

A Priori Estimates of Solution of Parametrized Singularly Perturbed Problem

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Abstract

In this paper, we consider a parameterized singularly perturbed second order quasilinear boundary value problem. Asymptotic estimates for the solution and its first and second derivatives have been established. The theoretical estimates have been justified by concrete example.

Keywords

Parameterized Problem, Asymptotic Bounds, Singular Perturbation, Boundary Layer

1. Introduction

In this paper, we are going to obtain the asymptotic bounds for the following parameterized singularly perturbed boundary value problem (BVP):

$$Lu := \varepsilon u'' - a(t)u' - f(t, u, \lambda) = 0, \quad 0 < t < T, \quad (1.1)$$

$$u(0) = \mu_0, \quad u'(0) = \mu_1, \quad u'(T) = \frac{\mu_2}{\varepsilon} \quad (1.2)$$

where $0 < \varepsilon \leq 1$ is the perturbation parameter, μ_i ($i = 0, 1, 2$) are given constants and $0 < \alpha \leq a(t) \leq a^*$ is a sufficiently smooth function in $[0, T]$. Further, the function $f(t, u, \lambda)$ is assumed to be sufficiently continuously differentiable for our purpose function in $\{0 \leq t \leq T, -\infty < u, \lambda < \infty\}$ and

$$0 \leq \frac{\partial f}{\partial u} \leq M_1^*, \quad 0 < m_1 \leq \left| \frac{\partial f}{\partial \lambda} \right| \leq M_1 < \infty. \quad (1.3)$$

By a solution of (1.1), (1.2), we mean pair $\{u(t), \lambda\} \in C^1[0, T] \times \mathbb{R}$ for which problem (1.1), (1.2) is satisfied. An overview of some existence and uniqueness results and applications of parameterized equations may be obtained, for example, in [1]-[10]. In [11]-[14] have also been considered some approxi-mating aspects of this kind of problems. The qualitative analysis of singular perturbation situations has always been far from trivial because of the boundary layer behavior of the solution. In singular perturbation cases, problems depend on a small parameter ε in such a way that the solution exhibits a multiscale character, *i.e.*, there are thin transition layers where the solution varies rapidly while away from layers it behaves regularly and varies slowly [15]-[18]. In this note, we establish the boundary layer behaviour for $u(t)$ of the solution of (1.1)-(1.2) and its first and second derivatives. Example that agrees with the analytical results is given.

2. The Continuous Problem

Lemma 2.1. Let $a(t) \geq \alpha > 0, b(t) \geq 0$ and $\Phi(t)$ be the continuous functions on $[0, T]$. Then, the solution of the boundary-value problem

$$Lv := \varepsilon v'' - a(t)v' - b(t)v = \Phi(t), 0 < t < T, \quad (2.1)$$

$$v(0) = \mu_0, v'(T) = \frac{\mu_2}{\varepsilon}, \quad (2.2)$$

satisfies the inequality

$$|v(t)| \leq |\mu_0| + |\mu_2| \omega(t) + \alpha^{-1} t \|\Phi\|_\infty, 0 \leq t \leq T, \quad (2.3)$$

where

$$\omega(t) = \alpha^{-1} \left(e^{\frac{-\alpha(T-t)}{\varepsilon}} - e^{\frac{-\alpha T}{\varepsilon}} \right).$$

Proof. Under the above conditions, the operator Lv admits the following maximum principle:

Suppose $v(t) \in C^2[0, T]$ be any function satisfying $Lv \leq 0$ ($0 < t < T$), $v(0) \geq 0$ and $v'(T) \geq 0$. Then, $v(t) \geq 0$ for all $t \in [0, T]$.

Now, for the barrier fonction

$$\Psi_\pm = \pm v(t) + |\mu_0| + |\mu_2| \omega(t) + \alpha^{-1} t \|\Phi\|_\infty,$$

taking also into consideration that, $\omega(t)$ is a solution of the problem

$$\varepsilon \omega'' - \alpha \omega' = 0, \omega(0) = 0, \omega'(T) = \frac{1}{\varepsilon},$$

it follows that,

$$\begin{aligned} L\Psi_\pm &= \pm \Phi(t) - b(t)|\mu_0| + |\mu_2|(\alpha - \alpha(t))\omega'(t) \\ &\quad - |\mu_2|b(t)\omega(t) - a(t)\alpha^{-1} \|\Phi\|_\infty - b(t)\alpha^{-1} t \|\Phi\|_\infty \\ &\leq \pm \Phi(t) - a(t)\alpha^{-1} \|\Phi\|_\infty \leq 0, \\ \Psi_\pm(0) &= \pm \mu_0 + |\mu_0| \geq 0 \\ \Psi'_\pm(T) &= \pm \frac{\mu_2}{\varepsilon} + \frac{1}{\varepsilon} |\mu_2| \geq 0 \end{aligned}$$

therefore $\Psi_\pm(t) \geq 0$, which immediayely leads to (2.3).

