Journal of Applied Mathematics and Physics, 2013, 1, 15-19
Published Online November 2013 (http://www.scirp.org/journal/jamp)
http://dx.doi.org/10.4236/jamp.2013.16004

Hybrid Post-Processing Procedure for Displacement-Based Plane Elements

Xiaoming Chen[1], Song Cen[2], Jianyun Sun[1], Yungui Li[1]
[1]China State Construction Technical Center, Beijing, China
[2]Department of Engineering Mechanics, School of Aerospace, Tsinghua University, Beijing, China
Email: chenxiaoming00@tsinghua.org.cn

Received August 2013

ABSTRACT

In the analysis of high-rise building, traditional displacement-based plane elements are often used to get the in-plane internal forces of the shear walls by stress integration. Limited by the singular problem produced by wall holes and the loss of precision induced by using differential method to derive strains, the displacement-based elements cannot always present accuracy enough for design. In this paper, the hybrid post-processing procedure based on the Hellinger-Reissner variational principle is used for improving the stress precision of two quadrilateral plane elements. In order to find the best stress field, three different forms are assumed for the displacement-based plane elements $AQ4\theta$ and $AQ4\theta\lambda$ with drilling DOF. Numerical results show that by using the proposed method, the accuracy of stress solutions of these two displacement-based plane elements can be improved.

Keywords: Finite Element; Displacement-Based Plane Element; Hybrid Post-Processing Procedure

1. Introduction

For traditional displacement-based elements derived from the principle of minimum potential energy, the trial functions are usually assumed as a polynomial of nodal displacements, and then the internal forces or stresses can be derived by stress-strain relationship. By this process of formulation, these kinds of elements can present accurate displacement results, otherwise the construction procedures of these formulations are also easier. But in most situations, the stress results are much more important than the displacements such as the designing of high-rise buildings. The reinforcements of shear walls are depended on the internal forces integrated from stresses. For the sake of differential with displacement, the accuracy of stresses is usually lower than the displacement results. Many efforts have been made for overcoming this shortage, such as the assumed strain formulations developed by MacNeal [1], Piltner and Taylor [2].

On the other hand, the hybrid/mixed elements are developed from the modified variational principle of Hu-Washisu, such as the hybrid method proposed by Pian [3], Pian and Sumihara [4]. It is different from the displacement-based elements. To formulate these elements, the internal force field or stress field should be assumed at first. Without the differential process which is needed for the displacement-based elements to derive strains, so they can exhibit better stress results. But they also present obvious disadvantages. For example, the construction procedures are much more complex, and the calculation of matrix inversion and condense are usually unavoidable, furthermore, the accuracy of displacement results usually are not as accurate as the displacement-based elements.

By combining the merits of both hybrid elements and displacement-based elements, a new method named hybrid post procedure was proposed by Cen [5]. In this method, the nodal displacements are derived from displacement-based plate elements at first, after this, a new internal force field which should satisfy the equilibrium Equation is assumed to substitute into the hybrid energy functional, by the principle of stationary, the undetermined parameters of the assumed internal force field can be solved out, and finally the new internal forces can be calculated easily. It is seemed very effective for improving the internal force accuracy of plate element. In this paper, this method is extended to improve the plane elements.

2. Hybrid Post-Processing Procedure

For the hybrid plate elements, the discrete energy functional can be written as follows [5,6]:

$$\begin{aligned}\Pi_R^e = & -\frac{1}{2}\iint_{A^e}\{\mathbf{M}\}^T[\mathbf{D}_b]^{-1}\{\mathbf{M}\}\,dA \\ & -\frac{1}{2}\iint_{A^e}\{\mathbf{T}\}^T[\mathbf{D}_s]^{-1}\{\mathbf{T}\}dA \\ & +\iint_{A^e}\{\mathbf{M}\}^T\{\boldsymbol{\kappa}\}\,dA+\iint_{A^e}\{\mathbf{T}\}^T\{\boldsymbol{\gamma}\}\,dA-\iint_{A^e} f_z w dA \\ & -\int_s(\bar{T}_n w+\bar{M}_{xn}\psi_x+\bar{M}_{yn}\psi_y)ds\end{aligned} \tag{1}$$

where $W_{\exp}$ is the work of external forces:

$$W_{\exp}=\iint_{A^e} f_z w dA+\int_s(\bar{T}_n w+\bar{M}_{xn}\psi_x+\bar{M}_{yn}\psi_y)ds \tag{2}$$

