An Optimal Inequality for One-Parameter Mean

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Received September 16, 2013; revised October 15, 2013; accepted October 21, 2013

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ABSTRACT

In the present paper, we answer the question: for $0 < \alpha < 1$ fixed, what are the greatest values $p(\alpha)$ and the least values $q(\alpha)$ such that the inequality

$$J_p(a,b) < A^{p}\ (a,b)G^{1-q}\ (a,b) < J_q(a,b)$$

holds for all $a, b > 0$ with $a \neq b$? where for $p \in R$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive real numbers $a$ and $b$ are defined by

$$J_p(a,b) = \begin{cases} a, & a \neq b, \\ p(a^{p+1} - b^{p+1}) \div (p+1)(a^p - b^p), & a \neq b, p \neq -1,0, \\ ab(\log a - \log b), & a \neq b, p = -1, \\ a-b \div \log a - \log b, & a \neq b, p = 0, \end{cases}$$

$$A(a,b) = \frac{a+b}{2}$$ and

$$G(a,b) = \sqrt{ab},$$

respectively.

$$J_p(a,b) = \begin{cases} a, & a \neq b, \\ p(a^{p+1} - b^{p+1}) \div (p+1)(a^p - b^p), & a \neq b, p \neq -1,0, \\ ab(\log a - \log b), & a \neq b, p = -1, \\ a-b \div \log a - \log b, & a \neq b, p = 0, \end{cases}$$

In [1], Gao and Niu found the greatest values $p,s_i$ and the least values $q,s_i$ such that the inequalities

$$J_p(a,b) \leq A^{p}\ (a,b)G^{\alpha}\ (a,b)H^{1-\alpha}\ (a,b) \leq J_q(a,b)$$

and

$$G_{s_i}(a,b) \leq A^{p}\ (a,b)G^{\alpha}\ (a,b)H^{1-\alpha}\ (a,b) \leq G_{s_i}(a,b)$$

hold for all $a, b > 0$ with $a \neq b$, where $\alpha + \beta \in (0,1)$,
and $G_{s,1}(a,b) = \left[ \left( \frac{a^r + b^r}{a + b} \right)^r \right]^{\frac{1}{r-1}}$, as the Gini mean.

In [2], Cheune and Qi proved the logarithmic convexity of the one-parameter mean values $J_p(a,b)$ and presented the monotonicity of $J(r) - J(r^e)$ for $r \in R$.

In [3], Wang, Qi and Chu obtained the greatest value $r_1$ and the least value $r_2$ such that the double inequality

$$J_{q_1}(a,b) \leq \alpha A(a,b) + (1-\alpha) H(a,b) \leq J_{q_2}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

In [4], Hu, Tu and Chu presented the greatest value $r_1$ and the least value $r_2$ such that the double inequality

$$J_{q_1}(a,b) \leq T(a,b) \leq J_{q_2}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$, where

$$T(a,b) = \frac{2ab}{2 \arctan \frac{(a-b)}{(a+b)}}$$

denotes the first Seiffert mean.

In [5], Long and Chu found the greatest value $p$ and the least value $q$ such that

$$J_p(a,b) \leq \alpha A(a,b) + (1-\alpha) H(a,b) \leq J_q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

In [6], the authors established Schur-convexities of two types of one-parameter mean values in $n$ variables, and obtained Schur-convexities of some well-known functions.

The purpose of this paper is to answer the question: for $0 < \alpha < 1$ fixed, what are the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that

$$J_p(a,b) < A^\alpha(a,b) G^{1-\alpha}(a,b) < J_q(a,b)$$

holds for all $a,b > 0$ with $a \neq b$?

2. A Preliminary Lemma

In order to prove the main theorem of this paper, we need the following lemma.

**Lemma 2.1.** For all $t > 1$, one has

$$m(t) = \frac{t(t+1) \log t}{2(t-1)} < 1.$$  \hspace{1cm} (2)

**Proof.** The logarithmic derivative of $m(t)$ is

$$\frac{m'(t)}{m(t)} = \left[ \log m(t) \right]' = \frac{n(t)}{t(t^2-1) \log t},$$  \hspace{1cm} (3)

where

$$n(t) = -t^2 + 4t + 1 \log t + 3(t^2-1), \lim_{t \to \infty} n(t) = 0.$$  \hspace{1cm} (4)

Simple calculations lead to

$$n'(t) = 5t^4 - 4t - 2(t+2) \log t, \lim_{t \to \infty} n'(t) = 0,$$

$$n^*(t) = -4t \log t, \lim_{t \to \infty} n(t) = 0,$$

$$n^*(t) = -\frac{2(t-1)^2}{t^2} < 0.$$  \hspace{1cm} (7)

(2) follows from (3)-(7) and the fact

$$\lim_{t \to \infty} m(t) = 1.$$

3. Main Result

The main result of this paper is the following theorem.

