Space of Operators and Property (MB)

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Abstract

In this paper, a new class of Banach spaces, termed as Banach spaces with property (MB), will be introduced. It is stated that a space X has property (MB) if every V-subset of X^* is an L-subset of X^*. We describe those spaces which have property (MB). Also, we show that if a Banach space X has property (MB) and Banach space Y does not contain ℓ_1, then every operator $T : X \to Y^*$ is completely continuous.

Keywords

L-Sets, V-Sets, Completely Continuous Operators, Unconditionally Converging Operators

1. Introduction

In 1992, Saab and Smith [1] studied Banach spaces with the property that every unconditionally converging operator from such a space to an arbitrary Banach space is weakly completely continuous. It is well known that every completely continuous operator is an unconditionally converging operator. In this paper, the converse of this fact by using some localized properties, e.g., L-sets, V-sets will be studied.

First, the property (MB) for Banach spaces is presented, then spaces which have property (MB) are considered. Next, it is demonstrated that X has property (MB) if and only if every unconditionally converging operator $T : X \to Y$ is completely continuous, for every Banach space Y.

In [2], it has shown that a Banach space X has the Reciprocal Dunford-Pettis property whenever it has property (V) but the converse is false. We investigate under what conditions this implication could be reversed.

Also, it will verified that property (MB) implies the Dunford-Pettis property. In addition, we show that a Banach space with property (V) and the Dunford-Pettis

property, as well has the property (MB).

The notions of a relatively compact sets, L-sets, V-sets have a significant role in this study.

Definitions and Notation

In this article, real Banach spaces will be symbolized by X and Y, the unit ball of X will be denoted by B_X, and the continuous linear dual of X will be denoted by X^*. A continuous and linear map T from X to Y is called an operator, and the adjoint of T is denoted by T^*. Also $L(X,Y)$ will stand for the set of all operators .

If T is an operator from X to Y and $T(\sum x_n)$ is an unconditionally convergent series in Y whenever the series $\sum x_n$ in X is weakly unconditionally converging, then T is called an unconditionally converging operator. A completely continuous operator is an operator $T : X \to Y$ which takes weakly null sequences in X into norm convergent sequences in Y.

Recall that a series $\sum x_n$ is weakly unconditionally converging if and only if $\sum |x^*(x_n)| < \infty$ for each $x^* \in X^*$ and the series $\sum x_n$ is unconditionally converging if and only if every rearrangment converges in the norm topology of X.

Suppose S is a bounded set in X, then it is stated that S is weakly precompact if every single sequence from S includes a weakly Cauchy subsequence. Rosentlal's ℓ_1 Theorem states that "A Banach space X does not contain copies of ℓ_1 if and only if B_X is a weakly precompact set in X."

The reader is referred to Diestel [3] or Dunford-Schwartz [4] for undefined notation and terminology.

2. Main Results

Let K be a bounded set in X^*, then K is said to be an L-set, if for all weakly null sequence (x_n) in X we have the following ,

$$\limsup_n \left\{ \left| x^*(x_n) \right| : x^* \in K \right\} = 0,$$

e.g., see [5] [6] [7]. Moreover, any L-subset of X^* is relatively compact if and only if ℓ_1 does not embed in X [5] [6].

Closely related to the notion of L-set is the idea of a V-set. Let K be a bounded set in X^*, then K is called a V-set, if for all weakly unconditionally converging series $\sum x_n$ in X, we have the following

$$\limsup_n \left\{ \left| x^*(x_n) \right| : x^* \in K \right\} = 0.$$

It can easily be derived directly from the definitions that every L-subset of X^* is a V-subset of X^*. Next, equivalent characterizations of Banach spaces such that the converse statement holds are defined.

Initially, new property for Banach spaces will be introduced.

Definition 2.1. Let X be a Banach space. Then we say that X has property (MB) if each V-set in X^* is also an L-set in X^*.

Theorem 3.1 (ii) in [2], plays a consistent and vital position in this research. It states that; A necessary and sufficient condition for an operator $T : X \to Y$ to be completely continuous is that $T^* \left(B_{Y^*} \right)$ is an L-subset of X^*.

In the following, necessery and sufficient condition are given that every V-subset of X^* is an L-subset of X^*.