Assume the new internal force field as follows:

$$\begin{aligned}&\{\mathbf{M}\}=[\mathbf{P}_M]\{\boldsymbol{\alpha}_M\}\\&\{\mathbf{T}\}=\begin{Bmatrix}T_x\\T_y\end{Bmatrix}=\begin{Bmatrix}M_{x,x}+M_{xy,y}\\M_{xy,x}+M_{y,y}\end{Bmatrix}=[\mathbf{P}_T]\{\boldsymbol{\alpha}_M\}\end{aligned} \tag{3}$$

where

$$\begin{aligned}&\{\mathbf{M}\}=[M_x \quad M_y \quad M_{xy}]^T\\&[\mathbf{P}_M]=\begin{bmatrix}1&\xi&\eta&\xi\eta&0&0&0&0&0&0&0&0\\0&0&0&0&1&\xi&\eta&\xi\eta&0&0&0&0\\0&0&0&0&0&0&0&0&1&\xi&\eta&\xi\eta\end{bmatrix}\\&[\mathbf{P}_T]=\begin{bmatrix}0&j_{11}&j_{12}&j_{11}\eta+j_{12}\xi&0&0&0&0&0&j_{21}&j_{22}&j_{21}\eta+j_{22}\xi\\0&0&0&0&0&j_{21}&j_{22}&j_{21}\eta+j_{22}\xi&0&j_{11}&j_{12}&j_{11}\eta+j_{12}\xi\end{bmatrix}\\&\{\boldsymbol{\alpha}_M\}=\begin{bmatrix}\alpha_1&\alpha_2&\cdots&\alpha_{11}&\alpha_{12}\end{bmatrix}^T\end{aligned} \tag{4}$$

and j_{11}, j_{12}, j_{21}, j_{22} are components of the inversion matrix of Jacobian .

After substituting Equation (3) into Equation (1), the new form of energy functional can be written as:

$$\begin{aligned}\Pi_R^e = & -\frac{1}{2}\{\boldsymbol{\alpha}_M\}^T\left(\begin{aligned}&\iint_{A^e}[\mathbf{P}_M]^T[\mathbf{D}_b]^{-1}[\mathbf{P}_M]dA\\&+\iint_{A^e}[\mathbf{P}_T]^T[\mathbf{D}_s]^{-1}[\mathbf{P}_T]dA\end{aligned}\right)\{\boldsymbol{\alpha}_M\}\\ & +\{\boldsymbol{\alpha}_M\}^T\iint_{A^e}([\mathbf{P}_M]^T[\mathbf{B}_b]\\ & +[\mathbf{P}_T]^T[\mathbf{B}_s])dA\cdot\{\mathbf{q}\}^e-W_{\exp}\end{aligned} \tag{5}$$

By the principle of stationary:

$$\frac{\partial\Pi_R^e}{\partial\{\boldsymbol{\alpha}_M\}}=0 \tag{6}$$

the parameters of the assumed internal force field can be written as:

$$\{\boldsymbol{\alpha}_M\}=-[\mathbf{K}_{MM}]^{-1}[\mathbf{K}_{Mq}]\{\mathbf{q}\}^e \tag{7}$$

where

$$\begin{aligned}[\mathbf{K}_{MM}]=&-\iint_{A^e}([\mathbf{P}_M]^T[\mathbf{D}_b]^{-1}[\mathbf{P}_M]\\&+[\mathbf{P}_T]^T[\mathbf{D}_s]^{-1}[\mathbf{P}_T])dA\\ [\mathbf{K}_{Mq}]=&\iint_{A^e}([\mathbf{P}_M]^T[\mathbf{B}_b]+[\mathbf{P}_T]^T[\mathbf{B}_s])dA\end{aligned} \tag{8}$$

Substitute Equation (7) into Equation (3), the new internal force can be obtained.