Remark 1. The inequality (2.3) yields.

$$|u(t)| \leq |\mu_0| + |\mu_2| \omega(t) + \alpha^{-1} T \|\Phi\|_\infty. \quad (2.4)$$

Theorem 2.1. For $\rho = 1 - \alpha^{-1} m_1^{-1} M_1 M_1^* T$ and under conditions (1.3), the solution $\{u(t), \lambda\}$ of the problem

(1.1), (1.2), satisfies,

$$|\lambda| \leq c_0 \quad (2.5)$$

$$|u| \leq c_1 \quad (2.6)$$

where

$$c_0 = \rho^{-1} \left[\frac{|\mu_1| a^*}{m_1 (1 - e^{-a^* T})} + \frac{|\mu_2|}{m_1 (e^{a^* T} - 1)} + m_1^{-1} M_1^* (|\mu_0| + \alpha^{-1} |\mu_2| + \alpha^{-1} T \|F\|_\infty) \right],$$

$$c_0 = |\mu_0| + \alpha^{-1} |\mu_2| + \alpha^{-1} T \|F\|_\infty + \alpha^{-1} c_0 M_1 T, \quad F(t) = f(t, 0, 0)$$

and

$$|u^{(k)}(t)| \leq C \left(1 + \frac{1}{\varepsilon^k} e^{-\frac{\alpha t}{\varepsilon}} \right), \quad k = 1, 2, t \in [0, T], \quad (2.7)$$

provided $a \in C^1[0, T]$ and $\left| \frac{\partial f}{\partial t} \right| \leq C$ for $t \in [0, T]$ and $|u| \leq c_1, |\lambda| \leq c_0$.

Proof. We rewrite Equation (1.1) in form

$$\varepsilon u'' - a(t)u' - b(t)u - \lambda c(t) - F(t) = 0, \quad t \in [0, T], \quad (2.8)$$

where, $b(t) = \frac{\partial f}{\partial u}(t, \tilde{u}, \tilde{\lambda})$, $c(t) = \frac{\partial f}{\partial \lambda}(t, \tilde{u}, \tilde{\lambda})$, $\tilde{u} = \gamma u$, $\tilde{\lambda} = \gamma \lambda$ ($0 < \gamma < 1$) intermediate values.

From (2.8) for the first derivate, we have

$$u'(t) = \frac{\mu_2}{\varepsilon} e^{-\frac{1}{\varepsilon} \int_0^t a(\tau) d\tau} - \frac{1}{\varepsilon} \int_0^t b(\tau) u(\tau) e^{-\frac{1}{\varepsilon} \int_0^\tau a(\tau) d\tau} d\tau - \frac{\lambda}{\varepsilon} \int_0^t c(\tau) e^{-\frac{1}{\varepsilon} \int_0^\tau a(\tau) d\tau} d\tau - \frac{1}{\varepsilon} \int_0^t F(\tau) e^{-\frac{1}{\varepsilon} \int_0^\tau a(\tau) d\tau} d\tau, \quad (2.9)$$

from which, after using the initial condition $u'(0) = \mu_1$, it follows that,

$$\lambda = \frac{-\mu_1}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau} + \frac{\frac{\mu_2}{\varepsilon} e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau}}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau} - \frac{\frac{1}{\varepsilon} \int_0^T b(\tau) u(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau} - \frac{\frac{1}{\varepsilon} \int_0^T F(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau}. \quad (2.10)$$

Applying the mean value theorem for integrals, we deduce that,

$$\left| \frac{\frac{1}{\varepsilon} \int_0^T F(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau} \right| \leq \frac{\|F\|_\infty \left| \int_0^T e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau \right|}{m_1 \left| \int_0^T e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau \right|} \leq m_1^{-1} \|F\|_\infty \quad (2.11)$$

and

$$\left| \frac{\frac{1}{\varepsilon} \int_0^T b(\tau) u(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau}{\frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau} \right| \leq \frac{M_1^* \|u\|_\infty \left| \int_0^T e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau \right|}{m_1 \left| \int_0^T e^{-\frac{1}{\varepsilon} \int_0^T a(\tau) d\tau} d\tau \right|} \leq m_1^{-1} M_1^* \|u\|_\infty \quad (2.12)$$

Also, for first and second terms in right side of (2.10) for $\varepsilon \leq 1$ values, we have

$$\frac{\frac{|\mu_1|}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^{\tau} a(t) dt} d\tau}{\left| \frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^{\tau} a(t) dt} d\tau \right|} + \frac{\frac{|\mu_2|}{\varepsilon} e^{-\frac{1}{\varepsilon} \int_0^T a(t) dt}}{\left| \frac{1}{\varepsilon} \int_0^T c(\tau) e^{-\frac{1}{\varepsilon} \int_0^{\tau} a(t) dt} d\tau \right|} \leq \frac{|\mu_1|}{m_1 (a^*)^{-1} \left(1 - e^{-\frac{a^* T}{\varepsilon}}\right)} + \frac{|\mu_2|}{m_1 \varepsilon \left(e^{\frac{\alpha T}{\varepsilon}} - 1\right)}. \quad (2.13)$$

