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The Periodic Solitary Wave Solutions for the (2 + 1)-Dimensional Fifth-Order KdV Equation

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Abstract

The (2 + 1)-dimensional fifth-order KdV equation is an important higher-dimensional and higher-order extension of the famous KdV equation in fluid dynamics. In this paper, by constructing new test functions, we investigate the periodic solitary wave solutions for the (2 + 1)-dimensional fifth-order KdV equation by virtue of the Hirota bilinear form. Several novel analytic solutions for such a model are obtained and verified with the help of symbolic computation.

Keywords

(2 + 1)-Dimensional Fifth-Order KdV Equation, Periodic Solitary Wave Solutions, Hirota Bilinear Form

1. Introduction

The soliton equations play a very important role in the study of nonlinear phenomena in different fields such as the fluid physics, nonlinear optics, plasma physics and so on [1] [2]. The researches on the explicit analytic solutions for the soliton equations can help understand the nonlinear dynamics better. With the development of soliton theory, there are many systematic approaches solving different kinds of soliton solutions, such as the inverse scattering transformation [1] [2], the Darboux transformation [3], the variable seperation method [4], the bilinear method and so on [5]-[7]. Among those methods, the bilinear method is a powerful and direct approach to find soliton solutions for the nonlinear partial differential equations. Besides the soliton solutions, Dai has presented that the periodic solitary wave solutions for the soliton equations can be obtained by suitable test functions using the bilinear form [8].

In this paper, we will consider the (2 + 1)-dimensional fifth-order KdV equation

$$36u_t = -u_{xxxxx} - 15(uu_{xx})_x - 45u^2u_x + 5u_{xxy} + 15uu_y + 15u_x \int u_y \mathrm{d}x + 5\int u_{yy} \mathrm{d}x, \tag{1}$$

with $u = u(x,y,t)$, which is a (2 + 1)-dimensional analogue of the Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation [9]. When $u_y = 0$, it can be reduced to the CDGKS equation. Equation (1) was first proposed by Konopelchenko and Dubovsky [10] [11]. In Ref. [12], the quasi-periodic solutions for Equation (1) have been obtained in terms of the Riemann theta functions. The symmetry transformations for Equation (1) have been given based on its Lax pair [13]. In this paper, with the help of symbolic computation, some novel periodic solitary wave solutions for Equation (1) will be derived based on the bilinear form.

2. Bilinear Form

According to the leading order analysis in the Painlevé test, we can find the dependent variable transformation for Equation (1),

$$u = 2(\ln f)_{xx}, \tag{2}$$

with $f = f(x,y,t)$. Substituting Transformation (2) into Equation (1), the following bilinear form can be obtained,

$$\left(36D_xD_t + D_x^6 - 5D_x^3D_y - 5D_y^2\right) f \cdot f = 0, \tag{3}$$

where D_x, D_y and D_t are the bilinear derivative operators [14] defined as,

$$D_x^m D_y^n D_t^l (\alpha\cdot\beta) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l \alpha(x,y,t)\beta(x',y',t')\Big|_{x'=x,y'=y,t'=t}.$$

3. Periodic Solitary Wave Solutions

In this section, according to different test functions for f, we will derive the periodic solitary wave solutions for Equation (1).

3.1. Single Periodic Solitary Wave Solutions

Taking f in Equation (3) as the following form

$$f = a_1\cos(k_1x + h_1y + w_1t) + a_2\cosh(k_2x + h_2y + w_2t) + \mathrm{e}^{-k_3x - h_3y - w_3t} + a_3\mathrm{e}^{k_3x + h_3y + w_3t}. \tag{4}$$

Substituting Solution (4) into the bilinearized Equation (3), and equating the coefficients of different triangle and exponential functions to be zero, we can obtain the following equations,

$$-k_1^6 + 15k_3^2k_1^4 - 5h_1k_1^3 - 15k_3^4k_1^2 + 15h_3k_3k_1^2 + 15h_1k_3^2k_1 - 36w_1k_1 + k_3^6 - 5h_3k_3^3 + 5h_1^2 - 5h_3^2 + 36k_3w_3 = 0, \tag{5}$$