Compared with plate elements, it is much easier to extend this method to plane elements. The according hybrid discrete energy functional of plane elements can be written as [6]:

$$\Pi_R^e=\iint_{V^e}\left[\{\boldsymbol{\sigma}\}^T D\{\mathbf{u}\}-\frac{1}{2}\{\boldsymbol{\sigma}\}^T[\mathbf{D}]^{-1}\{\boldsymbol{\sigma}\}\right]dV \tag{9}$$

Similar to Equation (4), a new form of stress field of displacement-based plane element can be assumed as:

$$\{\boldsymbol{\sigma}\}=[\mathbf{P}_\sigma]\{\boldsymbol{\alpha}_\sigma\} \tag{10}$$

Substitute Equation (10) into Equation (9), the parameters α_{i} in $[\mathbf{P}_\sigma]$ can be obtained by using the principle of stationary. The according matrix named $[\mathbf{K}_{\sigma\sigma}]$ and $[\mathbf{K}_{\sigma q}]$ for plane elements are as follows:

$$\begin{aligned}[\mathbf{K}_{\sigma\sigma}]&=-\iint_{A^e}[\mathbf{P}_\sigma]^T[\mathbf{D}]^{-1}[\mathbf{P}_\sigma]tdA\\ [\mathbf{K}_{\sigma q}]&=\iint_{A^e}[\mathbf{P}_\sigma]^T[\mathbf{B}_b]tdA\end{aligned} \tag{11}$$

With different matrix of $[\mathbf{P}_\sigma]$, the stress field will be different too. This new method will be used to try to improve the stress accuracy of plane element named AQ4θ and AQ4$\theta\lambda$ [7].

3. Introduction of AQ4θ and AQ4$\theta\lambda$

AQ4θ and AQ4$\theta\lambda$ are two plane elements with drilling DOF formulated using quadrilateral area coordinate methods presented by Long *et al*. [8,9]. They have the merits of high accuracy and robust against mesh distortions. After having been programmed into the software for the analysis of high-rise buildings, the results show that better accuracy is still needed for the stress of shear walls with holes.

The definition of DOF of these two elements is:

$$\{\mathbf{q}\}^e=\left[\{\mathbf{q}_1\}^T \quad \{\mathbf{q}_2\}^T \quad \{\mathbf{q}_3\}^T \quad \{\mathbf{q}_4\}^T\right]^T \tag{12}$$

where

$$\{\mathbf{q}_i\}=[u_i \quad v_i \quad \theta_i]^T \ (i=1,2,3,4) \tag{13}$$

and θ_i is the additional rigid rotation at element node.

The displacements of element are as follows:

$$\{\mathbf{u}\}=\{\mathbf{u}^0\}+\{\mathbf{u}^\theta\} \tag{14}$$

where $\{\mathbf{u}^0\}$ is a polynomial about u_i and v_i, $\{\mathbf{u}^\theta\}$ is the additional displacement field induced only by the rigid rotation at nodes denoted as θ_i.

