [4]. An iterative method is applied to computing the generalized inverse in [13]. In this paper, we develop a high order iterative method to solve the problem (1). The proposed method can be implemented with any initial $x_0 \in T$ and it has higher-order accuracy. The necessary and sufficient condition of convergence analysis also is given, which is different the condition given in [14]. The stability of our scheme is also considered.

The paper is organized as follows. In Section 2, an iterative method for the general restricted linear equation is developed. The convergence analysis of our method is considered, an error estimate is also given in Section 3. In Section 4, some numerical examples are presented to test the effectiveness of our method.

2. Preliminaries and Iterative Scheme

In this section, we develop an iterative method for computing the solution of the general restricted linear Equation (1).

Lemma 1 ([1]) Let $A \in \mathbb{C}^{m \times n}$ and T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, with $\dim T = \dim S^{\perp} = t \leq r$. Then A has a {2}-inverse (or an outer inverse) X such that $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ if and only if

$$AT \oplus S = \mathbb{C}^m,$$

in which case X is unique (denoted by $A_{T,S}^{(2)}$).

Proposition 2 ([2]) Let $A \in \mathbb{C}^{m \times n}$ and T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Assume that the condition (2) is satisfied, then the unique solution of (1) can be expressed by

$$x = A_{T,S}^{(2)} b. \tag{3}$$

Let L and M be complementary subspaces of $\mathbb{C}^m$, *i.e.*, $L \oplus M = \mathbb{C}^m$, the projection P_L be a linear transformation such that $P_L x = x, x \in L$ and $P_L y = 0, y \in M$.

Lemma 3 ([12]) Assume that $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times n}$ with $m \leq n$. Then the n eigenvalues BA are the m eigenvalues of AB together with $n - m$ zeros.

In this paper, we construct our iterative scheme as follows:

$$\begin{cases} Z_k = \left[tI - C_t^2 Z_{k-1} A + \cdots + (-1)^{t-1} (Z_{k-1} A)^{t-1} \right] Z_{k-1}, \\ x_k = x_{k-1} + Z_k (b - A x_{k-1}), \end{cases} \tag{4}$$

where $k = 1, 2, 3, \cdots$, $t \in \mathbb{N}$, and $t \geq 2$. Here, we take the initial value $Z_0 = \beta Y$ in our scheme (4), where β is a relaxation factor. Thus, if $t = 2$, then (4) degenerates to the non-stationary Richardson iterative method given in [4].

Lemma 4 Let $A \in \mathbb{C}^{m \times n}$, T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Assume that $Z_0 = \beta Y$ and $\mathcal{R}(Y) \subseteq T$, where β is a nonzero constant and $Y \in \mathbb{C}^{n \times m}$. For any initial $x_0 \in T$, the iterative scheme (4) converges to some solution of (1) if and only if

$$\rho(P_T - Z_0 A) < 1,$$

where a projection P_T from $\mathbb{C}^m$ onto T.

Proof. The proof can be given as following the line of in [4]. □

3. Convergence Analysis

Now, we consider the convergence analysis of our iterative method (4).

Theorem 5 Let $A \in \mathbb{C}^{m \times n}$, T and S be subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Assume that $AT \oplus S = \mathbb{C}^m$ and $Y \in \mathbb{C}^{nm}$ satisfies $\mathcal{R}(Y) \subseteq T$ and $\mathcal{N}(Y) \subseteq S$, where $\dim T = \dim S^{\perp}$. If $b \in AT$, for the given initial value $Z_0 = \beta Y, \beta \neq 0$ and $x_0 \in T$, then the sequence $\{x_k\}$ generated by iteration (4) converges to the unique solution of (1) if and only if $\rho(P_T - Z_0 A) < 1$, where P_T is a projection. In this case, we have

$$\lim_{k \to \infty} x_k = (I - P_T + Z_0 A)^{-1} Z_0 b. \tag{5}$$

Further, we have

$$\|x_k - x_\infty\| \leq \|q\|^{\frac{t(t^k - 1)}{t-1}} \left(\|x_0 - Z_0 b\| + \frac{\|q\|}{1 - \|q\|} \|Z_0 b\| \right). \tag{6}$$

where $q = P_T - A Z_0$.