$$-6k_3k_1^5 + 20k_3^3k_1^3 - 5h_3k_1^3 - 15h_1k_3k_1^2 - 6k_3^5k_1 + 15h_3k_3^2k_1 - 36w_3k_1 + 5h_1k_3^3 + 10h_1h_3 - 36k_3w_1 = 0, \tag{6}$$

$$-k_2^6 - 15k_3^2k_2^4 + 5h_2k_2^3 - 15k_3^4k_2^2 + 15h_3k_3k_2^2 + 15h_2k_3^2k_2 - 36w_2k_2 - k_3^6 + 5h_3k_3^3 + 5h_2^2 + 5h_3^2 - 36k_3w_3 = 0, \tag{7}$$

$$-6k_3k_2^5 - 20k_3^3k_2^3 + 5h_3k_2^3 + 15h_2k_3k_2^2 - 6k_3^5k_2 + 15h_3k_3^2k_2 - 36w_3k_2 + 5h_2k_3^3 + 10h_2h_3 - 36k_3w_2 = 0, \tag{8}$$

$$-k_1^6 + 15k_2^2k_1^4 - 5h_1k_1^3 - 15k_2^4k_1^2 + 15h_2k_2k_1^2 + 15h_1k_2^2k_1 - 36w_1k_1 + k_2^6 - 5h_2k_2^3 + 5h_1^2 - 5h_2^2 + 36k_2w_2 = 0, \tag{9}$$

$$-6k_2k_1^5 + 20k_2^3k_1^3 - 5h_2k_1^3 - 15h_1k_2k_1^2 - 6k_2^5k_1 + 15h_2k_2^2k_1 - 36w_2k_1 + 5h_1k_2^3 + 10h_1h_2 - 36k_2w_1 = 0, \tag{10}$$

$$\begin{aligned} &a_1^2\left(-20h_1k_1^3 + 5h_1^2 - 4\left(4k_1^6 + 9w_1k_1\right)\right) + a_2^2\left(16k_2^6 - 20h_2k_2^3 + 36w_2k_2 - 5h_2^2\right) \\ &+4a_3\left(16k_3^6 - 20h_3k_3^3 + 36w_3k_3 - 5h_3^2\right) = 0. \end{aligned} \tag{11}$$

Solving the above system, two sets of single periodic solitary wave solutions can be got.

Case 1: $k_1 = k_2 = -k_3 = -k$, $w_1 = w_2 = k^5$, $w_3 = -k^5$, $h_1 = h_3 = -2k^3$, $h_2 = 2k^3$

Denoting

$$\theta_1 = kx + 2k^3y - k^5t, \quad \theta_2 = kx - 2k^3y - k^5t,$$

then

$$f=a_1\cos\theta_1+\left(1+\frac{a_2}{2}\right)\mathrm{e}^{-\theta_2}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_2}. \tag{12}$$

with k, a_1, a_2 and a_3 are arbitrary constants. Substituting f into the transformation (2), we can obtain the single periodic solitary wave solution

$$u=-2\frac{\left[-a_1k\sin\theta_1-\left(\frac{a_2}{2}+1\right)k\mathrm{e}^{-\theta_2}+\left(\frac{a_2}{2}+a_3\right)k\mathrm{e}^{\theta_2}\right]^2}{\left[a_1\cos\theta_1+\left(\frac{a_2}{2}+1\right)\mathrm{e}^{-\theta_2}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_2}\right]^2}+2\frac{-a_1k^2\cos\theta_1+\left(\frac{a_2}{2}+1\right)k^2\mathrm{e}^{-\theta_2}+\left(\frac{a_2}{2}+a_3\right)k^2\mathrm{e}^{\theta_2}}{a_1\cos\theta_1+\left(\frac{a_2}{2}+1\right)\mathrm{e}^{-\theta_2}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_2}}. \tag{13}$$

Case 2: $k_1=k_2=-k_3=-k$, $w_1=w_2=-\frac{k^5}{4}$, $w_3=\frac{k^5}{4}$, $h_1=h_3=k^3$, $h_2=-k^3$