In order to determine the displacement field $\{\mathbf{u}^0\}$, the shape functions in reference [10] is used, they are as follows:

$$u^0 = \sum_{i=1}^{4} N_i^0 u_i \qquad v^0 = \sum_{i=1}^{4} N_i^0 v_i \tag{15}$$

where:

$$N_i^0 = -\frac{1}{2} g_{i+2} + L_i + L_{i+1} + \xi_i \eta_i g_{i+2} P \quad (i = 1,2,3,4) \tag{16}$$

and

$$P = \frac{3(L_3-L_1)(L_4-L_2)-(g_2-g_3)(L_3-L_1)-(g_1-g_2)(L_4-L_2)-\frac{1}{2}(g_2g_4-g_1g_3)}{1+g_1g_3+g_2g_4} \tag{17}$$

Assume the rotational displacement field as polynomials of quadrilateral area coordinates:

$$\begin{aligned} u_\theta &= \alpha_1 + \alpha_2(L_3-L_1) + \alpha_3(L_4-L_2) \\ &\quad + \alpha_4(L_3-L_1)(L_4-L_2) \\ &\quad + \alpha_5 L_1 L_3 + \alpha_6 L_2 L_4 \\ v_\theta &= \beta_1 + \beta_2(L_3-L_1) + \beta_3(L_4-L_2) \\ &\quad + \beta_4(L_3-L_1)(L_4-L_2) \\ &\quad + \beta_5 L_1 L_3 + \beta_6 L_2 L_4 \end{aligned} \tag{18}$$

with the conforming Equations as follows, the rotational displacement can be obtained.

$$\begin{aligned} &\sum_{i=1}^{4} (u_\theta^i - \bar{u}_\theta^i) = 0 \\ &\sum_{i=1}^{4} \xi_i \eta_i (u_\theta^i - \bar{u}_\theta^i) = 0 \\ &\int_{l_{ij}} (u_\theta - \bar{u}_\theta) d\bar{s} = 0 \quad (ij = 12,23,34,41) \end{aligned} \tag{19}$$

where

$$\begin{aligned} \begin{Bmatrix} \bar{u}_{\theta 12} \\ \bar{v}_{\theta 12} \end{Bmatrix} &= \begin{Bmatrix} -b_4 \\ -c_4 \end{Bmatrix} L_1 L_3 \left(\frac{1}{g_1 g_2^2} L_1 \theta_1 - \frac{1}{g_1^2 g_2} L_3 \theta_2 \right) \\ \begin{Bmatrix} \bar{u}_{\theta 23} \\ \bar{v}_{\theta 23} \end{Bmatrix} &= \begin{Bmatrix} -b_1 \\ -c_1 \end{Bmatrix} L_2 L_4 \left(\frac{1}{g_2 g_3^2} L_2 \theta_2 - \frac{1}{g_2^2 g_3} L_4 \theta_3 \right) \\ \begin{Bmatrix} \bar{u}_{\theta 34} \\ \bar{v}_{\theta 34} \end{Bmatrix} &= \begin{Bmatrix} -b_2 \\ -c_2 \end{Bmatrix} L_1 L_3 \left(\frac{1}{g_3 g_4^2} L_3 \theta_3 - \frac{1}{g_3^2 g_4} L_1 \theta_4 \right) \\ \begin{Bmatrix} \bar{u}_{\theta 41} \\ \bar{v}_{\theta 41} \end{Bmatrix} &= \begin{Bmatrix} -b_3 \\ -c_3 \end{Bmatrix} L_2 L_4 \left(\frac{1}{g_4 g_1^2} L_4 \theta_4 - \frac{1}{g_4^2 g_1} L_2 \theta_1 \right) \end{aligned} \tag{20}$$

and $b_i = y_{i+1} - y_{i+2}$, $c_i = x_{i+2} - x_{i+1}$

Then the stiffness matrix of element AQ4θ can be solved out easily, and the strain matrix is:

$$\{\varepsilon\} = [\boldsymbol{B}_q]\{\boldsymbol{q}\}^e \tag{21}$$

where

$$[\boldsymbol{B}_q] = [\boldsymbol{B}_1 \quad \boldsymbol{B}_2 \quad \boldsymbol{B}_3 \quad \boldsymbol{B}_4] \tag{22}$$