Proof. For any $x \in T \cap N(A)$, we have $Ax = 0$. By $AT \oplus S = \mathbb{C}^m$ and Lemma 1, there exists a matrix X such that $\mathcal{R}(X) = T$ and $XAX = X$. Now, assume that $y \in \mathbb{C}^m$ satisfies $x = Xy$, we have

$$x = Xy = XAXy = XAx = 0, T \cap N(A) = \{0\}.$$

If $b \in AT$, then (2) is satisfied. Therefore, by ([4], Lemma 1.1), the scheme (4) converges to the unique solution of (1).

Since $\mathcal{R}(Y) \subseteq T$, $P_T Z_0 = Z_0$, and then by (4), we obtain $P_T Z_k = Z_k$. Since $x_0 \in T$, $P_T x_k = x_k$ by (4b), and therefore

$$x_k = x_{k-1} + Z_k (b - A x_{k-1}) = Z_k b + (P_T - Z_k A) x_{k-1}. \tag{7}$$

If $P_T W = W$, then

$$(I - Z_{k-1} A) W = (P_T - Z_{k-1} A) W. \tag{8}$$

By (4) and (8), we obtain

$$Z_k A = \sum_{i=1}^{t} (-1)^{i-1} C_t^i (Z_{k-1} A)^i = \sum_{i=0}^{t-1} (P_T - Z_{k-1} A)^i Z_{k-1} A,$$

$$(P_T - Z_k A) W = (P_T - Z_{k-1} A)^t W = (P_T - Z_0 A)^{t^k} W. \tag{9}$$

By induction on k, it leads to

$$
\begin{aligned}
Z_k A &= \sum_{i=0}^{t-1} \left(P_T - Z_0 A \right)^{it^{k-1}} Z_{k-1} A \\
&= \sum_{i_1=0}^{t-1} \left(P_T - Z_0 A \right)^{i_1 t^{k-1}} \sum_{i_2=0}^{t-1} \left(P_T - Z_0 A \right)^{i_2 t^{k-2}} Z_{k-2} A \\
&= S_k Z_0 A,
\end{aligned}
\tag{10}
$$

where $S_k := \sum_{i=0}^{t^k-1} \left(P_T - Z_0 A \right)^i$. From $b \in AT$, $w \in T$, we have $b = Aw$ and it implies that $Z_k b = Z_k Aw = S_k Z_0 Aw = S_k Z_0 b$. By (9), we have

$$
\begin{aligned}
x_k &= S_k Z_0 b + \left(P_T - Z_0 A \right)^{t^k} x_{k-1} \\
&= S_k Z_0 b + \sum_{i=0}^{k-1} \prod_{j=0}^{i} \left(P_T - Z_0 A \right)^{t^{k-j}} S_{k-1-i} Z_0 b + \prod_{i=0}^{k-1} \left(P_T - Z_0 A \right)^{t^{k-i}} x_0 \\
&= \sum_{i=0}^{k} \left(P_T - Z_0 A \right)^{\frac{t^{k+1}-t^{k+1-i}}{t-1}} S_{k-i} Z_0 b + \left(P_T - Z_0 A \right)^{\frac{t^{k+1}-t}{t-1}} x_0.
\end{aligned}
\tag{11}
$$

Note that $\left(P_T - Z_0 A \right) S_k = S_k \left(P_T - Z_0 A \right)$, $\left[I - \left(P_T - Z_0 A \right) \right] S_k = I - \left(P_T - Z_0 A \right)^{t^k}$. From (11), we obtain

$$
\begin{aligned}
& \left[I - \left(P_T - Z_0 A \right) \right] x_k \\
&= \left[I - \left(P_T - Z_0 A \right)^{\frac{t^{k+1}-t}{t-1}} \right] Z_0 b + \left[I - \left(P_T - Z_0 A \right) \right] \left(P_T - Z_0 A \right)^{\frac{t^{k+1}-t}{t-1}} x_0.
\end{aligned}
\tag{12}
$$