**Theorem 3.1.** Let $0 < \alpha < 1$. Then for any $a,b > 0$ with $a \neq b$, we have

$$J_{\alpha^{-1}}(a,b) < A^\alpha(a,b) G^{1-\alpha}(a,b) < J_{\alpha^{-1}}(a,b).$$  \hspace{1cm} (8)

Moreover, the bounds $J_{\alpha^{-1}}(a,b)$ and $J_{\alpha^{-1}}(a,b)$ are optimal.

**Proof.** It is no loss of generality to assume that $a > b$.

Let $t^2 = \frac{a}{b} > 1$, $p \in \left\{ \frac{\alpha-1}{2}, \frac{3\alpha-1}{2} \right\}$ and

$$f(t) = \frac{J_p(t^2,1)}{A^\alpha(t^2,1) G^{1-\alpha}(t^2,1)},$$

then

$$\frac{f'(t)}{f(t)} = \left[ \log f(t) \right]' = \frac{g_i(t)}{t(t^2+1)(t^2+1)(t^{2p+2}-1)},$$  \hspace{1cm} (9)

where

$$g_i(t) = (1-\alpha)t^{2p+4} + (\alpha+1)t^{2p+2} + (\alpha-2p-1)t^{2p+4} + (2p+1-\alpha) t^{2p} - (\alpha+1)t^2 + \alpha - 1$$

$$+ (1-\alpha)x^{2p+4} + (\alpha+1)x^{2p+2} + (\alpha-2p-1)x^{2p+4} + (2p+1-\alpha)x^p - (\alpha+1)x + \alpha - 1 = h_i(x),$$  \hspace{1cm} (10)

where $x = t^2 > 1$. Simple calculations lead to

$$\lim_{x \to \infty} h_i(x) = 0,$$

$$h_i'(x) = 2(p+1)(1-\alpha)x^{2p+1} + (2p+1)(\alpha+1)x^{2p} + (p+2)(\alpha-2p-1)x^{2p+1} + p(2p+1-\alpha)x^{2p+1} - \alpha - 1,$$  \hspace{1cm} (12)
\[
\lim_{t \to +\infty} h_1'(x) = 0,
\]
\[
h_n'(x) = x^{n-2} h_2(x),
\]
where
\[
h_2(x) = 2(2p+1)(p+1)(1-\alpha)x^{p+2} + 2p(2p+1)(\alpha+1)x^{p+1} + (p+1)(p+2)(x-2p-1)x^2 + (p-1)p(2p+1-\alpha),
\]
\[
\lim_{t \to +\infty} h_2(x) = 0,
\]
\[
h_n'(x) = 2(p+1)xh_3(x),
\]
where
\[
h_3(x) = (p+2)(2p+1)(1-\alpha)x^p + p(2p+1)(\alpha+1)x^{p-1} + (p+2)(\alpha-2p-1),
\]
\[
\lim_{t \to +\infty} h_3(x) = p(2p-3\alpha+1),
\]
\[
h_n'(x) = p(2p+1)x^{p-2}h_4(x)
\]
where
\[
h_4(x) = (p+2)(1-\alpha)+x+(p-1)(\alpha+1)
\]
\[
\lim_{t \to +\infty} h_4(x) = 2p-3\alpha+1,
\]
\[
h'_n(x) = (p+2)(1-\alpha).
\]

We now distinguish between two cases.

**Case 1.** \( p = \frac{3\alpha - 1}{2} \). We first consider the case \( \alpha = \frac{1}{3} \) since in this case the one-parameter mean \( J_1(a,b) \) has different expression from others. The result

\[
A^p(t,1)/G^\alpha(t,1) < J_0(t,1)
\]

follows from Lemma 2.1 since
\[
A^p(t,1)/G^\alpha(t,1)/J_0(t,1) = m^3(t,1) < 1,
\]

In the following we assume \( \alpha \neq \frac{1}{3} \).