and

$$[\boldsymbol{B}_i] = \begin{bmatrix} \frac{\partial N_i^0}{\partial x} & 0 & \frac{\partial N_{u\theta i}}{\partial x} \\ 0 & \frac{\partial N_i^0}{\partial y} & \frac{\partial N_{v\theta i}}{\partial y} \\ \frac{\partial N_i^0}{\partial y} & \frac{\partial N_i^0}{\partial x} & \frac{\partial N_{u\theta i}}{\partial y} + \frac{\partial N_{v\theta i}}{\partial x} \end{bmatrix} \tag{23}$$

After substituting Equation (21) into the stress-strain relationship, the stress of AQ4θ can be obtained.

AQ4θ is an improved element based on AQ4θ by adding a displacement field which is induced by internal parameters.

The additional displacement fields mentioned above are as follows:

$$\begin{aligned} u_\lambda &= \alpha_1 + \alpha_2(L_3-L_1) + \alpha_3(L_4-L_2) \\ &\quad + \alpha_4 L_1 L_3 + \lambda_1 L_2 L_4 + \lambda_2 (L_3-L_1)(L_4-L_2) \\ v_\lambda &= \beta_1 + \beta_2(L_3-L_1) + \beta_3(L_4-L_2) \\ &\quad + \beta_4 L_1 L_3 + \lambda_1' L_2 L_4 + \lambda_2' (L_3-L_1)(L_4-L_2) \end{aligned} \tag{24}$$

where $\lambda_1, \lambda_1', \lambda_2, \lambda_2'$ are the internal parameters.

In order to solve the undetermined parameters, the conforming Equations are taken as:

$$\int_{l_{ij}} \{\mathbf{u}_\lambda\} ds = 0 \tag{25}$$

Substitute Equation (24) into (25), the shape functions of $\{\mathbf{u}_\lambda\}$ can be formulated out, they are:

$$\begin{aligned} N_{\lambda 1} &= \frac{(6g_1g_2g_3g_4 - g_1g_3 - g_2g_4)}{6(1-g_1g_4-g_2g_3)} \\ &\quad + \frac{g_1-g_2}{6}(L_3-L_1) \\ &\quad + \frac{(g_1-g_4)(g_1g_4+g_2g_3)}{6(1-g_1g_4-g_2g_3)}(L_4-L_2) \\ &\quad + \frac{g_1g_4+g_2g_3}{1-g_1g_4-g_2g_3} L_1L_3 + L_2L_4 \\ N_{\lambda 2} &= \frac{1}{3}(g_3-g_2)(L_3-L_1) - 2(g_1-g_2)(L_4-L_2) \\ &\quad + (L_3-L_1)(L_4-L_2) \end{aligned} \tag{26}$$

For element AQ4$\theta\lambda$, $N_{\lambda 1}$ is taken as the final internal shape function, then:

$$\{\mathbf{u}_\lambda\} = [\mathbf{N}_\lambda]\{\boldsymbol{\lambda}\} \tag{27}$$

where

$$\{\boldsymbol{\lambda}\} = [\lambda_1 \quad \lambda_1']^T$$
$$[\mathbf{N}_\lambda] = \begin{bmatrix} N_{\lambda 1} & 0 \\ 0 & N_{\lambda 1} \end{bmatrix} \tag{28}$$

Through the condense calculation, stiffness matrix of AQ4$\theta\lambda$ can be written as:

$$[\boldsymbol{k}]^e = [\boldsymbol{k}_{qq}] - [\boldsymbol{k}_{\lambda q}]^T [\boldsymbol{k}_{\lambda\lambda}]^{-1} [\boldsymbol{k}_{\lambda q}] \tag{29}$$