If $\rho\left(P_T - Z_0 A \right) < 1$, then $I - \left(P_T - Z_0 A \right)$ is invertible and it implies that x_k converges as $k \to \infty$. For convenience, let its limit denote by x_∞. Thus, we have $\left[I - \left(P_T - Z_0 A \right) \right] x_\infty = Z_0 b$. Since T is closed and $x_k \in T$, we have $x_\infty \in T$ and $P_T x_\infty = x_\infty$. Thus, $Z_0 \left(A x_\infty - b \right) = 0$ and $A x_\infty - b \in \mathcal{N}\left(Z_0 \right) \bigcap AT = \mathcal{N}\left(Y \right) \bigcap AT \subseteq S \bigcap AT = \{0\}$. Note that x_∞ is the unique solution of (1) and $x_\infty = \left[I - \left(P_T - Z_0 A \right) \right]^{-1} Z_0 b$. From (4), it follows that

$$
x_{k+1} - x_\infty = \left(P_T - Z_{k+1} A \right)\left(x_0 - x_\infty \right) = \left(P_T - Z_0 A \right)^{\frac{t\left(t^{k+1}-1 \right)}{t-1}} \left(x_0 - x_\infty \right).
\tag{13}
$$

From Lemma 4, we have $\rho\left(P_T - Z_0 A \right) < 1$ and

$$
x_k - x_\infty = \left(P_T - Z_0 A \right)^{\frac{t\left(t^k-1 \right)}{t-1}} \left(\left(x_0 - Z_0 b \right) - \left(I - \left(I - P_T + Z_0 A \right)^{-1} \right) Z_0 b \right).
$$

Therefore,

$$
\begin{aligned}
\left\| x_k - x_\infty \right\| &\leq \left\| q \right\|^{\frac{t\left(t^k-1 \right)}{t-1}} \left(\left\| x_0 - Z_0 b \right\| + \left\| I - \left(I - q \right)^{-1} \right\| \left\| Z_0 b \right\| \right) \\
&\leq \left\| q \right\|^{\frac{t\left(t^k-1 \right)}{t-1}} \left(x_0 - Z_0 b + \frac{\left\| q \right\|}{1 - \left\| q \right\|} Z_0 b \right),
\end{aligned}
$$

where $q = P_T - A Z_0$. □

Remark If $\mathcal{N}\left(Y \right) \subseteq S$ in Theorem 5 is removed and $t = 2$, then the result degenerates into that given in ([4], Theorem 3.2). However, the sequence $\{x_k\}$ given in (4) does not converse to $A_{T,S}^{(2)} b$, the unique solution of (1) is given by

Proposition 2. Here, it can be tested by the following example:

Let A and b of the general restricted linear Equation (1) be

$$A = \begin{bmatrix} 2 & 2.5 & 0.2 & 0.3 & 0 \\ 0 & 1.5 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0 \\ 0 & 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}_4^{6\times5}, \quad b = \begin{bmatrix} 7 \\ 3 \\ 0.2 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}. \tag{14}$$

The matrix Y is

$$Y = \begin{bmatrix} 1.2 & 2 & 0.2 & -2 & 1 & 0 \\ 0 & 2 & 5 & -2 & 0 & 0 \\ 0 & 0 & 0.25 & 0.1 & 0 & 0 \\ 0 & -0.1 & 0 & 1.3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that $R(Y) \subset T$, but $\mathcal{N}(Y) \nsubseteq S$. If take $\beta = 0.16$, then $\rho(P_T - Z_0 A) < 1$. Here, we choose $t = 2$ in (4). Thus, it can be seen as the method given in [4]. The errors $\|Ax_k - b\|_2$, $\|x_k - x_{k-1}\|_2$, and $\left\|x_k - A_{T,S}^{(2)} b\right\|_2$ of (4) with $t = 2$ and $\beta = 0.16$ are presented in Table 1. Numerical results given in Table 1 show that $\|Ax_k - b\|_2 \to 0$, but $\left\|x_k - A_{T,S}^{(2)} b\right\|_2 \nrightarrow 0$. Thus, the limit of x_k is not the solution of (1) presented by Proposition 2.

