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Some New Fixed Point Theorems for Fuzzy Iterated Contraction Maps in Fuzzy Metric Spaces

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Abstract

The purpose of this paper is to introduce the notion of fuzzy iterated contraction maps in fuzzy metric spaces and establish some new fixed point theorems for fuzzy iterated contraction maps in fuzzy metric spaces.

Keywords

Fixed Point, Iterated Contraction Map, Fuzzy Metric Space

1. Introduction

George and Veeramani [1] slightly modified the concept of fuzzy metric space introduced by Kramosil and Michalek, defined a Hausdorff topology and proved some known results in 1994. Rheinboldt [2] initiated the study of iterated contraction in 1969. The concept of iterated contraction proves to be very useful in the study of certain iterative process and has wide applicability in metric spaces. In this paper we establish some new fixed point theorems for fuzzy iterated contraction maps in fuzzy metric spaces.

2. Preliminaries

Definition 2.1 ([1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(FM-1) $M(x, y, t) > 0$;

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$;

(FM-3) $M(x, y, t) = M(y, x, t)$;

(FM-4) $M(x, y, t) * M(x, z, t+s)$;

(FM-5) $M(x,y,\bullet):(0,\infty)\to(0,1)$ is continuous.

Definition 2.2 ([1]). A map $T:X\to X$, satisfying $M(Tx,Ty,t)\ge M\left(x,y,\frac{t}{k}\right)$, for all $x,y\in X, t>0, 0<k<1$, is called a contraction map.

Definition 2.3 ([3]). If $(X,M,*)$ is a fuzzy metric space such that $M(Tx,T^2x,t)\ge M\left(x,Tx,\frac{t}{k}\right)$ for all $x\in X, t>0, 0<k<1$, then T is said to be a fuzzy iterated contraction map.

3. Main Results

In this part, we prove some new fixed point theorems for fuzzy iterated maps under different settings. According to these theorems, some useful corollaries are obtained.

Theorem 3.1 If $T:C\to C$ is a fuzzy iterated contraction and is continuous, where C is closed subset of a metric space X, then T has a fixed point provided that $T(C)$ is compact.

Proof: We show that the sequence $\{x_n\}$ has a convergent subsequence. Using iterated contraction and continuity of T we get a fixed point.

Definition 3.1 Let X be a metric space and $T:X\to X$. Then T is said to be a fuzzy iterated nonexpansive map if $M(Tx,T^2x,t)\ge M(x,Tx,t)$ for all $x\in X$.

The following is a fixed point theorem for the fuzzy iterated nonexpansive map.

Theorem 3.2 Let X be a metric space and $T:X\to X$ a fuzzy iterated nonexpansive map satisfying the following:

If $x=Tx$, then $M(Tx,T^2x,t)\ge M(x,Tx,t)$,

If for some $x\in X$, the sequence of iterates $x_{n+1}=Tx_n$ has a convergent subsequence converging to y say and T is continuous at y. Then T has a fixed point.

Proof: The sequence $\{M(x_{n+1},x_n,t)\}$ is a nondecreasing sequence of reals. It is bounded above by 1, and therefore has a limit. Since the subsequence converges to y and T is continuous on X, so $T(x_{n_i})$ converges to Ty and $T^2(x_{n_i})$ converges to T^2y.

Thus $M(y,Ty,t)=\lim M(x_{n_i},x_{n_{i+1}},t)=\lim M(x_{n_{i+1}},x_{n_{i+2}},t)=M(Ty,T^2y,t)$.

If $y\ne Ty$, then $M(Ty,T^2y,t)>M(y,Ty,t)$, since T is a fuzzy iterative contractive map.

Consequently, $M(y,Ty,t)=M(Ty,T^2y,t)>M(y,Ty,t)$, a contradiction and hence $Ty=y$.

Note 3.1 If C is compact, then condition 2) of Theorem 3.2 is satisfied, and hence the result.