where

$$\begin{aligned} [\mathbf{k}_{qq}] &= \iint [\mathbf{B}_q]^T [\mathbf{D}][\mathbf{B}_q] t dA \\ [\mathbf{k}_{\lambda\lambda}] &= \iint [\mathbf{B}_\lambda]^T [\mathbf{D}][\mathbf{B}_\lambda] t dA \\ [\mathbf{k}_{\lambda q}] &= \iint [\mathbf{B}_\lambda]^T [\mathbf{D}][\mathbf{B}_q] t dA \end{aligned} \tag{30}$$

and $[\mathbf{B}_\lambda]$ is the strain matrix of additional displacement field. The according stress fields are as follows:

$$\{\boldsymbol{\varepsilon}_\lambda\} = [\mathbf{B}_\lambda]\{\boldsymbol{\lambda}\} \tag{31}$$

Combined Equation (21) with (31), the stress field of AQ4$\theta\lambda$ can be obtained.

4. Hybrid Post-Processing of AQ4*θ* and AQ4*θλ*

In the formulating of stress in elements AQ4θ and AQ4$\theta\lambda$, the strain matrix $[\mathbf{B}_q]$ and $[\mathbf{B}_\lambda]$ are the differential results of displacements to coordinates. The differential calculation lowered the accurate order of stress. To avoid the differential, Hybrid Post-processing procedures can be used to improve the stress accuracy. Based on this theory, three forms of stress fields are presented for these two displacement-based elements.

4.1. Stress Field I

The first form of stress field is assumed as Equation (32). It is the stress field of a hybrid element developed by Pian and Wu [6]:

$$\{\boldsymbol{\sigma}\} = \begin{bmatrix} 1 & & & a_1^2\eta & a_3^2\xi \\ & 1 & & b_1^2\eta & b_3^2\xi \\ & & 1 & a_1b_1\eta & a_3b_3\zeta \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_5 \end{Bmatrix} \tag{32}$$

where

$$\begin{aligned} a_1 &= \frac{1}{4}\sum_{i=1}^{4}\xi_i x_i \quad & a_2 &= \frac{1}{4}\sum_{i=1}^{4}\xi_i\eta_i x_i \quad & a_3 &= \frac{1}{4}\sum_{i=1}^{4}\eta_i x_i \\ b_1 &= \frac{1}{4}\sum_{i=1}^{4}\xi_i y_i \quad & b_2 &= \frac{1}{4}\sum_{i=1}^{4}\xi_i\eta_i y_i \quad & b_3 &= \frac{1}{4}\sum_{i=1}^{4}\eta_i y_i \end{aligned} \tag{33}$$

4.2. Stress Field II

In order to raise the complete order of stress field, the second form is assumed as a polynomial of analytical trial function method presented by Fu and Long [11]:

$$\{\boldsymbol{\sigma}\} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2y & 0 & 2x & 0 & 6xy \\ 0 & 1 & 0 & 2x & 0 & 2y & 0 & 6xy & 0 \\ -1 & 0 & 0 & 0 & 0 & -2x & -2y & -3x^2 & -3y^2 \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_9 \end{Bmatrix} \tag{34}$$

4.3. Stress Field III

Try to make the three stress components to be independent, the third form of stress field is assumed as follows:

$$\{\boldsymbol{\sigma}\} = \begin{bmatrix} 1 & x & y & xy & & & & & & & & \\ & & & & 1 & x & y & xy & & & & \\ & & & & & & & & 1 & x & y & xy \end{bmatrix} \begin{Bmatrix} \beta_1 \\ \vdots \\ \beta_{12} \end{Bmatrix} \tag{35}$$

5. Numerical Examples

5.1. Strict Patch Test

The constant strain/stress patch test using irregular mesh is shown in **Figure 1**. Let Young's modulus $E = 1000$, Poisson's ratio $\mu = 0.25$, and thickness of the patch $t = 1$. After modified by three different forms of Hybrid Post-processing, these two elements can produce exact solutions without any problem.