Now, the main theorem, which establishes Banach spaces with property (MB) will be determined as the resulting.

Theorem 2.2. The following statements are equivalent about a Banach space X.

(i) X has property (MB).

(ii) Let Y be any Banach space, then an operator T from X to Y is completely continuous whenever T is unconditionally converging.

(iii) If an operator T from X to ℓ_∞ is unconditionally converging, then it is also completely continuous.

Proof. $(i) \Rightarrow (ii)$ Assume that Y is a Banach space and operator $T : X \to Y$ is unconditionally converging. Now let the series $\sum x_n$ in X be weakly unconditionally converging. Thus $\sum T(x_n)$ is an unconditionally converging in Y and $\left\| T(x_n) \right\| \to 0$. Hence for each weakly unconditionally converging series $\sum x_n$ in X, $\lim_n \left\| T(x_n) \right\| = 0$, and thus we have the following,

$$\limsup_n \left\{ \left| T^* y^* (x_n) \right| : y^* \in B_{Y^*} \right\} = 0.$$

This follows that $T^* \left(B_{Y^*} \right)$ is a V-set in X^*, and as X has property (MB), then it is also an L-set in X^*. Therefore the operator T is completely continuous [2].

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Assume that A is a V-set in X^* and let $\left(x_n^* \right)$ be some sequence in A. Let us define $T : \ell_1 \to X^*$ by $T(b) = \sum_n b_n x_n^*$ and suppose that the series $\sum x_n$ is weakly unconditionally converging in X. See that $T^* \big|_X : X \to \ell_\infty$, and $T^*(x) = \left(x_i^* (x) \right)$. As A is a V-set in, we have the following,

$$\limsup_n \left| x_i^* (x_n) \right| = 0.$$

So $\lim_n \left\| T^*(x_n) \right\| = 0$, which implies that the operator $T^* \big|_X$ is unconditionally converging. Finally, by assumption we conclude that $T^* \big|_X$ is a completely continuous operator. Now assume (x_n) is a weakly null sequence in X and $y \in B_{\ell_1}$. Then $\left| T(y)(x_n) \right| = \left| T^*(x_n)(y) \right| \le \left\| T^*(x_n) \right\| \to 0$. Hence $T\left(B_{\ell_1} \right)$ is an L-set in X^* which follows that A is also an L-set in X^*. Therefore X has property (MB) and the proof is complete. □

Remembrance that if for every Banach space Y, each operator $T : X \to Y$ is completely continuous whenever T is weakly compact operator, then X is said to have the Dunford-Pettis property. Equivalently, X has Dunford-Pettis property if and only if $\left(x_n^* (x_n) \right) \to 0$ whenever $\left(x_n^* \right)$ is weakly null in X^* and (x_n) is weakly Cauchy in X [8]. A Banach space X has property (V) if each operator $T : X \to Y$ is weakly compact for every Banach space Y whenever T is unconditionally converging [9].

Examples of Banach spaces which satisfy both property (V) and the Dunford-Pettis property are c_0 and l^∞.

Theorem 2.2 has some corollaries which they are proved at this time. The first part is connected to Dunford-Pettis property.

Corollary 2.3. *Let Y be a Banach space. Then the following is given:*

(i) X has the Dunford-Pettis property provided that X has property (MB).

(ii) X has the Dunford-Pettis property if X^* has property (MB).

(iii) X has property (MB) whenever X has property (V) and the Dunford-Pettis property.

Proof. (i) First we assume that the operator $T : X \to Y$ is weakly compact. Then it is easily seen that T is an unconditionally converging operator. Now Theorem 2.2 implies that the operator T is completely continuous, as X has property (MB). Thus the proof is complete and X has the Dunford-Pettis property.

(ii) Note that (i) concludes that X^* has the Dunford-Pettis property, as it has property (MB). Hence X has the Dunford-Pettis property [8] and we get the result.