Theorem 6 Under the same conditions as in Theorem 5. The iterative scheme (4) is stable for solving (1), where $\|P_T - Z_0 A\| < 1$.

Proof. Let Δ_k and δ_k be numerical perturbations of x_k and Z_k given in (4), respectively. Thus, we can express as $\tilde{x}_k = x_k + \delta_k$, $\tilde{Z}_k = Z_k + \Delta_k$. If $P_T \Delta_k = \Delta_k$, $P_T \delta_k = \delta_k$, then $P_T \tilde{x}_k = \tilde{x}_k$, $P_T \tilde{Z}_k = \tilde{Z}_k$. Here, we formally neglect quadratic terms containing $\Delta_k A \delta_h$, $\Delta_k A \Delta_k$. Since $Z_k A \tilde{Z}_k = \tilde{Z}_k A Z_k$, we get

$$Z_k A \Delta_k = \Delta_k A Z_k, \quad (Z_k A + \Delta_k A)^t = (Z_k A)^t + t \Delta_k A (Z_k A)^{t-1}.$$

By (4), we derive

$$\begin{aligned} \tilde{Z}_k &= Z_k + \sum_{i=2}^{t} (-1)^{i-1} C_t^i (Z_{k-1}A)^{i-1} \Delta_{k-1} + t\Delta_{k-1} \\ &\quad + \sum_{i=2}^{t} (-1)^{i-1} (i-1) C_t^i \Delta_{k-1} A (Z_{k-1}A)^{i-2} Z_{k-1} \\ &= Z_k + t(P_T - Z_{k-1}A)^{t-1} \Delta_{k-1}. \end{aligned} \tag{15}$$

From (9) and (4), we have $\Delta_k = t^k (P_T - Z_0 A)^{t^k-1} \Delta_0$ and

Table 1. Error results of (4) with $t = 2$, $\beta = 0.16$.

	$\|Ax_k - b\|_2$	$\|x_k - x_{k-1}\|_2$	$\left\|x_k - A_{T,S}^{(2)} b\right\|_2$
$k = 11$	3.3871e−10	8.3702e−09	21.8430
$k = 12$	6.8861e−16	3.5821e−15	21.8430

$$\tilde{x}_k = x_k + t^k \left(P_T - Z_0 A \right)^{t^k - 1} \Delta_0 \left(b - A x_{k-1} \right) + \left(I - Z_k A \right) \delta_{k-1}.$$

Therefore, we obtain

$$\delta_k = t^k \left(P_T - Z_0 A \right)^{t^k - 1} \Delta_0 \left(b - A x_{k-1} \right) + \left(P_T - Z_0 A \right)^{t^k} \delta_{k-1}. \tag{16}$$

By (4), we have $I - A Z_k = \left(I - A Z_0 \right)^{t^k}$. Similarly, we have

$$b - A x_k = \left(I - A Z_k \right) \left(b - A x_{k-1} \right) = \left(I - A Z_0 \right)^{\frac{t^{k+1} - t}{t - 1}} \left(b - A x_0 \right). \tag{17}$$

By (17) and (16), we derive

$$\delta_k = t^k \left(P_T - Z_0 A \right)^{t^k - 1} \Delta_0 \left(I - A Z_0 \right)^{\frac{t^k - t}{t - 1}} \left(b - A x_0 \right) + \left(P_T - Z_0 A \right)^{t^k} \delta_{k-1}. \tag{18}$$

Thus, by $\Delta_0 \left(I - A Z_0 \right) = \left(P_T - Z_0 A \right) \Delta_0$, we have

$$\begin{aligned}
\delta_k &= \left(t^k + t^{k-1} + \cdots + t \right) \left(P_T - Z_0 A \right)^{\frac{t^{k+1} - 2t + 1}{t - 1}} \Delta_0 \left(b - A x_0 \right) \\
&\quad + \left(P_T - Z_0 A \right)^{t^k + t^{k-1} + \cdots + t} \delta_0 \\
&= \frac{t^{k+1} - t}{t - 1} \left(P_T - Z_0 A \right)^{\frac{t^{k+1} - 2t + 1}{t - 1}} \Delta_0 \left(b - A x_0 \right) + \left(P_T - Z_0 A \right)^{\frac{t^{k+1} - t}{t - 1}} \delta_0.
\end{aligned} \tag{19}$$