From (21) we see that \( h_4'(x) > 0 \) for \( x > 1 \), which implies \( h_4(x) \) is strictly increasing for \( x > 1 \). From (20) we know that \( h_3'(x) > 0 \) for all \( x > 1 \). (18) implies
\[
\begin{cases}
0 < \alpha < \frac{1}{3}, \\
0 < \alpha < 1,
\end{cases}
\]
from which we know \( h_4(x) \) is strictly decreasing for
\[
\alpha \in \left(0, \frac{1}{3}\right) \text{ and strictly increasing for } \alpha \in \left(0, \frac{1}{3}\right).
\]

This result together with (17) implies \( h_1'(x) < 0 \) for \( \alpha \in \left(0, \frac{1}{3}\right) \) and \( h_1'(x) > 0 \) for \( \alpha \in \left(0, \frac{1}{3}\right) \). The same reasoning applies to \( h_1'(x), h_2'(x), h_3'(x), h_4'(x) \) as well, and using (15), (14), (12), (11), (9) and (8), we know
\[
g_1(t) < 0 \quad \text{for } \alpha \in \left(0, \frac{1}{3}\right) \quad \text{and} \quad g_1(t) > 0 \quad \text{for } \alpha \in \left(0, \frac{1}{3}\right)
\]

(8) implies \( f_1'(t) > 0 \) for all \( t > 1 \). Thus \( f_1(t) \) is strictly increasing for \( t > 1 \), which together with
\[
\lim_{t \to +\infty} f_1(t) = 1
\]

implies right-hand side inequality of (8).

**Case 2.** \( p = \frac{\alpha-1}{2} \). From (21) we know \( h_4'(x) > 0 \) for all \( x > 1 \), which implies that \( h_4(x) \) is strictly increasing for \( x > 1 \). By (20) one has \( h_4(1') = 2\alpha < 0 \), and by (19) one has
\[
\lim_{t \to +\infty} h_4(t) = +\infty.
\]

Thus there exists \( \xi_1 > 1 \) such that \( h_4(x) < 0 \) for \( x \in (1, \xi_1) \) and \( h_4(x) > 0 \) for \( x \in (\xi_1, +\infty) \). (18) implies \( h_4'(x) > 0 \) for \( x \in (1, \xi_1) \) and \( h_4'(x) < 0 \) for \( x \in (\xi_1, +\infty) \). Thus \( h_4(x) \) is strictly increasing for \( x \in (1, \xi_1) \) and strictly decreasing for \( x \in (\xi_1, +\infty) \). By (17) \( h_4(1') > 0 \) and by
\[
\lim_{t \to +\infty} h_4(x) = 0
\]

we know \( h_4(x) > 0 \) for \( x > 1 \). The same reasoning applies to \( h_1'(x), h_2'(x), h_3'(x), h_4'(x) \) and \( g_1(t) \) as well, and applying (9)-(16), we have \( g_1(t) > 0 \) for all \( t > 1 \). (9) implies \( f_1'(t) < 0 \), thus \( f_1(t) \) is strictly decreasing for \( t > 1 \). The left-hand side inequality of (8) follows from (22).

Next we prove that the bounds \( J_{\alpha,\xi}^p(a,b) \) and \( J_{\alpha,\xi}^p(a,b) \) are optimal.

For any \( \varepsilon > 0 \) and \( t > 0 \) sufficiently small,
\[
\frac{J_{\alpha,\xi}^p(1+t,1)}{J_{\alpha,\xi}^p(1+t,1)} \log \frac{A^p(1+t,1)}{A^p(t,1)} = \log \frac{4t + (3\alpha - 2\varepsilon - 1)t^2}{4t + (3\alpha - 2\varepsilon - 3)t^2} - \log \left(\frac{t + 2\varepsilon}{2}\right)^{1+p}
\]

\[
= 2t - \alpha \left(\frac{t}{2} - \frac{t^2}{8}\right) - \frac{1}{2} \alpha \left(\frac{t}{2} - \frac{t^2}{8}\right)
\]

\[
= \frac{(18 - \alpha)t - 4t + o(t)}{8} < 0.
\]

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This implies

\[ J_{\frac{1}{\alpha + \varepsilon}}(t,1) < A^\alpha(t,1)G^{1-\alpha}(t,1) \]

for \( t \) sufficiently close to 1.

For any \( \varepsilon > 0 \), since

\[ \lim_{t \to 0^+} \frac{J_{\frac{1}{\alpha + \varepsilon}}(t,1)}{A^\alpha(t,1)G^{1-\alpha}(t,1)} = +\infty, \]

then there exists \( T > 1 \) such that

\[ J_{\frac{1}{\alpha + \varepsilon}}(t,1) > A^\alpha(t,1)G^{1-\alpha}(t,1) \]

For \( t > T \).

4. Acknowledgements

This research is supported by NSF of Hebei Province (No. A2011201011).

REFERENCES