5.2. Cook's Skew Beam

This example was proposed by Cook *et al.* [12]. As shown in **Figure 2**, a skew cantilever beam subjected to distributed shear load along its free edge. The results of $\sigma_{\max}$ at point A and $\sigma_{\min}$ at point B are listed in **Tables 1** and **2**.

6. Conclusion

Using the traditional method to calculate stress of displacement-based elements, the accuracy will descend for the reason of differential when strains are derived from the displacement fields. For improving the stress accuracy, hybrid/mix elements are very effective. However, the formulations of the hybrid elements are more complicated than those displacement-based elements. Based on

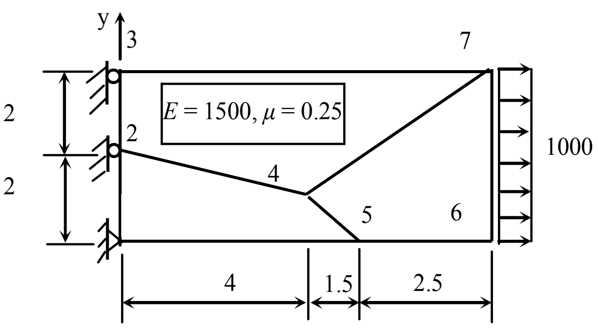

Figure 1. Patch test.

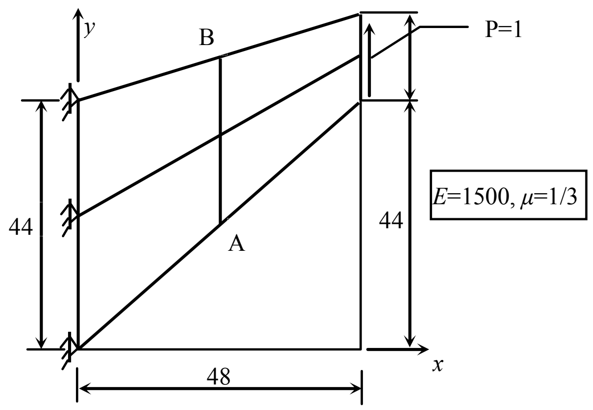

Figure 2. Cook's skew beam.

Table 1. Stress at point A and B of Cook's Beam of AQ4*θ*.

Method	σ_{Amax}			σ_{Bmin}		
	2 × 2	4 × 4	8 × 8	2 × 2	4 × 4	8 × 8
Field I	0.1791	0.2261	0.2338	−0.1700	−0.1929	−0.2002
Field II	0.1951	0.2298	0.2348	−0.1942	−0.1933	−0.2010
Field III	0.1914	0.2240	0.2319	−0.1769	−0.1938	−0.2009
Source Val.	0.1917	0.2241	0.2377	−0.1877	−0.1939	−0.2060
Ref. Val.		0.2362			−0.2023	

Table 2. Stress at point A and B of Cook's Beam of AQ4*θλ*.

Method	σ_{Amax}			σ_{Bmin}		
	2 × 2	4 × 4	8 × 8	2 × 2	4 × 4	8 × 8
Field I	0.1913	0.2271	0.2342	−0.1748	−0.1919	−0.2009
Field II	0.2145	0.2358	0.2364	−0.2084	−0.2032	−0.2027
Field III	0.2147	0.2358	0.2364	−0.2092	−0.2033	−0.2027
Source Val.	0.2498	0.2338	0.2358	−0.1729	−0.1896	−0.2018
Ref. Val.		0.2362			−0.2023	

the Hellinger-Reissner variational principle, hybrid post-process procedure can take advantage of the merits of these two kinds of elements to establish the relationship between the displacement and the stress or internal force fields. In this paper, based on this theory, three forms of stress fields are used to improve the stress of plane elements with drilling DOF. Through the numerical results, for element AQ4θ, only the second form of stress field is effective, but for AQ4θλ, except for the first form of stress, the other two forms can present better results than the source elements. It is proved that the method of hybrid post-process procedure is workable.

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