If $\left\| P_T - Z_0 A \right\| < 1$, then $\lim\limits_{k \to \infty} \frac{t^{k+1} - t}{t - 1} \left\| P_T - Z_0 A \right\|^{\frac{t^{k+1} - 2t + 1}{t - 1}} = 0$ and for any k,

$$\left\| \delta_k \right\| \le K_0 \left(\left\| \Delta_0 \right\| \left\| \left(b - A x_0 \right) \right\| + \left\| \delta_0 \right\| \right),$$

where $K_0 = \max \left\{ \frac{t^{k+1} - t}{t - 1} \left\| P_T - Z_0 A \right\|^{\frac{t^{k+1} - 2t + 1}{t - 1}}, \left\| P_T - Z_0 A \right\|^{\frac{t^{k+1} - t}{t - 1}} \right\}$. It follows that the iterative method (4) is asymptotically stable. □

4. Numerical Examples

In the section, we give an example to test the accuracy of our scheme (4), which is implemented by our main code given in Appendix, and make a comparison with the method given in [4]. We also apply our scheme to solve the restricted linear system (1) with taking different t and intial value.

Example 1 Consider the restricted linear system (1) with a coefficient matrix being random $A = \mathrm{randn}(n)$ of order $n \times n$, where $n = 800$, 900, 1000, 2000 of index one and random vectors $b \in AT$. Let $E = \mathrm{randn}(n)$ be a random matrix. Take $Y = A E E^{\mathrm{T}}$, $Z_0 = \beta Y$, and a random vector $x_0 \in T$. Here, we make a comparison the mean CPU time(MCT) and error bounds of our scheme (4) with those given by the method of [4]. The stopping criteria used is given as in [4] by

$$\max \left\{ \left\| A x_k - b \right\|_2, \left\| x_k - x_{k-1} \right\|_2 \right\} \le 5.26\mathrm{e} - 15.$$

Numerical results given in Table 2 and Figure 1 show that the accuracy of our method is similar to those given in [4] and our method cost less time (MCT) than the method of [4]. We can see that, to obtain the similar accuracy, the MCT of our scheme is similar to those given in [4] from Figure 2.

Table 2. The mean CPU time (MCT) and error in Example 1.

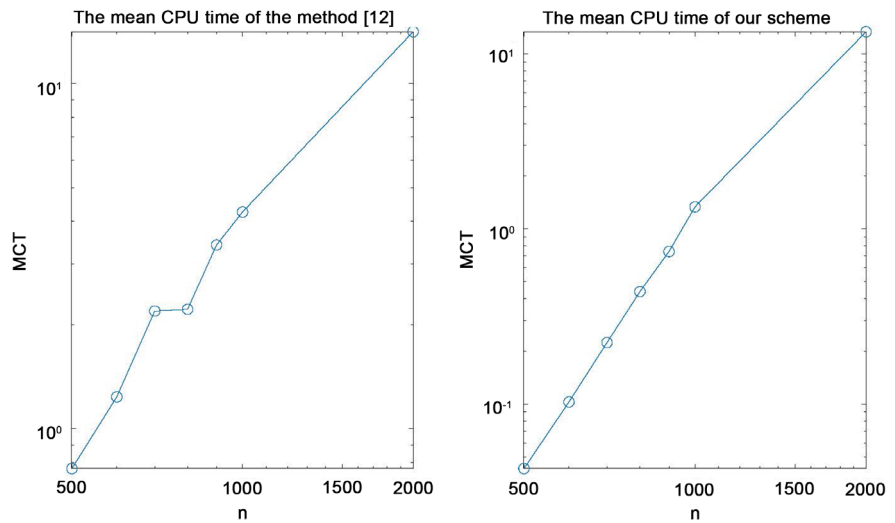| $n \times n$	t	method of [4] MCT [4]	$\|Ax_k - b\|_2$	$\|x_k - x_{k-1}\|_2$
800×800		2.221434	1.4613e−12	3.4439e−12
900×900		3.412709	1.5764e−13	2.2203e−12
1000×1000		4.249560	9.7132e−13	7.3896e−12
2000×2000		14.14631	4.5823e−13	3.3269e−12
$n \times n$	t	Our Scheme MCT	$\|Ax_k - b\|_2$	$\|x_k - x_{k-1}\|_2$
800×800	6	0.438568	9.2913e−13	8.3698e−13
900×900	7	0.742810	1.3444e−12	1.2432e−12
1000×1000	8	1.336598	1.4786e−12	6.8270e−13
2000×2000	10	13.361577	4.3165e−12	3.0732e−12