If C is a closed subset of a metric space X and $T:C\to C$ a fuzzy iterated contraction. If the sequence $\{x_n\}$ converges to y, where T is continuous at y, then $Ty=y$, that is, T has a fixed point.

The following theorem deals with two metrics on X.

Theorem 3.3 Let $T:X\to X$ satisfy the following:

1) X is complete with metric M and $M(x,Tx,t) \le \delta(x,Tx,t)$ for all x, $Tx \in X$,

2) T is a fuzzy iterated contraction with respect to δ,

Then for $x \in X$, the sequence of iterates $x_n = T^n x$ converges to $y \in X$. If T is continuous at y, then T has a fixed point, say $Ty = y$.

Proof: It is easy to show that $\{x_n\}$ is a Cauchy sequence with respect to δ. Since $M(x,Tx,t) \le \delta(x,Tx,t)$, therefore $\{x_n\}$ is a Cauchy sequence with respect to M. The sequence $\{x_n\}$ converges to y in (X,M) since it is complete. The function T is continuous at y,

$y = \lim x_n = \lim f\left(f^{n-1}x\right) = \lim fx_{n-1} = f \lim x_{n-1} = Ty$. Hence $Ty = y$.

Theorem 3.4 Let $T: X \to X$ be a continuous fuzzy iterated contraction map such that:

if $x = Tx$, then $M\left(Tx, T^2x, t\right) > M(x,Tx,t)$, and

the sequence $x_{n+1} = T(x_n)$ has a convergent subsequence converging to y.

Then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof: It is easy to see that the sequence $\{M(x_n, x_{n+1}, t)\}$ is a nondecreasing and bounded above by 1. Let $\{x_n\}$ be a subsequence of $\{x_n\}$ converging to y.

Then, $M(y,Ty,t) = \lim M\left(x_{n_i}, x_{n_{i+1}}, t\right) = \lim M\left(x_{n_{i+1}}, x_{n_{i+2}}, t\right) = M\left(Ty, T^2y, t\right)$.

If $y \ne Ty$, then, $M\left(Ty, T^2y, t\right) > M(y,Ty,t)$, since T is a fuzzy iterative contractive map.

Consequently, $M(y,Ty,t) = M\left(Ty, T^2y, t\right) > M(y,Ty,t)$.

This is a contradiction so $y = Ty$. Since $M(x_{n+1}, y, t) > M(x_n, y, t)$ for all n, so $\{x_n\}$ converges to y.

Corollary 3.1 Let T be a map of a fuzzy metric space X into itself such that

1) T is a nonexpansive map on X, that is, $M(Tx,Ty,t) \ge M(x,y,t)$ for all $x, y \in X$,

2) if $x \ne Tx$, then $M\left(Tx, T^2x, t\right) > M(x,Tx,t)$,

3) the sequence $x_{n+1} = T(x_n)$ has a convergent subsequence converging to y.

Then the sequence $\{x_n\}$ converges to a fixed point of T.

Proof: It is easy to prove by **Theorem 3.4.**

Note 3.2 If $T: X \to X$ is a fuzzy contractive map and $T(X)$ compact, then T has a unique fixed point.

It is easy to see that the sequence of iterates $\{x_n\}$ converges to a unique fixed point of T. However, for nonexpansive map, a sequence of iterates need not converge to a fixed point of T.

Note 3.3 If $H = aTx + (1-a)x$, $0 < a < 1$, then the fixed point of T is the same as of H.

Let $Ty = y$. Then $Hy = aTy + (1-a)y = y$, that is, $Hy = y$ since $Ty = y$.

Let $Hy = y$. Then we show that $y = Ty$. Here $Hy = aTy + (1-a)y = y$.

Then $Ty = y$, that is, T has a fixed point y. In case the sequence $x_{n+1} = Hx_n$ converges to y a fixed point of H, then $y = Ty$.

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