(iii) Let the operator $T : X \to Y$ be unconditionally converging. Now as X has property (MB) and the Dunford-Pettis property, then T is weakly compact which implies that it is also completely continuous. Finally Theorem 2.2 gives the result, that is, X has property (MB). $\square$

Note that ℓ^∞ has both the Dunford-Pettis property and property (V) which implies that it also has property (MB). The converse of Corollary 2.3 (iii) does not hold. In general, it is not true that if X has property (MB), then it also has property (V). For example, ℓ^1 does not have property (V) although it has property (MB) (otherwise its dual ℓ^∞ would be weakly sequentially complete [9] which contradicts that ℓ^∞ contains a copy of c_0).

A Banach space X is said to have the Reciprocal Dunford-Pettis property if for every Banach space Y, every completely continuous operator $T : X \to Y$ is weakly compact. In [2], Bator, Lewis, and Ochoa, showed that property (V) implies the Reciprocal Dunford-Pettis property, but generally, the converse of this fact is not true. To see, this inference could be reversed, we begin with a space X containing property (MB).

Corollary 2.4. *If X has property (MB) and the Reciprocal Dunford-Pettis property, then X has property (V).* $\square$

Proof. Suppose $T : X \to Y$ is an unconditionally converging operator. Since X has property (MB), then by Theorem 2.2, T is completely continuous. As X has the Reciprocal Dunford-Pettis property, then T is weakly compact. Hence X has property (V).

Soyabs, in [10], introduced a property called the (D) property. A Banach space X has the (D) property if every linear operator $T : Y \to X^*$ is weakly compact for every Banach space Y whose dual does not contain an isomorphic copy of ℓ_∞. If Banach space X has (V) property, then it has the (D) property. Also, E. Saab and P. Saab [11] have introduced the property (W). A Banach space X has the (W) property if

every operator $T : X \to X^{*}$ is weakly compact. If Banach space X has (D) property, then it has the (W) property [10], therefore, we immediatly have the following result.

Corollary 2.5. *Let X be a Banach space with the Reciprocal Dunford-Pettis property and the property (MB). Then we have the following:*

(i) Every operator $T : Y \to X^{*}$ is weakly compact for every Banach space Y whose dual does not contain an isomorphic copy of ℓ_{∞}.

(ii) Every operator $T : X \to X^{*}$ is weakly compact.

Remembering that, Banach spaces which does not contain a copy of ℓ_{1}, have the Reciprocal Dunford-Pettis property. Certainly, Odell's result ([12], p. 377) implies that, for any Banach space X, each operator T from X to Y is compact provided it is completely continuous if and only if ℓ_{1} does not embed in X. Also, Rosenthal and Dor's Theorem [3], if X is a Banach space and (x_{n}) is a sequence in B_{X} such that fails to have a weak Cauchy subsequence, then (x_{n}) has a subsequence which is equivalent to the unit vector of ℓ_{1}, is required for the following consequence. Note that Banach space c has the property (MB) but does not contain ℓ_{1}, see [13] (Corollary 2.3 (i)).

Theorem 2.6. *Suppose that X has property (MB). Then we have the following:*

(i) If X does not contain ℓ_{1}, then every unconditionally converging operator $T : X \to Y$ is compact for every Banach space Y.

(ii) If Y does not contain ℓ_{1}, then every operator $T : X \to Y^{*}$ is completely continuous.

Proof. (i) Suppose $T : X \to Y$ is an unconditionally converging operator. By Theorem 2.2, T is completely continuous, since X has property (MB). As X does not contain ℓ_{1}, T is a compact operator ([12], p. 377).

(ii) Suppose by contradiction that $T : X \to Y^{*}$ is not completely continuous operator. Let (x_{n}) be a weakly null sequence in X and $\|Tx_{n}\| \geq \varepsilon$ for some $\varepsilon > 0$. Choose $(y_{n}) \in B_{Y}$ so that $(Tx_{n})(y_{n}) > \varepsilon/2$. Since $\ell_{1} \nrightarrow Y$, by Rosenthal-Dor Theorem, we may assume that (y_{n}) is weakly Cauchy. By corollary 2.3, X has the Dunford-Pettis property, hence $(T^{*}y_{n})(x_{n}) \to 0$, that is a contradiction. $\square$

Theorem 2.6 has an immediate consequence; thus the proof is omitted.

Corollary 2.7. *Suppose that X has property (MB), then every operator $T : X \to Y$ is completely continuous for every separable dual space Y.*

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