Figure 1. Error in example 1.

Example 2 Consider the general restricted linear Equation (1), where A and b is given as in (14). Here, we use the scheme (4) to solve the example. Let $T = \mathbb{C}^5$, $S = \text{span}\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\text{T}} \right\}$. Take

$$Y = \begin{bmatrix} 1.2 & 2 & 0.2 & -2 & 1 & 0 \\ 0 & 2 & 5 & -2 & 0 & 0 \\ 0 & 0 & 0.25 & 0.1 & 0 & 0 \\ 0 & -0.1 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}, \quad P_T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Obviously, $AT \oplus S = \mathbb{C}^6$, $\mathcal{R}(Y) \subseteq T$, $\mathcal{N}(Y) \subseteq S$, $b \in AT$, and $x_0 \in T$. To verify the accuracy of our method, we present the generalized inverse $A_{T,S}^{(2)}$ as
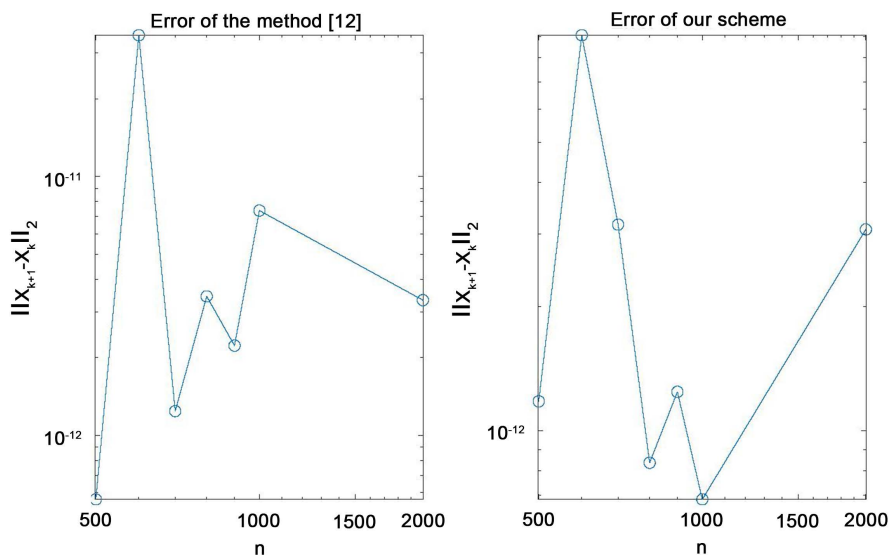
Figure 2. The MCT in Example 1.

Table 3. Error for (4) in Example 2 with $t = 2,3$.

t	β	k	$\left\|Ax_k - b\right\|_2$	$\left\|x_k - x_{k-1}\right\|_2$	$\left\|x_k - A_{T,S}^{(2)}b\right\|_2$
	0.13	11	2.0035e−08	3.1905e−08	3.1905e−08
		12	9.9495e−16	6.1815e−16	3.6906e−15
2	0.14	11	5.1443e−09	8.1923e−09	8.1923e−09
		12	5.9511e−17	2.2204e−16	1.8243e−15
	0.15	11	1.3203e−09	2.1026e−09	2.1026e−09
		12	8.8861e−16	4.5776e−16	3.9315e−15
t	β	k	$\left\|Ax_k - b\right\|_2$	$\left\|x_k - x_{k-1}\right\|_2$	$\left\|x_k - A_{T,S}^{(2)}b\right\|_2$
	0.13	11	0	0	4.1466e−15
		12	0	0	4.1466e−15
3	0.14	11	0	0	2.0562e−15
		12	0	0	2.0562e−15
	0.15	11	0	0	3.9440e−15
		12	0	0	3.9440e−15