Denoting

$$\theta_3=kx-k^3y+\frac{k^5}{4}t,\quad \theta_4=kx+k^3y+\frac{k^5}{4}t,$$

then

$$f=a_1\cos\theta_3+\left(1+\frac{a_2}{2}\right)\mathrm{e}^{-\theta_4}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_4}, \tag{14}$$

with k, a_1, a_2 and a_3 are arbitrary constants. Substituting f into the transformation (2), the following single periodic solitary wave solution can be obtained,

$$u=-2\frac{\left[-a_1k\sin\theta_3-\left(\frac{a_2}{2}+1\right)k\mathrm{e}^{-\theta_4}+\left(\frac{a_2}{2}+a_3\right)k\mathrm{e}^{\theta_4}\right]^2}{\left[a_1\cos\theta_3+\left(\frac{a_2}{2}+1\right)\mathrm{e}^{-\theta_4}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_4}\right]^2}+2\frac{-a_1k^2\cos\theta_3+\left(\frac{a_2}{2}+1\right)k^2\mathrm{e}^{-\theta_4}+\left(\frac{a_2}{2}+a_3\right)k^2\mathrm{e}^{\theta_4}}{a_1\cos\theta_3+\left(\frac{a_2}{2}+1\right)\mathrm{e}^{-\theta_4}+\left(\frac{a_2}{2}+a_3\right)\mathrm{e}^{\theta_4}}. \tag{15}$$

3.2. Two Periodic Solitary Wave Solutions

Taking f as the following form

$$f=a_1\cos\left(k_1x+h_1y+w_1t\right)+a_2\sin\left(k_2x+h_2y+w_2t\right)+\mathrm{e}^{-k_3x-h_3y-w_3t}+a_3\mathrm{e}^{k_3x+h_3y+w_3t}. \tag{16}$$

Substituting f into the bilinearized Equation (3), and equating the coefficients of different triangle and exponential functions to be zero, we can obtain the following equations,

$$-k_1^6+15k_3^2k_1^4-5h_1k_1^3-15k_3^4k_1^2+15h_3k_3k_1^2+15h_1k_3^2k_1-36w_1k_1+k_3^6-5h_3k_3^3+5h_1^2-5h_3^2+36k_3w_3=0,$$

$$-6k_3k_1^5+20k_3^3k_1^3-5h_3k_1^3-15h_1k_3k_1^2-6k_3^5k_1+15h_3k_3^2k_1-36w_3k_1+5h_1k_3^3+10h_1h_3-36k_3w_1=0,$$

$$-6k_3k_2^5+20k_3^3k_2^3-5h_3k_2^3-15h_2k_3k_2^2-6k_3^5k_2+15h_3k_3^2k_2-36w_3k_2+5h_2k_3^3+10h_2h_3-36k_3w_2=0,$$

$$-k_2^6+15k_3^2k_2^4-5h_2k_2^3-15k_3^4k_2^2+15h_3k_3k_2^2+15h_2k_3^2k_2-36w_2k_2+k_3^6-5h_3k_3^3+5h_2^2-5h_3^2+36k_3w_3=0,$$

$$-6k_2k_1^5-20k_2^3k_1^3-5h_2k_1^3-15h_1k_2k_1^2-6k_2^5k_1-15h_2k_2^2k_1-36w_2k_1-5h_1k_2^3+10h_1h_2-36k_2w_1=0,$$

$$-k_1^6-15k_2^2k_1^4-5h_1k_1^3-15k_2^4k_1^2-15h_2k_2k_1^2-15h_1k_2^2k_1-36w_1k_1-k_2^6-5h_2k_2^3+5h_1^2+5h_2^2-36k_2w_2=0,$$

$$\begin{aligned}&\left(-20h_1k_1^3+5h_1^2-4\left(4k_1^6+9w_1k_1\right)\right)a_1^2+a_2^2\left(-16k_2^6-20h_2k_2^3-36w_2k_2+5h_2^2\right)\\&+4a_3\left(16k_3^6-20h_3k_3^3+36w_3k_3-5h_3^2\right)=0\end{aligned}.$$

Solving the above equations, we can obtain some novel two periodic solitary solutions of Equation (1).