It then follows from (2.11)-(2.13),

$$|\lambda| \leq \frac{a^* |\mu_1|}{m_1 (1 - e^{-a^* T})} + \frac{|\mu_2|}{m_1 (e^{\alpha T} - 1)} + m_1^{-1} M_1^* \|u\|_{\infty} + m_1^{-1} \|F\|_{\infty} \quad (2.14)$$

Further from (2.4) by taking $\Phi(t) = \lambda c(t) + F(t)$ we get

$$\|u\|_{\infty} \leq |\mu_0| + \alpha^{-1} |\mu_2| + \alpha^{-1} T \|F\|_{\infty} + \alpha^{-1} M_1 T |\lambda|. \quad (2.15)$$

The inequalities (2.14), (2.15) immediately leads to (2.5), (2.6). After taking into consideration the uniformly boundnees in ε of $u(t)$ and λ , it then follows from (2.9) that,

$$|u'(t)| \leq \frac{|\mu_2|}{\varepsilon} e^{-\frac{1}{\varepsilon} \int_0^t a(\tau) d\tau} + \frac{1}{\varepsilon} c_1 M_1^* \int_t^T e^{-\frac{1}{\varepsilon} \int_t^{\tau} a(t) dt} d\tau + \frac{1}{\varepsilon} c_0 M_1 \int_t^T e^{-\frac{1}{\varepsilon} \int_t^{\tau} a(t) dt} d\tau + \frac{1}{\varepsilon} \|F\|_{\infty} \int_t^T e^{-\frac{1}{\varepsilon} \int_t^{\tau} a(t) dt} d\tau,$$

which proves (2.7) for $k = 1$. To obtain (2.7) for $k = 2$, first from (1.1) we have

$$|u''(T)| \leq \frac{a(T)u'(T) + F(T, u(T), \lambda)}{\varepsilon}$$

from which after taking into consideration here $u'(T) = \frac{\mu_2}{\varepsilon}$ and (2.5) we obtain

$$|u''(T)| \leq \frac{C}{\varepsilon^2}. \quad (2.16)$$

Next, differentiation (1.1) gives

$$\varepsilon v'' - a(t)v' - Q(t) = 0, \quad 0 < t < T, \quad (2.17)$$

$$v'(T) = O(\varepsilon^{-2}) \quad (2.18)$$

with

$$v(t) = u'(t),$$

$$Q(t) = a'(t)v(t) + \frac{\partial f}{\partial t}(t, u(t), \lambda) + \frac{\partial f}{\partial u}(t, u(t), \lambda),$$

and due to our assumptions clearly,

$$|Q(t)| \leq C \left(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha(T-t)}{\varepsilon}}\right).$$

Consequently, from (2.17), (2.18) we have

$$|v'(t)| \leq \frac{C}{\varepsilon^2} e^{-\frac{\alpha(T-t)}{\varepsilon}} + \frac{1}{\varepsilon} \int_t^T |Q(t)| e^{-\frac{\alpha(T-t)}{\varepsilon}} = \frac{C}{\varepsilon^2} e^{-\frac{\alpha(T-t)}{\varepsilon}} + C \left(1 + \frac{1}{\varepsilon}\right) \int_t^T e^{-\frac{\alpha(T-t)}{\varepsilon}}$$

which proves (2.7) for $k = 2$. □

Example. Consider the following parameterized singular perturbation problem:

$$\varepsilon u'' - a(t)u' + \lambda + \tanh(1 - t + \lambda) = f(t), \quad 0 < t < 1,$$

$$u(0) = \mu_0, \quad u'(0) = \mu_1, \quad u'(T) = \frac{\mu_2}{\varepsilon}$$

with

$$\mu_1 = \frac{2}{2+\varepsilon} \left[\frac{e+1+\varepsilon}{\varepsilon \left(e^{\frac{2}{\varepsilon}} - 1 \right)} - \frac{e}{2} \right], \quad \mu_2 = \frac{2e+2+\varepsilon \left(1+e^{\frac{2}{\varepsilon}} \right)}{1-e^{\frac{2}{\varepsilon}}}$$

and $f(t)$ selected so that the solution is

$$u(t) = \gamma_1 + \gamma_2 e^{-\frac{2}{\varepsilon}(1-t)} + \frac{1}{2+\varepsilon} e^{1-t}, \quad \lambda = 0.5$$

where,

$$\gamma_1 = -\frac{e+(1+\varepsilon)e^{-\frac{2}{\varepsilon}}}{(2+\varepsilon)\left(1-e^{-\frac{2}{\varepsilon}}\right)}, \quad \gamma_2 = \frac{1+e+\varepsilon}{(2+\varepsilon)\left(1-e^{-\frac{2}{\varepsilon}}\right)}.$$

First and second derivatives have the form

$$u(t) = \left(\frac{2}{\varepsilon}\right)^k \gamma_2 e^{-\frac{2}{\varepsilon}(1-t)} + \frac{1}{2+\varepsilon} e^{1-t}, \quad k = 1, 2.$$

Therefore, we observe here the accordance in our theoretical results described above.

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