$$A_{T,S}^{(2)} = \begin{bmatrix} 0.5 & -0.7 & -0.5 & 0.6 & 0 & 0 \\ 0 & 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To ensure $\rho(P_T - AZ_0) < 1$, we take parameter $\beta = 0.13, 0.14, 0.15$ in **Table 3** and $\beta = 0.13, 0.15, 0.16$ in **Table 4**, respectively. We present the errors $Ax_k - b$, $x_k - x_{k+1}$, and $x_k - A_{T,S}^{(2)}b$ in 2-norm as $t = 2,3$ in **Table 3** and $t = 5,8$ in **Table 4**, respectively.

Table 4. Error for (4) in Example 2 with $t = 5, 8$.

t	β	k	$\|Ax_k - b\|_2$	$\|x_k - x_{k-1}\|_2$	$\left\|x_k - A_{T,S}^{(2)}b\right\|_2$
	0.13	$k = 5$	4.5219e−08	7.2011e−08	7.2011e−08
		$k = 6$	8.8861e−16	4.5776e−16	3.8357e−15
5	0.15	$k = 5$	3.3786e−09	5.3804e−09	5.3804e−09
		$k = 6$	0	0	3.8794e−15
	0.16	$k = 5$	9.2293e−10	1.4698e−09	1.4698e−09
		$k = 6$	2.7756e−17	1.1102e−16	2.6990e−15
t	β	k	$\|Ax_k - b\|_2$	$\|x_k - x_{k-1}\|_2$	$\left\|x_k - A_{T,S}^{(2)}b\right\|_2$
	0.13	$k = 5$	8.8818e−16	4.4409e−16	4.4270e−15
		$k = 6$	0	0	4.0550e−15
8	0.15	$k = 5$	0	0	3.0562e−15
		$k = 6$	0	0	3.7659e−15
	0.16	$k = 5$	8.8816e−16	4.5776e−16	2.2834e−15
		$k = 6$	0	0	2.7240e−15

From the numerical results given in Table 3 and Table 4, we can see that the scheme (4) has high order accuracy and these results given with $t = 3, 8$ are better than those obtained by $t = 2, 5$, respectively.

5. Conclusion

The high order iterative method has been derived for solving the general restricted linear equation. The convergence and stability of our method also have derived. Numerical experiments have presented to demonstrate the efficiency and accuracy.

Acknowledgements

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Appendix

```
function hocigrlscm( A, X, P, Z, b, ktmax )
```

$zn = size(A,1); \quad id = speye(z_n);$

fprintf(' |Num. of iterative| $\quad \|Ax_k - b\|_2 \quad \|x_{k+1} - x_k\|_2 \quad$ |CPU time| $\backslash n'$)

fprintf('-----------------------BEGING--------------------------- $\backslash n'$)

$it\max = 1e2; \quad tol = 5.26e-15; \quad cnvs = 0;$

for $it = 1 : it\max,$

tic; $\quad za = A * Z; \quad X_0 = X; \quad ZY = 0;$

for $t_k = 2 : kt\max$

$zc = (-1)^{(t_k-1)} * nchoosek(kt\max, t_k);$

$ZY = ZY + zc * (za)^{(t_k-1)}; \quad \% \text{ compute } Z_{k+1}$

end

$Z = Z * (kt\max * id + ZY);$

$X = X + Z * (b - A * X); \quad \% X_k \, -- > X_{k+1}$

clear $ZY;$

$ermk = X - X_0; \quad L = A * X - b;$

$ermk = sqrt\left(ermk(:)' * ermk\right);$

$erml = sqrt(L' * L); \quad itm = toc;$

if $ermk < tol, \quad cnvs = cnvs + 1; \text{break, end}$

end $\% \text{ END it}$