Case 1: $k_1=k_3=-k_2=-k, w_1=w_3=-\frac{k^5}{4}, w_2=\frac{k^5}{4}, h_2=h_3=-k^3, h_1=k^3$

Denoting

$$\zeta_1=-kx+k^3y-\frac{k^5}{4}t,\ \ \zeta_2=kx-k^3y+\frac{k^5}{4}t,\ \ \zeta_3=-kx-k^3y-\frac{k^5}{4}t, \tag{17}$$

then

$$f=a_1\cos\zeta_1+e^{-\zeta_3}+a_2\sin\zeta_2+a_3e^{\zeta_3}, \tag{18}$$

with k, a_1, a_2 and a_3 are arbitrary constants. Substituting f into the transformation (2), we can obtain the two periodic solitary wave solution

$$\begin{aligned}u=&2\frac{k^2e^{-\zeta_3}-a_1k^2\cos\zeta_1-a_2k^2\sin\zeta_2+a_3k^2e^{\zeta_3}}{a_1\cos\zeta_1+e^{-\zeta_3}+a_2\sin\zeta_2+a_3e^{\zeta_3}}\\&-2\frac{\left(ke^{-\zeta_3}-a_1k\sin\zeta_2+a_2k\cos\zeta_1-a_3ke^{\zeta_3}\right)^2}{\left(a_1\cos\zeta_1+e^{-\zeta_3}+a_2\sin\zeta_2+a_3e^{\zeta_3}\right)^2}\end{aligned}. \tag{19}$$

Case 2: $k_1=k_3=-k_2=-k$, $w_1=w_3=k^5$, $w_2=-k^5$, $h_2=h_3=2k^3$, $h_1=-2k^3$

Denoting

$$\zeta_4=-kx-2k^3y+k^5t,\ \ \zeta_5=kx+2k^3y-k^5t,\ \ \zeta_6=-kx+2k^3y+k^5t,$$

then

$$f=a_1\cos\zeta_4+e^{-\zeta_6}+a_2\sin\zeta_5+a_3e^{\zeta_6}. \tag{20}$$

with k, a_1, a_2 and a_3 are arbitrary constants. Substituting f into the transformation (2), we can obtain the two periodic solitary wave solution

$$\begin{aligned}u=&2\frac{k^2e^{-\zeta_6}-a_1k^2\cos\zeta_4-a_2k^2\sin\zeta_5+a_3k^2e^{\zeta_6}}{a_1\cos\zeta_4+e^{-\zeta_6}+a_2\sin\zeta_5+a_3e^{\zeta_6}}\\&-2\frac{\left(k_2e^{-\zeta_6}-a_1k\sin\zeta_5+a_2k\cos\zeta_4-a_3ke^{\zeta_6}\right)^2}{\left(a_1\cos\zeta_4+e^{-\zeta_6}+a_2\sin\zeta_5+a_3e^{\zeta_6}\right)^2}\end{aligned}. \tag{21}$$

Case 3: $k_1=k_3=-k_2=-k$, $h_1=-h_2$, $a_3=0$,

$$h_3=-2k^3-h_2\pm2\sqrt{3}\sqrt{k^6+h_2k^3},\quad w_1=\frac{16k^6+20h_2k^3-5h_2^2}{36k},\ \ w_2=-w_1,$$

$$w_3=\frac{-44k^6-40h_2k^3\pm20\sqrt{3}\sqrt{k^6+h_2k^3}k^3-5h_2^2\pm20\sqrt{3}h_2\sqrt{k^6+h_2k^3}}{36k},$$

which can give two sets of solution. Denoting

$$\vartheta_1=-kx-h_2y+\frac{\left(16k^6+20h_2k^3-5h_2^2\right)}{36k}t,\ \ \vartheta_2=-\vartheta_1,\ \ \vartheta_3=-kx+h_3y+w_3t.$$

Then f can be obtained in the following form,

$$f = a_1 \cos \vartheta_1 + a_2 \sin \vartheta_2 + e^{-\vartheta_3}, \qquad (22)$$

According to Transformation (2), we can yield another two periodic solitary wave solutions for Equation (1).

4. Conclusion

As the important (2 + 1)-dimensional higher-order generalization of the KdV equation, the solutions for the (2 + 1)-dimensional fifth-order KdV equation are good at understanding the nonlinear phenomena in the fluid dynamics. In this paper, the bilinear form for such an equation is derived based on a logarithm transformation. And then, by choosing two kinds of test functions, we have derived six new sets of periodic solitary wave solutions and verified them using the symbolic computation. It is hoped that the results obtained in this paper can be of help for the study of (2 + 1)-dimensional fifth-order KdV equation and the potential real application.

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